Research Article

On Interval-Valued Pseudolinear Functions and Interval-Valued Pseudolinear Optimization Problems

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Some basic characterizations of an interval-valued pseudolinear function are derived. By means of the properties of interval-valued pseudolinearity, a class of interval-valued pseudolinear programs is considered, and the solution set of the interval-valued pseudolinear optimization problem is characterized.

1. Introduction

In recent years, many approaches for interval-valued optimization problems have been explored in considerable details; see examples in [1–3]. Several authors have been interested in the optimality conditions and duality results for the interval-valued optimization problems. Wu has extended the concept of convexity for real-valued functions to LU-convexity for interval-valued functions, and then he has established the Karush-Kuhn-Tucker (KKT) optimality conditions [4] and duality theory [5] for an optimization problem with an interval-valued objective function under the assumption of LU-convexity. Sun and Wang [6] have derived the Fritz John type and Kuhn-Tucker type necessary and sufficient optimality conditions for nondifferentiable interval-valued programming. Jayswal et al. [7] have discussed Mond-Weir and Wolfe type duality theorems for the interval-valued programming problems under the conditions of generalized convexity. Chalco-Cano et al. [8] have obtained KKT optimality conditions for interval-valued programming problems by using a more general concept of gH-derivative. Bhurjee and Panda [9, 10] have introduced the parametric form of interval-valued functions and studied the solution of convex interval-valued programming problems. Jana and Panda [11] have studied the preferable efficient solutions of the problem of interval-valued vector optimization. Zhang et al. [12] have

extended the concepts of preinvexity and invexity to interval-valued functions and derived the KKT optimality conditions for LU-preinvex and invex optimization problems with an interval-valued objective function.

Interval-valued linear optimization [13, 14] is a class of important and simple interval-valued optimization problems. Recently, Hladík [15] has proposed a method for testing basis stability of interval-valued linear optimization problems; Hladík's studies have shown that if some basis stability criterion holds true, then the problems become much more easy to solve. In [16], Hladík has discussed lower and upper bound approximation for the best case optimal value of the interval-valued linear optimization problems and proposed an algorithm for computing the best case optimal value. Li et al. [17, 18] and Luo et al. [19] have established some necessary and sufficient conditions for checking weak and strong optimality of given feasible solutions for the interval-valued linear optimization problems.

In this paper, we consider and give some characterizations for a class of generalized interval-valued linear function, which is interval-valued pseudolinear function. Then, by means of the basic properties of interval-valued pseudolinearity, a class of interval-valued pseudolinear programs is considered, and the solution set of the interval-valued pseudolinear optimization problem is characterized. It can be shown that the interval-valued linear optimization problems

[13, 14] and the interval-valued linear fractional programs are included in interval-valued pseudolinear programs. These programs arise in many practical applications.

2. Preliminaries

In this section, we recall some basic concepts with regard to interval-valued functions.

Let R^n be n -dimensional Euclidean space, and let R_+^n be its nonnegative orthant. Let us denote by $\mathcal{I}$ the class of all closed intervals in R . $A = [a^L, a^U] \in \mathcal{I}$ denotes a closed interval, where a^L and a^U mean the lower and upper bounds of A , respectively. The closed interval A also can be expressed in terms of a parameter in several ways. Throughout this paper, we consider specific parametric representations of the interval as $A = [a^L, a^U] = \{a(t) = a^L + t(a^U - a^L) \mid t \in [0, 1]\}$.

Let $A = [a^L, a^U] = \{a(t_1) \mid t_1 \in [0, 1]\}$ and $B = [b^L, b^U] = \{b(t_2) \mid t_2 \in [0, 1]\}$ be in $\mathcal{I}$, and the interval operations can be performed with respect to parameters [9, 10] as follows:

- (a) $A \oplus B = \{a(t_1) + b(t_2) \mid t_1, t_2 \in [0, 1]\}$;
- (b) $A \ominus B = \{a(t_1) - b(t_2) \mid t_1, t_2 \in [0, 1]\}$;
- (c) $A \odot B = \{a(t_1) \cdot b(t_2) \mid t_1, t_2 \in [0, 1], \text{ and } kA = \{ka(t_1) \mid t_1 \in [0, 1]\}\}$;
- (d) $A \oslash B = \{a(t_1)/b(t_2) \mid t_1, t_2 \in [0, 1], b(t_2) \neq 0\}$.

Let $\mathbf{C}_v^k$ be k dimensional column whose elements are intervals. That is, $\mathbf{C}_v^k \in \mathcal{I}^k$, $\mathbf{C}_v^k = (\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_k)^T$, and $\mathbf{C}_j = [c_j^L, c_j^U] = c_j^L + t_j(c_j^U - c_j^L)$, $t_j \in [0, 1]$, $j = 1, 2, \dots, k$. The interval-valued function in parametric form is as follows.

Definition 1 (see [9]). For $c(t) = (c_1(t_1), c_2(t_2), \dots, c_k(t_k))^T \in \mathbf{C}_v^k$, let $f_{c(t)} : R^n \rightarrow R$. For a given interval vector $\mathbf{C}_v^k$, an interval-valued function $\mathbf{F}_{\mathbf{C}_v^k} : R^n \rightarrow \mathcal{I}$ is defined as

$$\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) = \{f_{c(t)}(\mathbf{x}) \mid f_{c(t)} : R^n \rightarrow R, c(t) \in \mathbf{C}_v^k\}. \quad (1)$$

For every fixed $\mathbf{x}$, if $f_{c(t)}(\mathbf{x})$ is continuous in t , then $\min_{t \in [0, 1]^k} f_{c(t)}(\mathbf{x})$ and $\max_{t \in [0, 1]^k} f_{c(t)}(\mathbf{x})$ exist. In that case

$$\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) = \left[\min_{t \in [0, 1]^k} f_{c(t)}(\mathbf{x}), \max_{t \in [0, 1]^k} f_{c(t)}(\mathbf{x}) \right]. \quad (2)$$

If $f_{c(t)}(\mathbf{x})$ is linear in t , then $\min_{t \in [0, 1]^k} f_{c(t)}(\mathbf{x})$ and $\max_{t \in [0, 1]^k} f_{c(t)}(\mathbf{x})$ exist in the set of vertices of $\mathbf{C}_v^k$. If $f_{c(t)}(\mathbf{x})$ is monotonically increasing in t , then $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) = [f_{c(0)}(\mathbf{x}), f_{c(1)}(\mathbf{x})]$.

The interval-valued function $\mathbf{F}_{\mathbf{C}_v^k} : R^n \rightarrow \mathcal{I}$ is differentiable at $\mathbf{x}^*$ if $f_{c(t)}(\mathbf{x})$ is differentiable at $\mathbf{x}^*$ for every $t \in [0, 1]^k$. The partial derivatives of $\mathbf{F}_{\mathbf{C}_v^k}$ at $\mathbf{x}^*$ may be calculated as follows:

$$\begin{aligned} & \frac{\partial \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}^*)}{\partial x_i} \\ &= \left\{ \frac{\partial f_{c(t)}(\mathbf{x}^*)}{\partial x_i} \mid \text{for every } t \in [0, 1]^k, c(t) \in \mathbf{C}_v^k \right\}. \end{aligned} \quad (3)$$

If $f_{c(t)}(\mathbf{x}^*)$ is continuous in t , then

$$\begin{aligned} & \frac{\partial \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}^*)}{\partial x_i} \\ &= \left[\min_{t \in [0, 1]^k} \frac{\partial f_{c(t)}(\mathbf{x}^*)}{\partial x_i}, \max_{t \in [0, 1]^k} \frac{\partial f_{c(t)}(\mathbf{x}^*)}{\partial x_i} \right]. \end{aligned} \quad (4)$$

The gradient of $\mathbf{F}_{\mathbf{C}_v^k}$ at $\mathbf{x}^*$ is an interval vector:

$$\begin{aligned} & \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}^*) \\ &= \left(\frac{\partial \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}^*)}{\partial x_1}, \frac{\partial \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}^*)}{\partial x_2}, \dots, \frac{\partial \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}^*)}{\partial x_n} \right)^T. \end{aligned} \quad (5)$$

In this paper, we denote the point $\lambda \mathbf{y} + (1 - \lambda)\mathbf{x}$, $\lambda \in [0, 1]$, by $\mathbf{x}\lambda\mathbf{y}$. For the two interval-valued functions of $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) = \{f_{c(t_1)}(\mathbf{x}) \mid f_{c(t_1)} : R^n \rightarrow R, c(t_1) \in \mathbf{C}_v^k\}$ and $\mathbf{G}_{\mathbf{C}_v^k}(\mathbf{x}) = \{g_{c(t_2)}(\mathbf{x}) \mid g_{c(t_2)} : R^n \rightarrow R, c(t_2) \in \mathbf{C}_v^k\}$, we say that $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) = \mathbf{G}_{\mathbf{C}_v^k}(\mathbf{x})$ or $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) \leq \mathbf{G}_{\mathbf{C}_v^k}(\mathbf{x})$, which means

$$\begin{aligned} & f_{c(t_1)}(\mathbf{x}) = g_{c(t_2)}(\mathbf{x}) \\ & \text{or } f_{c(t_1)}(\mathbf{x}) \leq g_{c(t_2)}(\mathbf{x}) \end{aligned} \quad (6)$$

for all $t_1, t_2 \in [0, 1]^k$ and t_1 may or may not be equal to t_2 .

However, we say that $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) =_w \mathbf{G}_{\mathbf{C}_v^k}(\mathbf{x})$ or $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) \leq_w \mathbf{G}_{\mathbf{C}_v^k}(\mathbf{x})$, which means

$$\begin{aligned} & f_{c(t_1)}(\mathbf{x}) = g_{c(t_2)}(\mathbf{x}) \\ & \text{or } f_{c(t_1)}(\mathbf{x}) \leq g_{c(t_2)}(\mathbf{x}) \end{aligned} \quad (7)$$

for all $t_1 = t_2 \in [0, 1]^k$; that is to say, t_1 and t_2 are the same in both sides.

Definition 2 (see [9]). Suppose that $X \subseteq R^n$ is a convex set and for given $\mathbf{C}_v^k \in \mathcal{I}^k$, $\mathbf{F}_{\mathbf{C}_v^k} : R^n \rightarrow \mathcal{I}$. For $\mathbf{x}_1, \mathbf{x}_2 \in X$, $0 \leq \lambda \leq 1$.

(i) $\mathbf{F}_{\mathbf{C}_v^k}$ is said to be convex with respect to $\leq$ if

$$\mathbf{F}_{\mathbf{C}_v^k}(\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2) \leq \lambda \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}_1) \oplus (1 - \lambda) \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}_2), \quad (8)$$

which means

$$\begin{aligned} & f_{c(t')}(\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2) \\ & \leq \lambda f_{c(t'')}(\mathbf{x}_1) + (1 - \lambda) f_{c(t''')}(\mathbf{x}_2) \end{aligned} \quad (9)$$

for all $t', t'' \in [0, 1]^k$; t' may or may not be equal to t'' .

(ii) $\mathbf{F}_{\mathbf{C}_v^k}$ is said to be convex with respect to $\leq_w$ if

$$\begin{aligned} & \mathbf{F}_{\mathbf{C}_v^k}(\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2) \\ & \leq_w \lambda \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}_1) \oplus (1 - \lambda) \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}_2), \end{aligned} \quad (10)$$

which means

$$f_{c(t)}(\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2) \leq \lambda f_{c(t)}(\mathbf{x}_1) + (1 - \lambda) f_{c(t)}(\mathbf{x}_2) \quad (11)$$

for all $t \in [0, 1]^k$; t is same in both sides.

It can be shown that $\mathbf{F}_{\mathbf{C}_v^k}$ is convex [9] with respect to $\leq_w$ if and only if $f_{c(t)}(\mathbf{x})$ is a convex function on X for every t . Similar result does not hold for convexity with respect to $\leq$.

Mangasarian [20] has introduced the following pseudoconvex real-valued functions.

Definition 3 (see [20]). The differentiable real-valued function $f : X \rightarrow \mathcal{R}$ on the open set $X \subseteq \mathcal{R}^n$ is pseudoconvex on X if

$$\mathbf{x}, \mathbf{y} \in X, \quad (\mathbf{x} - \mathbf{y})^T \nabla f(\mathbf{y}) \geq 0 \implies f(\mathbf{x}) - f(\mathbf{y}) \geq 0; \quad (12)$$

or equivalently,

$$\mathbf{x}, \mathbf{y} \in X, \quad f(\mathbf{x}) - f(\mathbf{y}) < 0 \implies (\mathbf{x} - \mathbf{y})^T \nabla f(\mathbf{y}) < 0. \quad (13)$$

Furthermore, we say that f is strictly pseudoconvex on X if

$$\mathbf{x}, \mathbf{y} \in X, \quad f(\mathbf{x}) - f(\mathbf{y}) \leq 0 \implies (\mathbf{x} - \mathbf{y})^T \nabla f(\mathbf{y}) < 0. \quad (14)$$

We can also define the following pseudoconvex interval-valued function.

Definition 4. A differentiable interval-valued function $\mathbf{F}_{\mathbf{C}_v^k} : X \rightarrow \mathcal{I}$, on the convex set $X \subseteq \mathcal{R}^n$ for given $\mathbf{C}_v^k \in \mathcal{I}^k$, is pseudoconvex with respect to $\leq_w$ on X if

$$\mathbf{x}, \mathbf{y} \in X, \quad (\mathbf{x} - \mathbf{y})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{y}) \succeq_w 0 \implies \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) \succeq_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{y}); \quad (15)$$

or equivalently,

$$\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) \prec_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{y}) \implies (\mathbf{x} - \mathbf{y})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{y}) \prec_w 0. \quad (16)$$

An interval-valued function $\mathbf{F}_{\mathbf{C}_v^k}$ is pseudoconcave if $-\mathbf{F}_{\mathbf{C}_v^k}$ is pseudoconvex with respect to $\leq_w$ on X .

Remark 5. It is worth pointing out that the definition of pseudoconvex interval-valued function in [4] has some limitations, which define the pseudoconvex interval-valued function by using the endpoint functions f^L and f^U of the interval-valued function $f(\mathbf{x}) = [f^L, f^U]$. However, in some case, if the two endpoint functions f^L and f^U are all real-valued pseudoconvex functions, the interval-valued function may be not a pseudoconvex interval-valued function, which can be seen in the following example.

Example 6. The interval-valued function $f(x) = [f^L(x), f^U(x)]$, $x \in (0, 1)$, $f^L(x) = \min\{f_1(x), f_2(x)\}$, and $f^U(x) = \max\{f_1(x), f_2(x)\}$, where $f_1(x) = \log x$ and $f_2(x) = -2x^2$.

It can be shown that $f_1(x) = \log x$ is strictly increasing function, being $(f_1(x))' = 1/x > 0$, for all $x \in (0, 1)$;

therefore, $(y - x)(f_1(x))' \geq 0 \Leftrightarrow y - x \geq 0$. Thus, $y - x \geq 0 \Leftrightarrow y \geq x \Rightarrow f_1(y) \geq f_1(x)$. So $f_1(x)$ is pseudoconvex.

The function $f_2(x) = -2x^2$ is strictly decreasing on $(0, 1)$, as $(f_2(x))' = -4x < 0$. We have $(y - x)(f_2(x))' \geq 0 \Leftrightarrow y - x \leq 0$; $y - x \leq 0 \Leftrightarrow y \leq x \Rightarrow f_2(x) \leq f_2(y)$, so $f_2(x)$ is pseudoconvex.

The sum is $z(x) = \lambda f^L(x) + (1 - \lambda) f^U(x)|_{\lambda=1/2} \in [f^L(x), f^U(x)]$ and $z(x) = (1/2)(\log x - 2x^2)$, $1 > x > 0$. We have $z'(x) = (1/2)(\log x - 2x^2)' = (1 - 4x^2)/2x$, $z'(x) \geq 0 \Leftrightarrow 1 - 4x^2 \geq 0 \Rightarrow x \leq 1/2$. Therefore, $z(x)$ has a maximum point at $x = 1/2$, so it is not pseudoconvex.

For a real-valued differentiable function, which is said to be a pseudolinear function [21–23], if it is both pseudoconvex and pseudoconcave, similar to the case of real-valued functions, we can also define the following interval-valued pseudolinear functions.

Definition 7. A differentiable interval-valued function $\mathbf{F}_{\mathbf{C}_v^k} : X \rightarrow \mathcal{I}$, on the convex set $X \subseteq \mathcal{R}^n$ for given $\mathbf{C}_v^k \in \mathcal{I}^k$, is pseudolinear with respect to $\leq_w$ on X if it is both pseudoconvex and pseudoconcave on X .

This class of pseudolinear interval-valued functions includes many useful functions. For example, the following function is a pseudolinear interval-valued function.

Example 8. The interval-valued function $\mathbf{F}_{\mathbf{C}_v^2} = (1 + 2t_1)x + (1 + t_2)\ln x : X \rightarrow \mathcal{I}$, where $t = (t_1, t_2)^T$, $t_i \in [0, 1]$, $i = 1, 2$, and $X = \{x \in \mathcal{R} \mid x > 0\}$. It can be shown that this interval-valued function is a pseudolinear interval-valued function.

3. Characterizations of Pseudolinear Interval-Valued Functions

In this section, we provide some characterizations of the pseudolinear interval-valued function.

Theorem 9. Let a differentiable interval-valued function $\mathbf{F}_{\mathbf{C}_v^k} : X \rightarrow \mathcal{I}$, on the convex set $X \subseteq \mathcal{R}^n$ for given $\mathbf{C}_v^k \in \mathcal{I}^k$. Then, the following statements (i)–(iii) are equivalent.

- (i) $\mathbf{F}_{\mathbf{C}_v^k}$ is pseudolinear over X .
- (ii) For any $\mathbf{x}$ and $\mathbf{y}$ in X , $(\mathbf{y} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) =_w 0$ if and only if $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{y}) =_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})$.
- (iii) There exists an interval-valued function $\mathbf{P}_{\mathbf{C}_v^k}$ defined on $X \times X$ such that $\mathbf{P}_{\mathbf{C}_v^k}(\mathbf{x}, \mathbf{y}) \succ_w 0$ for given $\mathbf{C}_v^k \in \mathcal{I}^k$ and

$$\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{y}) =_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) \oplus \mathbf{P}_{\mathbf{C}_v^k}(\mathbf{x}, \mathbf{y}) (\mathbf{y} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) \quad (17)$$

for any $\mathbf{x}$ and $\mathbf{y}$ in X .

Proof. (i) $\Rightarrow$ (ii) Suppose that $\mathbf{F}_{\mathbf{C}_v^k}$ is pseudolinear interval-valued function for given $\mathbf{C}_v^k \in \mathcal{I}^k$ on X . It can be shown that $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{y}) =_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})$ implies $(\mathbf{y} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) =_w 0$.

Let $\mathbf{x}$ and $\mathbf{y}$ be two points of X such that $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{y}) =_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})$. For any $\lambda \in (0, 1)$, we denote the point $\lambda\mathbf{y} + (1 - \lambda)\mathbf{x}$ by $\mathbf{x}\lambda\mathbf{y}$, and it can be shown that $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\mathbf{y}) =_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})$.

If $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\mathbf{y}) >_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})$, then $(\mathbf{x} - \mathbf{x}\lambda\mathbf{y})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\mathbf{y}) <_w 0$ since $\mathbf{F}_{\mathbf{C}_v^k}$ is pseudoconvex for given $\mathbf{C}_v^k \in \mathcal{J}^k$. But $\mathbf{y} - \mathbf{x}\lambda\mathbf{y} = -(1 - \lambda)(\mathbf{x} - \mathbf{x}\lambda\mathbf{y})/\lambda$, and then $(\mathbf{y} - \mathbf{x}\lambda\mathbf{y})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\mathbf{y}) >_w 0$. According to the pseudoconvexity of $\mathbf{F}_{\mathbf{C}_v^k}$, then $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{y}) \succeq_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\mathbf{y})$, which contradicts the assumption of $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\mathbf{y}) >_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) =_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{y})$. So the assumption of $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\mathbf{y}) >_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})$ is false. Similarly, it can be shown that $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\mathbf{y}) <_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})$ is false by the pseudoconcavity of the interval-valued function $\mathbf{F}_{\mathbf{C}_v^k}$. Then, we can show that $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\mathbf{y}) =_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})$ for given $\mathbf{C}_v^k \in \mathcal{J}^k$ on X and all λ in $(0, 1)$. So

$$\lim_{\lambda \rightarrow 0^+} \frac{\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\mathbf{y}) \ominus \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})}{\lambda} =_w \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})^T (\mathbf{y} - \mathbf{x}) =_w 0. \quad (18)$$

(ii) $\Rightarrow$ (iii) Let the interval-valued function $\mathbf{P}_{\mathbf{C}_v^k}(\mathbf{x}, \mathbf{y}) : X \times Y \rightarrow \mathcal{J}$ defined on $X \times X$, if $\nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})^T (\mathbf{y} - \mathbf{x}) =_w 0$ for $\mathbf{x}, \mathbf{y}$ in X and given $\mathbf{C}_v^k \in \mathcal{J}^k$ on X , and we define $\mathbf{P}_{\mathbf{C}_v^k}(\mathbf{x}, \mathbf{y}) =_w 1$ for given $\mathbf{C}_v^k \in \mathcal{J}^k$ on X ; if $\nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})^T (\mathbf{y} - \mathbf{x}) \neq_w 0$ for $\mathbf{x}, \mathbf{y}$ in X and given $\mathbf{C}_v^k \in \mathcal{J}^k$ on X , we define

$$\begin{aligned} \mathbf{P}_{\mathbf{C}_v^k}(\mathbf{x}, \mathbf{y}) &= \frac{\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{y}) \ominus \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})}{\nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})^T (\mathbf{y} - \mathbf{x})} \\ &= \left\{ \frac{f_{c(t)}(\mathbf{y}) - f_{c(t)}(\mathbf{x})}{\nabla f_{c(t)}(\mathbf{x})^T (\mathbf{y} - \mathbf{x})} \mid t \in [0, 1]^k, c(t) \in \mathbf{C}_v^k \right\}. \end{aligned} \quad (19)$$

We can show that $\mathbf{P}_{\mathbf{C}_v^k}(\mathbf{x}, \mathbf{y}) >_w 0$ for given $\mathbf{C}_v^k \in \mathcal{J}^k$.

Suppose that $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{y}) >_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})$ for given $\mathbf{C}_v^k \in \mathcal{J}^k$. For some $\lambda \in (0, 1)$, if $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\mathbf{y}) \leq_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})$, using the continuity of $\mathbf{F}_{\mathbf{C}_v^k}$, there exists $r \in (0, 1)$ such that $\lambda \leq r$ and $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}r\mathbf{y}) =_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})$. By (ii),

$$\lim_{\lambda \rightarrow 0^+} \frac{\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}r\mathbf{y}) \ominus \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})}{r} =_w \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})^T (\mathbf{y} - \mathbf{x}) =_w 0, \quad (20)$$

which contradicts the assumption of $\nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})^T (\mathbf{y} - \mathbf{x}) \neq_w 0$ for $\mathbf{x}, \mathbf{y}$ in X and given $\mathbf{C}_v^k \in \mathcal{J}^k$ on X . So $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\mathbf{y}) >_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})$ for given $\mathbf{C}_v^k \in \mathcal{J}^k$ and all $\lambda \in (0, 1)$; we also have

$$\lim_{\lambda \rightarrow 0^+} \frac{\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\mathbf{y}) \ominus \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})}{\lambda} =_w \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})^T (\mathbf{y} - \mathbf{x}) \succeq_w 0, \quad (21)$$

and since $\nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})^T (\mathbf{y} - \mathbf{x}) \neq_w 0$ for $\mathbf{x}, \mathbf{y}$ in X and given $\mathbf{C}_v^k \in \mathcal{J}^k$ on X , $\nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})^T (\mathbf{y} - \mathbf{x}) >_w 0$ holds. It can be shown that

$$\mathbf{P}_{\mathbf{C}_v^k}(\mathbf{x}, \mathbf{y}) =_w \frac{\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{y}) \ominus \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})}{\nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})^T (\mathbf{y} - \mathbf{x})} >_w 0 \quad (22)$$

for given $\mathbf{C}_v^k \in \mathcal{J}^k$ on X .

Similarly, we can show that $\mathbf{P}_{\mathbf{C}_v^k}(\mathbf{x}, \mathbf{y}) >_w 0$ if $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{y}) <_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})$ for given $\mathbf{C}_v^k \in \mathcal{J}^k$.

(iii) $\Rightarrow$ (i) It is obvious that

$$\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{y}) =_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) \oplus \mathbf{P}_{\mathbf{C}_v^k}(\mathbf{x}, \mathbf{y}) (\mathbf{y} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) \quad (23)$$

for any $\mathbf{x}$ and $\mathbf{y}$ in X , which implies $(\mathbf{y} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) =_w 0$ if and only if $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{y}) =_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})$. $\square$

Theorem 10. Let $\mathbf{F}_{\mathbf{C}_v^k} : X \rightarrow \mathcal{J}$ be an interval-valued function defined on the convex set $X \subseteq \mathbb{R}^n$ for given $\mathbf{C}_v^k \in \mathcal{J}^k$, which is once continuously differentiable on X . Then $\mathbf{F}_{\mathbf{C}_v^k}$ is pseudolinear on X if and only if $(\mathbf{y} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) =_w 0$ implies $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) =_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\mathbf{y})$ for any $\mathbf{x}$ and $\mathbf{y}$ in X and any $\lambda \in [0, 1]$.

Proof. Suppose that $\mathbf{F}_{\mathbf{C}_v^k}$ is pseudolinear on X . Let $\mathbf{x}$ and $\mathbf{y}$ in X be such that $(\mathbf{y} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) =_w 0$ for given $\mathbf{C}_v^k \in \mathcal{J}^k$. Then, for any $\lambda \in [0, 1]$,

$$(\mathbf{x}\lambda\mathbf{y} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) =_w \lambda (\mathbf{y} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) =_w 0. \quad (24)$$

By (ii) of Theorem 9, $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) =_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\mathbf{y})$ for any $\mathbf{x}$ and $\mathbf{y}$ in X and any $\lambda \in [0, 1]$.

Conversely, suppose that $(\mathbf{y} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) =_w 0$ implies $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) =_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\mathbf{y})$ for any $\mathbf{x}$ and $\mathbf{y}$ in X and any $\lambda \in [0, 1]$. If $\mathbf{F}_{\mathbf{C}_v^k}$ is not pseudolinear on X , then there exist $\mathbf{x}$ and $\mathbf{y}$ in X such that $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) =_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{y})$, but $(\mathbf{y} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) \neq_w 0$, which means that $(\mathbf{y} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) =_w 0$ and $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) =_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{y})$ cannot occur. Suppose that $(\mathbf{y} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) >_w 0$, which means $(\mathbf{y} - \mathbf{x})^T \nabla f_{c(t)}(\mathbf{x}) > 0$ for $c(t) \in \mathbf{C}_v^k$. We define $f(\lambda) = \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\mathbf{y})$, which is once continuously differentiable function. According to the assumption, $f(0) = f(1)$ and $f'(0) = \{(\mathbf{y} - \mathbf{x})^T \nabla f_{c(t)}(\mathbf{x}) > 0 \mid \forall c(t) \in \mathbf{C}_v^k\}$. So $f(\lambda)$ assumes a local maximum at some point $\lambda_0 \in (0, 1)$. Then

$$f'(\lambda_0) = (\mathbf{y} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda_0\mathbf{y}) = 0. \quad (25)$$

So $(\mathbf{x}\lambda_0\mathbf{y} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda_0\mathbf{y}) = \lambda_0 (\mathbf{y} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda_0\mathbf{y}) = 0$. By the assumption, $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) =_w \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\mathbf{y})$ for all $0 \leq \lambda \leq \lambda_0$, and then $(\mathbf{y} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) = 0$, which is a contradiction. $\square$

How to check for interval pseudolinearity can be shown by the following example.

Example 11. Consider the interval-valued function on $X = \{\pi/2 > x > 0\}$:

$$\begin{aligned} \mathbf{F}_{\mathbf{C}_v^2}(\mathbf{x}) &= \{f_{c(t)}(x) = (2 + t_1)x \\ &\quad + (1 + t_2) \sin x \mid \forall t_1, t_2 \in [0, 1]\}. \end{aligned} \quad (26)$$

From (ii) of Theorem 9, for any x and y in X , $(\mathbf{y} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^2}(\mathbf{x}) =_w 0$ means $(y - x)^T \nabla f_{c(t)}(x) = (y - x)[(2 + t_1) + (1 + t_2) \cos x] = 0$ for all $t_1, t_2 \in [0, 1]$. Since $\nabla f_{c(t)}(x) = (2 + t_1) + (1 + t_2) \cos x \neq 0$, if and only if $x = y$, so, for $f_{c(t)}(x) =$

$(2+t_1)x + (1+t_2)\sin x$ and $f_{c(t)}(y) = (2+t_1)y + (1+t_2)\sin y$, we have $f_{c(t)}(x) = f_{c(t)}(y)$ for all $t_1, t_2 \in [0, 1]$, which implies $F_{C_v^k}(y) = {}_wF_{C_v^k}(x)$.

Therefore, the interval-valued function is a pseudolinear interval-valued function.

The following interval-valued linear fractional function is a class of interval-valued pseudolinear functions:

$$F_{C_v^k}(x) = \frac{A^T x + C}{B^T x + D}, \quad (27)$$

where $x \in R^n$, A and B are all in $\mathcal{J}^n$, and C and D are interval number. We can show that $F_{C_v^k}(x)$ is pseudolinear on $S_F = \{x \in R^n \mid B^T x + D \succ_w 0\}$,

$$F_{C_v^k}(y) \ominus F_{C_v^k}(x) = {}_w \frac{B^T x + D}{B^T y + D} (y - x)^T \nabla F_{C_v^k}(x), \quad (28)$$

where $P_{C_v^k}(x, y) = (B^T x + D)/(B^T y + D) \succ_w 0$ if x and y are in S_F .

For interval-valued linear fractional function, we have

$$P_{C_v^k}(x, y) P_{C_v^k}(y, z) = P_{C_v^k}(x, z) \quad (29)$$

for any x, y , and z . It can be shown that (29) implies $P_{C_v^k}(x, x) = 1$ and $P_{C_v^k}(x, y) = 1/P_{C_v^k}(y, x)$ for any x and y in S_F . We can show the following conclusion.

Theorem 12. Let a pseudolinear interval-valued function $F_{C_v^k} : X \rightarrow \mathcal{J}$, on the convex set $X \subseteq R^n$ with proportional functional $P_{C_v^k}(x, y)$ for given $C_v^k \in \mathcal{J}^k$. If $P_{C_v^k}(x, y)$ satisfies condition (29) and $P_{C_v^k}(x, \cdot)$ is differentiable for each x in X , then there exists an interval-valued linear fractional function $G_{C_v^k}$ such that $X \subset S_G$ and $G_{C_v^k}(x) = {}_wF_{C_v^k}(x)$ for all x in X .

Proof. Suppose that $F_{C_v^k}$ is not constant on X and X contains more than one point. Take y and z in X such that $F_{C_v^k}(z) \neq F_{C_v^k}(y)$. Then

$$F_{C_v^k}(x) = {}_wF_{C_v^k}(y) \oplus P_{C_v^k}(y, x) (x - y)^T \nabla F_{C_v^k}(y) \quad (30)$$

for any x and y in X .

Since $P_{C_v^k}(x, \cdot)$ is differentiable for each x in X , then

$$\begin{aligned} \nabla F_{C_v^k}(x) &= {}_wP_{C_v^k}(y, x) \nabla F_{C_v^k}(y) \\ &\oplus (x - y)^T \nabla F_{C_v^k}(y) \nabla_2 P_{C_v^k}(y, x) \end{aligned} \quad (31)$$

for any x and y in X , where $\nabla_2 P$ denotes the gradient of P with respect to the second variable.

From (30), we have

$$\begin{aligned} F_{C_v^k}(x) \ominus F_{C_v^k}(z) &= {}_wP_{C_v^k}(z, x) \nabla F_{C_v^k}(z)^T (x - z), \\ F_{C_v^k}(x) \ominus F_{C_v^k}(z) \\ &= {}_wF_{C_v^k}(y) \oplus P_{C_v^k}(y, x) \nabla F_{C_v^k}(y)^T (x - y) \\ &\ominus [F_{C_v^k}(y) \oplus P_{C_v^k}(y, z) \nabla F_{C_v^k}(y)^T (z - y)] \\ &= {}_wP_{C_v^k}(y, x) \nabla F_{C_v^k}(y)^T (x - y) \\ &\ominus P_{C_v^k}(y, z) \nabla F_{C_v^k}(y)^T (z - y). \end{aligned} \quad (32)$$

By (31), we have

$$\begin{aligned} P_{C_v^k}(z, x) \nabla F_{C_v^k}(z)^T (x - z) &= {}_wP_{C_v^k}(z, x) \\ &\cdot [P_{C_v^k}(y, z) \nabla F_{C_v^k}(y) \\ &\oplus (z - y)^T \nabla F_{C_v^k}(y) \nabla_2 P_{C_v^k}(y, z)]^T (x - z), \end{aligned} \quad (33)$$

By (32)-(33), we have

$$\begin{aligned} [P_{C_v^k}(y, x) \ominus P_{C_v^k}(y, z)] \nabla F_{C_v^k}(y)^T (z - y) \\ &= {}_wP_{C_v^k}(y, x) \nabla F_{C_v^k}(y)^T [(x - y) - (x - z)] \\ &\ominus P_{C_v^k}(y, z) \nabla F_{C_v^k}(y)^T (z - y) \\ &= {}_w[P_{C_v^k}(z, x) \nabla F_{C_v^k}(y)^T (z - y) \nabla_2 P_{C_v^k}(y, z)]^T \\ &\cdot (x - z). \end{aligned} \quad (34)$$

Since $F_{C_v^k}(z) \neq F_{C_v^k}(y)$, we have $\nabla F_{C_v^k}(y)^T (z - y) \neq 0$. From (34), we have

$$\begin{aligned} P_{C_v^k}(y, x) \ominus P_{C_v^k}(y, z) \\ &= {}_w[P_{C_v^k}(z, x) \nabla_2 P_{C_v^k}(y, z)]^T (x - z). \end{aligned} \quad (35)$$

Multiplying (35) by $P_{C_v^k}(x, y)$, we obtain

$$1 \ominus P_{C_v^k}(x, z) = {}_w[P_{C_v^k}(z, y) \nabla_2 P_{C_v^k}(y, z)]^T (x - z). \quad (36)$$

So

$$\begin{aligned} P_{C_v^k}(z, x) &= {}_w \frac{1}{P_{C_v^k}(x, z)} \\ &= {}_w [1 \ominus P_{C_v^k}(z, y) \nabla_2 P_{C_v^k}(y, z)^T (x - z)]^{-1}. \end{aligned} \quad (37)$$

Substituting (37) into (30) and replacing y by z , it can be shown that

$$\begin{aligned} F_{C_v^k}(x) &= {}_wF_{C_v^k}(z) \oplus (x - z)^T \nabla F_{C_v^k}(z) \\ &\cdot [1 \ominus P_{C_v^k}(z, y) \nabla_2 P_{C_v^k}(y, z)^T (x - z)]^{-1} \end{aligned} \quad (38)$$

is an interval-valued linear fractional function $G_{C_v^k}$. $\square$

The result of Theorem 12 and (iii) of Theorem 10 can be shown by the following example.

Example 13. Consider the interval-valued linear fractional function

$$\mathbf{F}_{\mathbf{C}_v^l}(\mathbf{x}) = \frac{\mathbf{A}^T \mathbf{x} + C}{\mathbf{B}^T \mathbf{x} + D}, \quad (39)$$

where $\mathbf{x} \in R_+$, $\mathbf{A}^T = [2, 3] = 2 + t_1$, and $\mathbf{B}^T = [3, 4] = 3 + t_2$ are all in $\mathcal{J}^2$ and $C = [0, 1] = t_3$ and $D = [1, 2] = 1 + t_4$ are interval number, where $t_i \in [0, 1]$, $i = 1, 2, 3, 4$.

We denote that

$$\begin{aligned} \mathbf{P}_{\mathbf{C}_v^l}(\mathbf{x}, \mathbf{y}) &= \frac{\mathbf{B}^T \mathbf{x} + D}{\mathbf{B}^T \mathbf{y} + D} \\ &= \left\{ \frac{(3 + t_2)x + (1 + t_4)}{(3 + t_2)y + (1 + t_4)} > 0 \mid \forall t_i \in [0, 1] \right\}. \end{aligned} \quad (40)$$

It can be shown that $\mathbf{P}_{\mathbf{C}_v^l}(\mathbf{x}, \mathbf{y})$ satisfies (29), and the following is satisfied:

$$\mathbf{F}_{\mathbf{C}_v^l}(\mathbf{y}) = {}_w\mathbf{F}_{\mathbf{C}_v^l}(\mathbf{x}) \oplus \mathbf{P}_{\mathbf{C}_v^l}(\mathbf{x}, \mathbf{y}) (\mathbf{y} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^l}(\mathbf{x}) \quad (41)$$

for any $\mathbf{x}$ and $\mathbf{y}$ in R_+ . From (iii) of Theorem 9, the interval-valued function $\mathbf{F}_{\mathbf{C}_v^l}(\mathbf{x})$ is a pseudolinear interval-valued function.

According to Theorem 12, there exists an interval-valued linear fractional function $\mathbf{G}_{\mathbf{C}_v^l}(\mathbf{x}) = {}_w\mathbf{F}_{\mathbf{C}_v^l}(\mathbf{z}) \oplus (\mathbf{x} - \mathbf{z})^T \nabla \mathbf{F}_{\mathbf{C}_v^l}(\mathbf{z}) [1 \ominus \mathbf{P}_{\mathbf{C}_v^l}(\mathbf{z}, \mathbf{y}) \nabla_2 \mathbf{P}_{\mathbf{C}_v^l}(\mathbf{y}, \mathbf{z})^T (\mathbf{x} - \mathbf{z})]^{-1}$, such that $\mathbf{G}_{\mathbf{C}_v^l}(\mathbf{x}) = {}_w\mathbf{F}_{\mathbf{C}_v^l}(\mathbf{x})$.

For all $t_i \in [0, 1]$, $i = 1, 2, 3, 4$, we have

$$\begin{aligned} [1 \ominus \mathbf{P}_{\mathbf{C}_v^l}(\mathbf{z}, \mathbf{y}) \nabla_2 \mathbf{P}_{\mathbf{C}_v^l}(\mathbf{y}, \mathbf{z})^T (\mathbf{x} - \mathbf{z})]^{-1} &= \left\{ 1 \right. \\ &\quad \left. - \frac{(3 + t_2)z + (1 + t_4)}{(3 + t_2)y + (1 + t_4)} \right. \\ &\quad \left. \cdot \left[-\frac{(3 + t_2)[(3 + t_2)y + (1 + t_4)]}{[(3 + t_2)z + (1 + t_4)]^2} \right] (x - z) \mid \forall t_i \right. \\ &\quad \left. \in [0, 1] \right\} = \left\{ \frac{(3 + t_2)x + (1 + t_4)}{(3 + t_2)z + (1 + t_4)} \mid \forall t_i \in [0, 1] \right\} \\ &= {}_w\mathbf{P}_{\mathbf{C}_v^l}(\mathbf{x}, \mathbf{z}). \end{aligned} \quad (42)$$

Then, we get

$$\mathbf{G}_{\mathbf{C}_v^l}(\mathbf{x}) = {}_w\mathbf{F}_{\mathbf{C}_v^l}(\mathbf{z}) \oplus (\mathbf{x} - \mathbf{z})^T \nabla \mathbf{F}_{\mathbf{C}_v^l}(\mathbf{z}) \mathbf{P}_{\mathbf{C}_v^l}(\mathbf{x}, \mathbf{z}). \quad (43)$$

By (iii) of Theorem 9, we have $\mathbf{G}_{\mathbf{C}_v^l}(\mathbf{x}) = {}_w\mathbf{F}_{\mathbf{C}_v^l}(\mathbf{x})$.

4. The Solution Sets of Interval-Valued Pseudolinear Programs

In this section, we consider the following interval-valued programming:

$$\begin{aligned} \text{(IVOP)} \quad &\min \quad \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) \\ &\text{subject to} \quad \mathbf{x} = (x_1, \dots, x_n) \in X \subseteq R^n, \end{aligned} \quad (44)$$

where $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) : X \rightarrow \mathcal{J}$ is an interval-valued function for given $\mathbf{C}_v^k \in \mathcal{J}^k$ and the feasible set X is a convex subset of R^n . We denote the solution set of (IVOP) as

$$\bar{S} = \arg \min_{\mathbf{x} \in X} \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}), \quad (45)$$

which is nonempty. If $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})$ is pseudolinear, then the solution set $\bar{S}$ of (IVOP) is a convex set.

The following characterization of the solution set of the problem of (IVOP) can be obtained.

Theorem 14. Let $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) : X \rightarrow \mathcal{J}$ be a pseudolinear interval-valued function for given $\mathbf{C}_v^k \in \mathcal{J}^k$ and $\bar{\mathbf{x}} \in \bar{S}$. Then $\bar{S} = \tilde{S} = \hat{S} = \check{S}$, where

$$\tilde{S} = \{\mathbf{x} \in X \mid (\bar{\mathbf{x}} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) = {}_w0\}, \quad (46)$$

$$\hat{S} = \{\mathbf{x} \in X \mid (\bar{\mathbf{x}} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\bar{\mathbf{x}}) = {}_w0\}, \quad (47)$$

$$\check{S} = \{\mathbf{x} \in X \mid (\bar{\mathbf{x}} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\bar{\mathbf{x}}) = {}_w0, \forall \lambda \in [0, 1]\}. \quad (48)$$

Proof. It is obvious that $\mathbf{x} \in \bar{S}$ if and only if $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) = {}_w\mathbf{F}_{\mathbf{C}_v^k}(\bar{\mathbf{x}})$, which implies $f_{c(t)}(\mathbf{x}) = f_{c(t)}(\bar{\mathbf{x}})$ for $t \in [0, 1]^k$ and $c(t) \in \mathbf{C}_v^k$. From (ii) of Theorem 9, $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) = {}_w\mathbf{F}_{\mathbf{C}_v^k}(\bar{\mathbf{x}})$ if and only if

$$(\bar{\mathbf{x}} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) = {}_w0 = {}_w(\bar{\mathbf{x}} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\bar{\mathbf{x}}). \quad (49)$$

Therefore, $\bar{S} = \tilde{S} = \hat{S}$.

It can be seen that $\check{S} \subset \tilde{S} = \bar{S}$. Let $\mathbf{x} \in \bar{S} = \tilde{S}$, and then

$$(\bar{\mathbf{x}} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) = {}_w0. \quad (50)$$

For each $\lambda \in (0, 1)$, we get

$$(\bar{\mathbf{x}} - \mathbf{x}\lambda\bar{\mathbf{x}})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) = {}_w(1 - \lambda)(\bar{\mathbf{x}} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) = {}_w0. \quad (51)$$

Therefore, from (ii) of Theorem 9, we have

$$\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}) = {}_w\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\bar{\mathbf{x}}). \quad (52)$$

By the pseudolinearity of $\mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x})$ and (52), we obtain

$$(\bar{\mathbf{x}} - \mathbf{x})^T \nabla \mathbf{F}_{\mathbf{C}_v^k}(\mathbf{x}\lambda\bar{\mathbf{x}}) = {}_w0, \quad \forall \lambda \in (0, 1). \quad (53)$$

So $\mathbf{x} \in \check{S}$ and $\bar{S} \subset \check{S}$. Thus, $\bar{S} = \check{S}$. $\square$

Corollary 15. Let $F_{C_v^k}(\mathbf{x}) : X \rightarrow \mathcal{F}$ be a pseudolinear interval-valued function for given $C_v^k \in \mathcal{F}^k$ and $\bar{\mathbf{x}} \in \bar{S}$. Then $\bar{S} = \bar{S}^* = \hat{S}^* = \check{S}^*$, where

$$\begin{aligned}\bar{S}^* &= \{\mathbf{x} \in X \mid (\bar{\mathbf{x}} - \mathbf{x})^T \nabla F_{C_v^k}(\mathbf{x}) \succeq_w 0\}, \\ \hat{S}^* &= \{\mathbf{x} \in X \mid (\bar{\mathbf{x}} - \mathbf{x})^T \nabla F_{C_v^k}(\bar{\mathbf{x}}) \succeq_w 0\}, \\ \check{S}^* &= \{\mathbf{x} \in X \mid (\bar{\mathbf{x}} - \mathbf{x})^T \nabla F_{C_v^k}(\mathbf{x}\lambda\bar{\mathbf{x}}) \succeq_w 0, \forall \lambda \in [0, 1]\}.\end{aligned}\quad (54)$$

Proof. From (46), it is clear that $\bar{S} \subset \bar{S}^*$. Let $\mathbf{x} \in \bar{S}^*$; that is, for $\mathbf{x} \in X$,

$$(\bar{\mathbf{x}} - \mathbf{x})^T \nabla F_{C_v^k}(\mathbf{x}) \succeq_w 0. \quad (55)$$

Then, from (iii) of Theorem 9 and the pseudolinearity of $F_{C_v^k}(\mathbf{x})$, we have

$$\begin{aligned}F_{C_v^k}(\bar{\mathbf{x}}) &= {}_w F_{C_v^k}(\mathbf{x}) \\ &\oplus P_{C_v^k}(\mathbf{x}, \bar{\mathbf{x}}) (\bar{\mathbf{x}} - \mathbf{x})^T \nabla F_{C_v^k}(\mathbf{x}) \succeq_w F_{C_v^k}(\mathbf{x}).\end{aligned}\quad (56)$$

Then $\mathbf{x} \in \bar{S}$, which implies $\bar{S} = \bar{S}^*$. It can be also similarly proved that $\bar{S} = \hat{S}^*$.

We can also prove that $\bar{S} = \check{S}^*$ as follows. In fact, it is clear that

$$\bar{S} = \check{S} \subset \check{S}^* \subset \bar{S}^* = \bar{S}, \quad (57)$$

where $\check{S}$ is defined by (48). Thus, the conclusion follows. $\square$

Example 16. Consider the following pseudolinear interval-valued program:

$$\begin{aligned}\min \quad & F_{C_v^2}(\mathbf{x}) \\ &= \{(2+t_1)x + (1+t_2)\sin x \mid \forall t_i \in [0, 1], i = 1, 2\}\end{aligned}\quad (58)$$

subject to $x \in X \subseteq \mathbb{R}$,

where $F_{C_v^2}(\mathbf{x}) : X \rightarrow \mathcal{F}$ is an interval-valued function, $t = (t_1, t_2)^T$, $t_i \in [0, 1]$, $i = 1, 2$, and $X = \{x \in \mathbb{R} \mid \pi/2 > x > 0\}$. By Example 11, $F_{C_v^2}(\mathbf{x})$ is a pseudolinear interval-valued function and $\bar{S} = \{0\}$.

According to Theorem 14, we get $\bar{S} = \bar{S}^* = \hat{S}^* = \check{S}^*$.

5. Conclusions

In this paper, we introduced the concept of the pseudolinear interval-valued functions. We have obtained some basic characterizations of an interval-valued pseudolinear function. By means of the properties of interval-valued pseudolinearity, a class of interval-valued pseudolinear programs was considered, and the solution set of the interval-valued pseudolinear optimization problem was characterized.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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