Research Article

Sharp Power Mean Bounds for the One-Parameter Harmonic Mean

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We present the best possible parameters $\alpha = \alpha(r)$ and $\beta = \beta(r)$ such that the double inequality $M_\alpha(a, b) < H_r(a, b) < M_\beta(a, b)$ holds for all $r \in (0, 1/2)$ and $a, b > 0$ with $a \neq b$, where $M_p(a, b) = [(a^p + b^p)/2]^{1/p}$ ($p \neq 0$) and $M_0(a, b) = \sqrt{ab}$ and $H_r(a, b) = 2[ra + (1-r)b][rb + (1-r)a]/(a+b)$ are the power and one-parameter harmonic means of a and b , respectively.

1. Introduction

For $p \in \mathbb{R}$ and $a, b > 0$, the p th power mean $M_p(a, b)$ of a and b is defined by

$$M_p(a, b) = \left(\frac{a^p + b^p}{2} \right)^{1/p} \quad (1)$$

$$(p \neq 0), \quad M_0(a, b) = \sqrt{ab}.$$

It is well known that $M_p(a, b)$ is strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$, symmetric and homogeneous of degree 1. Many classical means are special cases of the power mean: for example, $M_{-1}(a, b) = 2ab/(a+b) = H(a, b)$ is the harmonic mean, $M_0(a, b) = \sqrt{ab} = G(a, b)$ is the geometric mean, $M_1(a, b) = (a+b)/2 = A(a, b)$ is the arithmetic mean, and $M_2(a, b) = \sqrt{(a^2 + b^2)/2} = Q(a, b)$ is the quadratic mean. The main properties of the power mean are given in [1]. Recently, the power mean has attracted the attention of many researchers. In particular, many remarkable inequalities for the power mean can be found in the literature [2–10].

Let $L(a, b) = (b-a)/(\log b - \log a)$, $P(a, b) = (a-b)/[2 \arcsin((a-b)/(a+b))]$, $I(a, b) = 1/e(b^b/a^a)^{1/(b-a)}$, $T(a, b) = (a-b)/[2 \arctan((a-b)/(a+b))]$, and $C(a, b) = (a^2 + b^2)/(a+b)$ be the logarithmic, first Seiffert, identric, second Seiffert, and contraharmonic means of two distinct

positive real numbers a and b , respectively. Then it is well known that the inequalities

$$H(a, b) < G(a, b) < L(a, b) < P(a, b) < I(a, b) < A(a, b) < T(a, b) < Q(a, b) < C(a, b) \quad (2)$$

hold for all $a, b > 0$ with $a \neq b$.

Lin [11] proved that the double inequality

$$M_p(a, b) < L(a, b) < M_q(a, b) \quad (3)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq 0$ and $q \geq 1/3$.

In [12], Pittenger presented the best possible parameters $\lambda = \lambda(p)$ and $\mu = \mu(p)$ such that the double inequality

$$M_\lambda(a, b) < L_p(a, b) < M_\mu(a, b) \quad (4)$$

holds for all $a, b > 0$ with $a \neq b$, where $L_p(a, b) = [(b^{p+1} - a^{p+1})/((p+1)(b-a))]^{1/p}$ ($p \neq 0, -1$), $L_0(a, b) = I(a, b)$, and $L_{-1}(a, b) = L(a, b)$ is the generalized logarithmic mean of a and b .

Jagers [13] and Seiffert [14] proved that the double inequalities

$$M_{1/2}(a, b) < P(a, b) < M_{2/3}(a, b),$$

$$M_1(a, b) < T(a, b) < M_2(a, b) \quad (5)$$

hold for all $a, b > 0$ with $a \neq b$.

In [15, 16], the authors proved that the double inequalities

$$M_{\log 2/(1+\log 2)}(a, b) < \frac{I(a, b) + L(a, b)}{2} < M_{1/2}(a, b), \quad (6)$$

$$M_0(a, b) < \sqrt{I(a, b)L(a, b)} < M_{1/2}(a, b)$$

hold for all $a, b > 0$ with $a \neq b$.

Costin and Toader [17] proved that the double inequality

$$M_{4/3}(a, b) < T(a, b) < M_{5/3}(a, b) \quad (7)$$

holds for all $a, b > 0$ with $a \neq b$.

In [18–20], the authors proved that the double inequalities

$$M_p(a, b) < P(a, b) < M_q(a, b), \quad (8)$$

$$M_r(a, b) < T(a, b) < M_s(a, b)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $p \leq \log \pi / \log 2$, $q \geq 2/3$, $r \leq \log 2 / (\log \pi - \log 2)$, and $s \geq 5/3$.

Čizmesija [21] proved that $p = -\alpha/2$ and $q = \log 2 / [\log 4 - \log(1-\alpha)]$ are the best possible parameters such that the double inequality $M_p(a, b) < [H(a, b) + \alpha G(a, b) + (1-\alpha)A(a, b)]/2 < M_q(a, b)$ holds for all $\alpha \in (0, 1)$ and $a, b > 0$ with $a \neq b$.

In [22, 23], the authors proved that the inequalities

$$\begin{aligned} M_p[\mathcal{K}(r), \mathcal{E}(r)] &> \frac{\pi}{2}, \\ M_\lambda[\mathcal{K}(r), \mathcal{K}(\sqrt{1-r^2})] \\ &\leq \mathcal{K}\left(\frac{\sqrt{2}}{2}\right) \leq M_\mu[\mathcal{K}(r), \mathcal{K}(\sqrt{1-r^2})] \end{aligned} \quad (9)$$

hold for all $r \in (0, 1)$ if and only if $p \geq -1/2$, $\lambda \leq \log 2 / (\log \pi - \log 2) - \log[\mathcal{K}(\sqrt{2}/2)] = -4.18, \dots$, and $\mu \geq 1 - 4\mathcal{K}^4(\sqrt{2}/2)/\pi^2 = -3.789, \dots$, where $\mathcal{K}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{-1/2} d\theta$ and $\mathcal{E}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{1/2} d\theta$ are, respectively, the complete elliptic integrals of the first and second kinds.

Let $p \in [0, 1]$ and N be the bivariate symmetric mean. Then, the one-parameter mean $N_p(a, b)$ was defined by Neuman [24] as follows:

$$N_p(a, b) = N[pa + (1-p)b, pb + (1-p)a]. \quad (10)$$

Let $p_1, q_1, p_2, q_2, \lambda_1, \lambda_2, \lambda_3, \lambda, \mu \in (0, 1/2)$ and $p_3, q_3 \in (1/2, 1)$. Then, the authors in [25–28] proved that the inequalities

$$\begin{aligned} H_{p_1}(a, b) &< P(a, b) < H_{q_1}(a, b), \\ L_{p_2}(a, b) &< P(a, b) < L_{q_2}(a, b), \\ H_{\lambda_1}(a, b) &> G(a, b), \\ H_{\lambda_2}(a, b) &> L(a, b), \\ G_{\lambda_3}(a, b) &> L(a, b), \\ C_{p_3}(a, b) &< T(a, b) < C_{q_3}(a, b), \\ G_\lambda(a, b) &< I(a, b) < G_\mu(a, b) \end{aligned} \quad (11)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $p_1 \leq 1/2 - \sqrt{1-2/\pi}/2$, $q_1 \geq 1/2 - \sqrt{6}/12$, $p_2 \leq \lambda_0$, $q_2 \geq 1/2 - \sqrt{2}/4$, $\lambda_1 \geq 1/2 - \sqrt{2}/4$, $\lambda_2 \geq 1/2 - \sqrt{3}/6$, $\lambda_3 \geq 1/2 - \sqrt{6}/6$, $p_3 \leq 1/2 + \sqrt{4/\pi-1}/2$, $q_3 \geq 1/2 + \sqrt{3}/6$, $\lambda \leq 1/2 - \sqrt{1-4/e^2}/2$, and $\mu \geq 1/2 - \sqrt{3}/6$, where λ_0 is the unique solution of the equation $\log[(1-\lambda)/\lambda] = \pi(1-2\lambda)$.

The main purpose of this paper is to present the best possible parameters $\alpha = \alpha(r)$ and $\beta = \beta(r)$ such that the double inequality $M_\alpha(a, b) < H_r(a, b) < M_\beta(a, b)$ holds for all $r \in (0, 1/2)$ and $a, b > 0$ with $a \neq b$.

2. Lemmas

In order to prove our main result we need three lemmas, which we present in this section.

Lemma 1. *The inequality*

$$-8r^2 + 8r - 1 + \frac{\log 2}{\log[2r(1-r)]} < 0 \quad (12)$$

holds for all $r \in (0, 1/2)$.

Proof. It is not difficult to verify that $\log[2r(1-r)] < 0$ for all $r \in (0, 1/2)$. Therefore, we only need to prove that $f(r) > 0$ for $r \in (0, 1/2)$, where $f(r) = (-8r^2 + 8r - 1) \log[2r(1-r)] + \log 2$. Simple computations lead to

$$\begin{aligned} f\left(\frac{1}{2}\right) &= 0, \\ f'(r) &= (1-2r)g(r), \end{aligned} \quad (13)$$

where

$$g(r) = 8 \log[2r(1-r)] - \frac{1}{r(1-r)} + 8, \quad (14)$$

$$g\left(\frac{1}{2}\right) = 4(1 - \log 4) < 0, \quad (15)$$

$$g'(r) = \frac{(1-2r)[8r(1-r)+1]}{r^2(1-r)^2} > 0$$

for all $r \in (0, 1/2)$.

Inequality (15) implies that $g(r) < 0$ for all $r \in (0, 1/2)$. Then, from (13) we clearly see that $f(r) > 0$ for all $r \in (0, 1/2)$. $\square$

Lemma 2. *The inequality*

$$\begin{aligned} -16r^2 + 16r - 2 - (8r^2 - 8r - 1) \frac{\log 2}{\log[2r(1-r)]} \\ + \left[\frac{\log 2}{\log[2r(1-r)]} \right]^2 < 0 \end{aligned} \quad (16)$$

holds for all $r \in (0, 1/2)$.

Proof. Let $p = -\log 2 / \log[2r(1-r)]$ and $q = -8r^2 + 8r - 1$. Then, it is not difficult to verify that

$$0 < p < 1 \quad (17)$$

for all $r \in (0, 1/2)$.

It follows from Lemma 1 that

$$p > q \quad (18)$$

for all $r \in (0, 1/2)$.

Inequalities (17) and (18) lead to

$$\begin{aligned} & -16r^2 + 16r - 2 - (8r^2 - 8r - 1) \frac{\log 2}{\log [2r(1-r)]} \\ & + \left[\frac{\log 2}{\log [2r(1-r)]} \right]^2 = 2q - (q+2)p + p^2 \quad (19) \\ & = -(p-q)(2-p) < 0 \end{aligned}$$

for all $r \in (0, 1/2)$. $\square$

Lemma 3. *The inequality*

$$\begin{aligned} & -6r^2 + 6r - (12r^2 - 12r + 1) \frac{\log 2}{\log [2r(1-r)]} \\ & + \left[\frac{\log 2}{\log [2r(1-r)]} \right]^2 > 0 \quad (20) \end{aligned}$$

holds for all $r \in (0, 1/2)$.

Proof. Let $p = -\log 2 / \log [2r(1-r)]$ and $q = -8r^2 + 8r - 1$. Then, it follows from (17) and (18) that

$$\begin{aligned} & -6r^2 + 6r - (12r^2 - 12r + 1) \frac{\log 2}{\log [2r(1-r)]} \\ & + \left[\frac{\log 2}{\log [2r(1-r)]} \right]^2 = \frac{3}{4}(q+1) \\ & + \left[1 - \frac{3}{2}(q+1) \right] p + p^2 = \frac{3}{4}(q+1)(1-2p) \quad (21) \\ & + p(p+1) > \frac{3}{4}(q+1)(1-2p) + p(q+1) \\ & = \frac{1}{4}(q+1)(3-2p) = 2r(1-r)(3-2p) > 0 \end{aligned}$$

for all $r \in (0, 1/2)$. $\square$

3. Main Results

Theorem 4. *The double inequality*

$$M_\alpha(a, b) < H_r(a, b) < M_\beta(a, b) \quad (22)$$

holds for all $r \in (0, 1/2)$ and $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq -8r^2 + 8r - 1$ and $\beta \geq -\log 2 / \log [2r(1-r)]$.

Proof. Without loss of generality, we assume that $a = x \in (1, \infty)$ and $b = 1$. Let $\log[H_r(x, 1)] - \log[M_p(x, 1)] = f(x)$, where

$$\begin{aligned} f(x) = \log & \frac{r(1-r)x^2 + (1-2r+2r^2)x + r(1-r)}{1+x} \\ & + \frac{(1+p)\log 2 - \log(1+x^p)}{p}. \quad (23) \end{aligned}$$

Then, simple computations lead to

$$f(1) = 0, \quad (24)$$

$$\begin{aligned} f'(x) & = \frac{g(x)}{[r(1-r)x^2 + (1-2r+2r^2)x + r(1-r)](1+x)(1+x^p)x^{1-p}}, \quad (25) \end{aligned}$$

where

$$\begin{aligned} g(x) = & (-r^2 + r)x^{3-p} + (-2r^2 + 2r)x^{2-p} \\ & + (3r^2 - 3r + 1)x^{1-p} \\ & + (-3r^2 + 3r - 1)x^2 + (2r^2 - 2r)x \\ & + r^2 - r, \quad (26) \end{aligned}$$

$$g(1) = 0, \quad (27)$$

$$g'(1) = -8r^2 + 8r - 1 - p, \quad (28)$$

$$g''(1) = -16r^2 + 16r - 2 + p(8r^2 - 8r - 1) + p^2, \quad (29)$$

$$g'''(x) = (1-p)x^{-p-2}h(x), \quad (30)$$

where

$$\begin{aligned} h(x) = & (-r^2 + r)(3-p)(2-p)x^2 \\ & - (-2r^2 + 2r)p(2-p)x \\ & + (3r^2 - 3r + 1)p(p+1), \quad (31) \end{aligned}$$

$$h(1) = 6(-r^2 + r) + p(12r^2 - 12r + 1) + p^2, \quad (32)$$

$$h'(x) = 2(r-r^2)(2-p)[(3-p)x - p]. \quad (33)$$

We divide the proof into two cases.

Case 1 ($p = -8r^2 + 8r - 1$). We divide the discussion into two subcases.

Subcase 1.1 ($p = -8r^2 + 8r - 1$ and $r \in (0, (2 - \sqrt{2})/4) \cup (2 - \sqrt{2})/4, 1/2)$). Then, we clearly see that $p \in (-1, 0) \cup (0, 1)$, and (28), (29), (32), and (33) lead to

$$g'(1) = g''(1) = 0, \quad (34)$$

$$h(1) = \frac{1}{4}(p+1)(3-2p) > 0, \quad (35)$$

$$\begin{aligned} h'(x) = & \frac{1}{4}(p+1)(2-p)[(3-p)x - p] \\ & > \frac{1}{4}(p+1)(2-p)(3-2p) > 0 \end{aligned} \quad (36)$$

for $x \in (1, \infty)$.

It follows easily from (24), (25), (27), (30), and (34)–(36) that

$$H_r(x, 1) > M_{-8r^2+8r-1}(x, 1) \quad (37)$$

for all $r \in (0, 1/2)$.

Subcase 1.2 ($p = -8r^2 + 8r - 1$ and $r = (2 - \sqrt{2})/4$). Then, $p = 0$, and (37) follows from

$$\begin{aligned} H_r(x, 1) - M_{-8r^2+8r-1}(x, 1) \\ = \frac{x^2 + 6x + 1}{4(1+x)} - \sqrt{x} = \frac{(\sqrt{x} - 1)^2}{4(1+x)}. \end{aligned} \quad (38)$$

Case 2 ($p = -\log 2 / \log[2r(1-r)]$). Then, we clearly see that $p \in (0, 1)$, and Lemmas 1–3 and (23), (26), (28), (29), and (32) lead to

$$\lim_{x \rightarrow +\infty} f(x) = 0, \quad (39)$$

$$\lim_{x \rightarrow +\infty} g(x) = +\infty, \quad (40)$$

$$\lim_{x \rightarrow +\infty} g'(x) = +\infty, \quad (41)$$

$$g'(1) < 0, \quad (42)$$

$$\lim_{x \rightarrow +\infty} g''(x) = +\infty, \quad (43)$$

$$g''(1) < 0, \quad (44)$$

$$h(1) > 0 \quad (45)$$

and (36) again holds.

It follows from (30), (36), and (45) that g'' is strictly increasing on $(1, \infty)$. Then, (43) and (44) lead to the conclusion that there exists $\mu_0 > 1$ such that g' is strictly decreasing on $(1, \mu_0]$ and strictly increasing on $[\mu_0, \infty)$.

From (41) and (42) together with the piecewise monotonicity of g' we clearly see that there exists $\mu_1 > 1$ such that g is strictly decreasing on $(1, \mu_1]$ and strictly increasing on $[\mu_1, \infty)$. Then, (25), (27), and (40) lead to the conclusion that there exists $\mu_2 > 1$ such that f is strictly decreasing on $(1, \mu_2]$ and strictly increasing on $[\mu_2, \infty)$. Therefore,

$$H_r(x, 1) < M_{-\log 2 / \log[2r(1-r)]}(x, 1) \quad (46)$$

for all $r \in (0, 1/2)$ follows from (24) and (39) together with the piecewise monotonicity of f .

Next, we prove that $\alpha = -8r^2 + 8r - 1$ and $\beta = -\log 2 / \log[2r(1-r)]$ are the best possible parameters such that the double inequality

$$M_\alpha(a, b) < H_r(a, b) < M_\beta(a, b) \quad (47)$$

holds for all $r \in (0, 1/2)$ and $a, b > 0$ with $a \neq b$.

Let $r \in (0, 1/2)$, $\alpha_0 = -8r^2 + 8r - 1$, $\beta_0 = -\log 2 / \log[2r(1-r)]$, $\varepsilon \in (0, \beta_0)$, and $t > 0$. Then, we have

$$\begin{aligned} H_r(1, 1+t) - M_{\alpha_0+\varepsilon}(1, 1+t) \\ = \frac{2[(1-r)(1+t)+r][r(1+t)+1-r]}{t+2} \end{aligned} \quad (48)$$

$$\begin{aligned} - \left[\frac{1 + (1+t)^{\alpha_0+\varepsilon}}{2} \right]^{1/(\alpha_0+\varepsilon)}, \\ \lim_{t \rightarrow +\infty} \frac{H_r(1, t)}{M_{\beta_0-\varepsilon}(1, t)} = 2^{\varepsilon/[\beta_0(\beta_0-\varepsilon)]} > 1. \end{aligned} \quad (49)$$

Let $t \rightarrow 0^+$ and make use of the Taylor expansion; then, (48) leads to

$$H_r(1, 1+t) - M_{\alpha_0+\varepsilon}(1, 1+t) = -\frac{\varepsilon}{8}x^2 + o(x^2). \quad (50)$$

Inequality (49) and equation (50) imply that for any $r \in (0, 1/2)$ and $\varepsilon \in (0, \beta_0)$ there exist $T = T(\varepsilon) > 1$ and $\delta = \delta(\varepsilon) \in (0, 1)$ such that $H_r(1, t) > M_{\beta_0-\varepsilon}(1, t)$ for $t > T$ and $H_r(1, 1+t) < M_{\alpha_0+\varepsilon}(1, 1+t)$ for $t \in (0, \delta)$. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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