Research Article

Hyers-Ulam Stability of the First-Order Matrix Differential Equations

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We prove the generalized Hyers-Ulam stability of the first-order linear homogeneous matrix differential equations $\vec{y}'(t) = \mathbf{A}(t)\vec{y}(t)$. Moreover, we apply this result to prove the generalized Hyers-Ulam stability of the n th order linear differential equations with variable coefficients.

1. Introduction

For a given positive integer n , let I be a nondegenerate interval of $\mathbb{R}$. We will consider the (linear) differential equation of n th order

$$\mathcal{F}(y^{(n)}, y^{(n-1)}, \dots, y', y, x) = 0 \quad (1)$$

defined on I , where $y : I \rightarrow \mathbb{C}$ is an n times continuously differentiable function.

For an arbitrary $\varepsilon > 0$, assume that an n times continuously differentiable function $y : I \rightarrow \mathbb{C}$ satisfies the differential inequality

$$|\mathcal{F}(y^{(n)}, y^{(n-1)}, \dots, y', y, x)| \leq \varepsilon \quad (2)$$

for all $x \in I$. If for each function $y : I \rightarrow \mathbb{C}$, satisfying inequality (2), there exists a solution $y_0 : I \rightarrow \mathbb{C}$ of the differential equation (1) such that

$$|y(x) - y_0(x)| \leq K(\varepsilon) \quad (3)$$

for any $x \in I$, where $K(\varepsilon)$ depends on ε only and satisfies $\lim_{\varepsilon \rightarrow 0} K(\varepsilon) = 0$, then we say that the differential equation (1) satisfies (or has) the Hyers-Ulam stability (or the local Hyers-Ulam stability if the domain I is not the whole space $\mathbb{R}$). If the above statement also holds when we replace ε and $K(\varepsilon)$ with some appropriate $\varphi(x)$ and $\Phi(x)$, respectively, then we say that the differential equation (1) satisfies the generalized

Hyers-Ulam stability (or the Hyers-Ulam-Rassias stability). For more detailed definition of the Hyers-Ulam stability, refer to [1].

Obłozza seems to be the first author who investigated the Hyers-Ulam stability of linear differential equations (see [2, 3]). Given real-valued constants a and b , let $g, r : (a, b) \rightarrow \mathbb{R}$ be continuous functions with $\int_a^b |g(x)| dx < \infty$. Assume that $\varepsilon > 0$ is an arbitrary real number. Obłozza proved that if a differentiable function $y : (a, b) \rightarrow \mathbb{R}$ satisfies the inequality $|y'(x) + g(x)y(x) - r(x)| \leq \varepsilon$ for all $x \in (a, b)$ and if a function $y_0 : (a, b) \rightarrow \mathbb{R}$ satisfies $y_0'(x) + g(x)y_0(x) = r(x)$ for all $x \in (a, b)$ and $y(\tau) = y_0(\tau)$ for some $\tau \in (a, b)$, then there exists a constant $\delta > 0$ such that $|y(x) - y_0(x)| \leq \delta$ for all $x \in (a, b)$.

Thereafter, Alsina and Ger [4] proved that if a differentiable function $y : (a, b) \rightarrow \mathbb{R}$ satisfies the differential inequality $|y'(x) - y(x)| \leq \varepsilon$, then there exists a function $y_0 : (a, b) \rightarrow \mathbb{R}$ such that $y_0'(x) = y_0(x)$ and $|y(x) - y_0(x)| \leq 3\varepsilon$ for all $x \in (a, b)$. This result of Alsina and Ger was generalized by Takahasi et al. [5]. Indeed, they proved the Hyers-Ulam stability of the Banach space valued differential equation $y'(x) = \lambda y(x)$ (see also [6–12]).

In this paper, we prove the generalized Hyers-Ulam stability of the first-order linear homogeneous matrix differential equations:

$$\vec{y}'(t) = \mathbf{A}(t)\vec{y}(t). \quad (4)$$

Moreover, we apply this result to prove the generalized Hyers-Ulam stability of the n th order linear differential equations with variable coefficients.

2. Preliminaries

In what follows, let $I = (a, b)$ be an arbitrary open interval with $-\infty \leq a < b \leq \infty$ and let n be a fixed integer larger than 1. The first-order linear inhomogeneous differential equation with matrix coefficient is of the form

$$\vec{y}'(t) = \mathbf{A}(t) \vec{y}(t) + \vec{f}(t), \quad (5)$$

where $\vec{y} : I \rightarrow \mathbb{C}^n$ is an unknown vector function to be determined, $\mathbf{A}(t)$ is an $(n \times n)$ matrix whose entries are complex-valued continuous functions on I , and $\vec{f} : I \rightarrow \mathbb{C}^n$ is a continuous column vector function.

According to [13, Section 15], if the coefficient matrix $\mathbf{A}(t)$ is an $(n \times n)$ matrix whose entries are complex-valued continuous functions, then there exist n linearly independent solutions $\vec{y}_1, \dots, \vec{y}_n$ to the homogeneous matrix differential equation (4) and each of its solutions can be written in the form

$$\vec{y}(t) = Y(t) \vec{\eta}, \quad (6)$$

where $\vec{\eta} = (\eta_1 \ \eta_2 \ \dots \ \eta_n)^{\text{tr}}$ is a constant column vector and $Y(t)$ is a fundamental matrix whose columns consist of n linearly independent solutions $\vec{y}_1(t), \dots, \vec{y}_n(t)$ to the homogeneous differential equation (4); that is, $Y(t)$ is an $(n \times n)$ matrix defined by

$$Y(t) = (\vec{y}_1(t) \ \vec{y}_2(t) \ \dots \ \vec{y}_n(t)). \quad (7)$$

In view of [13, Section 16], the general solution to the inhomogeneous differential equation (5) is given by

$$\vec{y}(t) = Y(t) \vec{\eta} + Y(t) \int_c^t Y^{-1}(\tau) \vec{f}(\tau) d\tau, \quad (8)$$

where $Y(t)$ is a fundamental matrix for the corresponding homogeneous differential equation (4) and c is an arbitrarily given real number included in I .

Let $(\mathbb{C}^n, \|\cdot\|_n)$ be a complex normed space, where each of its elements is a column vector, and let $\mathbb{C}^{n \times n}$ be a vector space consisting of all $(n \times n)$ complex matrices. We choose a norm $\|\cdot\|_{n \times n}$ on $\mathbb{C}^{n \times n}$ which is compatible with $\|\cdot\|_n$; that is, both norms obey

$$\begin{aligned} \|\mathbf{A}\mathbf{B}\|_{n \times n} &\leq \|\mathbf{A}\|_{n \times n} \|\mathbf{B}\|_{n \times n}, \\ \|\mathbf{A}\vec{y}\|_n &\leq \|\mathbf{A}\|_{n \times n} \|\vec{y}\|_n \end{aligned} \quad (9)$$

for all $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n \times n}$ and $\vec{y} \in \mathbb{C}^n$.

Some of the most important matrix norms are induced by p -norms. For $1 \leq p \leq \infty$, the matrix norm induced by the p -norm,

$$\|\mathbf{A}\|_p := \sup_{\vec{x} \neq \vec{0}} \frac{\|\mathbf{A}\vec{x}\|_p}{\|\vec{x}\|_p}, \quad (10)$$

is called the matrix p -norm. For example, we get

$$\|\mathbf{A}\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|, \quad (11)$$

$$\|\mathbf{A}\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|.$$

It is known that the matrix p -norm, together with the p -norm, satisfies the conditions in (9), where

$$\|\vec{x}\|_1 = \sum_{j=1}^n |x_j|, \quad (12)$$

$$\|\vec{x}\|_\infty = \max_{1 \leq j \leq n} |x_j|$$

for any $\vec{x} \in \mathbb{C}^n$.

3. Generalized Hyers-Ulam Stability of (4)

We prove the main theorem of this paper concerning the generalized Hyers-Ulam stability of the first-order linear homogeneous matrix differential equation (4).

Theorem 1. Assume that $I = (a, b)$ is an arbitrary open interval with $-\infty \leq a < b \leq \infty$. Given a fixed integer $n > 1$, let $(\mathbb{C}^n, \|\cdot\|_n)$ and $(\mathbb{C}^{n \times n}, \|\cdot\|_{n \times n})$ be complex normed spaces with the property (9), whose elements are column vectors, respectively, $(n \times n)$ complex matrices. Assume that the coefficient matrix $\mathbf{A}(t)$ is an $(n \times n)$ matrix whose entries are complex-valued continuous functions on I . Let c be an arbitrary real number included in I and $\varphi : I \rightarrow [0, \infty)$ a continuous function. If a continuously differentiable vector function $\vec{y} : I \rightarrow \mathbb{C}^n$ satisfies the inequality

$$\|\vec{y}'(t) - \mathbf{A}(t) \vec{y}(t)\|_n \leq \varphi(t) \quad (13)$$

for all $t \in I$, then there exists a solution $\vec{y}_c : I \rightarrow \mathbb{C}^n$ to the homogeneous matrix differential equation (4) such that

$$\|\vec{y}(t) - \vec{y}_c(t)\|_n \leq \|Y(t)\|_{n \times n} \left\| \int_c^t Y^{-1}(\tau) \varphi(\tau) d\tau \right\| \quad (14)$$

for all $t \in I$, where $Y(t)$ is a fundamental matrix of the homogeneous matrix differential equation (4).

Proof. If we define a continuous vector function $\vec{f} : I \rightarrow \mathbb{C}^n$ by $\vec{f}(t) := \vec{y}'(t) - \mathbf{A}(t) \vec{y}(t)$, then it follows from (13) that

$$\|\vec{f}(t)\|_n = \|\vec{y}'(t) - \mathbf{A}(t) \vec{y}(t)\|_n \leq \varphi(t), \quad (15)$$

$$\vec{y}'(t) = \mathbf{A}(t) \vec{y}(t) + \vec{f}(t) \quad (16)$$

for all $t \in I$.

In view of (5) and (8), we have

$$\vec{y}(t) = Y(t) \vec{\eta} + Y(t) \int_c^t Y^{-1}(\tau) \vec{f}(\tau) d\tau \quad (17)$$

for all $t \in I$, where $\vec{\eta} = (\eta_1 \cdots \eta_n)^T$ is a constant column vector and $Y(t)$ is a fundamental matrix whose columns consist of n linearly independent solutions to the homogeneous equation (4) (see [13, Section 16]).

If we define a continuously differentiable vector function $\vec{y}_c : I \rightarrow \mathbb{C}^n$ by $\vec{y}_c(t) := Y(t)\vec{\eta}$ for each $t \in I$, then $\vec{y}_c$ is a solution to the homogeneous matrix differential equation (4) in view of (6). (We know that every function $\vec{y}_c(t) = Y(t)\vec{\eta}$ is a solution to (4).) Moreover, by (9) and (15), we have

$$\begin{aligned} \|\vec{y}(t) - \vec{y}_c(t)\|_n &= \left\| Y(t) \int_c^t Y^{-1}(\tau) \vec{f}(\tau) d\tau \right\|_n \\ &\leq \|Y(t)\|_{n \times n} \left\| \int_c^t Y^{-1}(\tau) \vec{f}(\tau) d\tau \right\|_n \\ &\leq \|Y(t)\|_{n \times n} \left| \int_c^t \|Y^{-1}(\tau)\|_{n \times n} \|\vec{f}(\tau)\|_n d\tau \right| \\ &\leq \|Y(t)\|_{n \times n} \left| \int_c^t \|Y^{-1}(\tau)\|_{n \times n} \varphi(\tau) d\tau \right| \end{aligned} \quad (18)$$

for any $t \in I$. $\square$

If we set $n = 2$, $\mathbf{A}(t) \equiv \mathbf{A}$ (a constant matrix), and $I = (a, b)$ with $-\infty \leq a < 0 < b \leq \infty$, then Theorem 1 yields the following corollary.

Corollary 2. Let $(\mathbb{C}^2, \|\cdot\|_2)$ and $(\mathbb{C}^{2 \times 2}, \|\cdot\|_{2 \times 2})$ be complex normed spaces with property (9), whose elements are column vectors, respectively, (2×2) complex matrices. Assume that the coefficient matrix $\mathbf{A}$ is given by

$$\mathbf{A} := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad (19)$$

where a_{ij} 's are complex numbers with

$$\begin{aligned} a_{11}a_{22} - a_{12}a_{21} &\neq 0, \\ \frac{(a_{11} + a_{22})^2}{a_{11}a_{22} - a_{12}a_{21}} &\neq 4. \end{aligned} \quad (20)$$

Assume that $I = (a, b)$ is an arbitrary open interval with $-\infty \leq a < b \leq \infty$ and c is an arbitrary real number included in I . Let $\varphi : I \rightarrow [0, \infty)$ be a continuous function. If a continuously differentiable vector function $\vec{y} : I \rightarrow \mathbb{C}^2$ satisfies the inequality

$$\|\vec{y}'(t) - \mathbf{A}\vec{y}(t)\|_2 \leq \varphi(t) \quad (21)$$

for all $t \in I$, then there exists a solution $\vec{y}_c : I \rightarrow \mathbb{C}^2$ to the homogeneous matrix differential equation (4) with $\mathbf{A}(t) \equiv \mathbf{A}$ such that

$$\begin{aligned} \|\vec{y}(t) - \vec{y}_c(t)\|_2 &\leq \|\mathbf{P}\|_{2 \times 2} \|\mathbf{P}^{-1}\|_{2 \times 2} \|Z(t)\|_{2 \times 2} \\ &\quad \cdot \left| \int_c^t \|Z^{-1}(\tau)\|_{2 \times 2} \varphi(\tau) d\tau \right| \end{aligned} \quad (22)$$

for all $t \in I$, where $Z(t)$ and $Z^{-1}(t)$ are given in (33) and (34).

Proof. The complex numbers

$$\begin{aligned} \lambda_1 &= \frac{a_{11} + a_{22} + \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}}{2}, \\ \lambda_2 &= \frac{a_{11} + a_{22} - \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}}{2} \end{aligned} \quad (23)$$

are the eigenvalues of $\mathbf{A}$ and the vectors

$$\begin{aligned} \vec{v}_1 &= \begin{pmatrix} 2a_{12} \\ -a_{11} + a_{22} + \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}} \end{pmatrix} \\ &= \begin{pmatrix} 2a_{12} \\ 2\lambda_1 - a_{11} \end{pmatrix}, \\ \vec{v}_2 &= \begin{pmatrix} 2a_{12} \\ -a_{11} + a_{22} - \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}} \end{pmatrix} \\ &= \begin{pmatrix} 2a_{12} \\ 2\lambda_2 - a_{11} \end{pmatrix} \end{aligned} \quad (24)$$

are the eigenvectors of $\mathbf{A}$ corresponding to λ_1 and λ_2 , respectively.

In view of [14, Theorems 1.4.5 and 7.1.4], the conditions in (20) imply that $\mathbf{A}$ is invertible and $0 \neq \lambda_1 \neq \lambda_2 \neq 0$. Hence, the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

If we set

$$\mathbf{P} := (\vec{v}_1 \quad \vec{v}_2), \quad (25)$$

then $\mathbf{P}$ is invertible and

$$\begin{aligned} \mathbf{P}^{-1} &= \frac{1}{4a_{12}\sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}} \\ &\quad \cdot \begin{pmatrix} a_{11} - a_{22} + \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}} & 2a_{12} \\ -a_{11} + a_{22} + \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}} & -2a_{12} \end{pmatrix} \\ &= \frac{1}{2a_{12}(2\lambda_1 - a_{11} - a_{22})} \begin{pmatrix} \lambda_1 - a_{22} & a_{12} \\ \lambda_1 - a_{11} & -a_{12} \end{pmatrix}. \end{aligned} \quad (26)$$

Due to a well known theorem of linear algebra, it holds that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} =: \mathbf{D}, \quad (27)$$

$$\text{or } \mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}.$$

If we define a vector function $\vec{z} : I \rightarrow \mathbb{C}^2$ by

$$\vec{z}(t) := \mathbf{P}^{-1}\vec{y}(t) \quad (28)$$

for all $t \in I$, then $\vec{z}(t)$ is continuously differentiable, and it follows from (21) and (27) that

$$\begin{aligned} \|\vec{z}'(t) - \mathbf{D}\vec{z}(t)\|_2 &= \|\mathbf{P}^{-1}\vec{y}'(t) - (\mathbf{P}^{-1}\mathbf{P})\mathbf{D}\mathbf{P}^{-1}\vec{y}(t)\|_2 \\ &= \|\mathbf{P}^{-1}\vec{y}'(t) - \mathbf{P}^{-1}\mathbf{A}\vec{y}(t)\|_2 \\ &\leq \|\mathbf{P}^{-1}\|_{2 \times 2} \|\vec{y}'(t) - \mathbf{A}\vec{y}(t)\|_2 \\ &\leq \|\mathbf{P}^{-1}\|_{2 \times 2} \varphi(t) \end{aligned} \quad (29)$$

for all $t \in I$.

According to Theorem 1, there exists a vector function $\vec{z}_c : I \rightarrow \mathbb{C}^2$ satisfying

$$\vec{z}'_c(t) = \mathbf{D}\vec{z}_c(t), \quad (30)$$

$$\begin{aligned} \|\vec{z}(t) - \vec{z}_c(t)\|_2 \\ \leq \|Z(t)\|_{2 \times 2} \left| \int_c^t \|Z^{-1}(\tau)\|_{2 \times 2} \|\mathbf{P}^{-1}\|_{2 \times 2} \varphi(\tau) d\tau \right| \end{aligned} \quad (31)$$

for every $t \in I$, where $Z(t)$ is a fundamental matrix of the homogeneous matrix differential equation (30).

If we set

$$\begin{aligned} \vec{z}_1(t) &:= \begin{pmatrix} e^{\lambda_1 t} \\ 0 \end{pmatrix}, \\ \vec{z}_2(t) &:= \begin{pmatrix} 0 \\ e^{\lambda_2 t} \end{pmatrix}, \end{aligned} \quad (32)$$

then $\vec{z}_1$ and $\vec{z}_2$ are the solutions to (30) which are linearly independent. Therefore, the fundamental matrix $Z(t)$ of the matrix differential equation (30) may be expressed as

$$Z(t) = (\vec{z}_1(t) \ \vec{z}_2(t)) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}. \quad (33)$$

Then, we have

$$Z^{-1}(t) = \begin{pmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{pmatrix} \quad (34)$$

for any $t \in I$.

If we set $\vec{y}_c(t) := \mathbf{P}\vec{z}_c(t)$ for all $t \in I$, then it follows from (27) and (30) that

$$\begin{aligned} \vec{y}'_c(t) &= \mathbf{P}\vec{z}'_c(t) = \mathbf{P}\mathbf{D}\vec{z}_c(t) = (\mathbf{P}\mathbf{D}\mathbf{P}^{-1})\mathbf{P}\vec{z}_c(t) \\ &= \mathbf{A}\vec{y}_c(t) \end{aligned} \quad (35)$$

for all $t \in I$; that is, $\vec{y}_c(t)$ is a solution to the matrix differential equation (4) with $\mathbf{A}(t) \equiv \mathbf{A}$.

Finally, it follows from (28) and (31) that

$$\begin{aligned} \|\vec{y}(t) - \vec{y}_c(t)\|_2 &= \|\mathbf{P}\vec{z}(t) - \mathbf{P}\vec{z}_c(t)\|_2 \leq \|\mathbf{P}\|_{2 \times 2} \\ &\cdot \|\vec{z}(t) - \vec{z}_c(t)\|_2 \leq \|\mathbf{P}\|_{2 \times 2} \|\mathbf{P}^{-1}\|_{2 \times 2} \|Z(t)\|_{2 \times 2} \\ &\cdot \left| \int_c^t \|Z^{-1}(\tau)\|_{2 \times 2} \varphi(\tau) d\tau \right| \end{aligned} \quad (36)$$

for all $t \in I$. □

4. Applications

We now consider the linear n th order differential equations with variable coefficients

$$\begin{aligned} y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) \\ + a_0(t)y(t) = 0, \end{aligned} \quad (37)$$

where $a_0, a_1, \dots, a_{n-1} : I \rightarrow \mathbb{C}$ are continuous functions.

In this section, we set

$$\mathbf{A}(t) := \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0(t) & -a_1(t) & -a_2(t) & -a_3(t) & \cdots & -a_{n-2}(t) & -a_{n-1}(t) \end{pmatrix} \quad (38)$$

for all $t \in I$.

Theorem 3. Assume that $I = (a, b)$ is an arbitrary open interval with $-\infty \leq a < b \leq \infty$. For a fixed integer $n > 1$, assume that continuous functions $a_0, a_1, \dots, a_{n-1} : I \rightarrow \mathbb{C}$ are given. Let c be an arbitrary real number included in I and $\varphi : I \rightarrow [0, \infty)$ a continuous function. If an n times

continuously differentiable function $y : I \rightarrow \mathbb{C}$ satisfies the inequality

$$\begin{aligned} |y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + a_{n-2}(t)y^{(n-2)}(t) + \cdots \\ + a_1(t)y'(t) + a_0(t)y(t)| \leq \varphi(t) \end{aligned} \quad (39)$$

for all $t \in I$, then there exists a solution $y_c : I \rightarrow \mathbb{C}$ to the homogeneous matrix differential equation (37) such that

$$|y(t) - y_c(t)| \leq \|U(t)\|_\infty \left| \int_c^t \|U^{-1}(\tau)\|_\infty \varphi(\tau) d\tau \right| \quad (40)$$

for all $t \in I$, where $U(t)$ is a fundamental matrix of the first-order linear homogeneous matrix differential equation (46).

Proof. If we define continuous functions $u_1, \dots, u_n : I \rightarrow \mathbb{C}$ by

$$\begin{aligned} u_1(t) &= y(t), \\ u_2(t) &= y'(t), \\ &\vdots \\ u_n(t) &= y^{(n-1)}(t), \end{aligned} \quad (41)$$

then it follows from (39) and (41) that

$$\begin{aligned} |u'_1(t) - u_2(t)| &= 0, \\ |u'_2(t) - u_3(t)| &= 0, \\ &\vdots \\ |u'_{n-1}(t) - u_n(t)| &= 0 \end{aligned} \quad (42)$$

and moreover

$$\begin{aligned} &|u'_n(t) + a_{n-1}(t)u_n(t) + a_{n-2}(t)u_{n-1}(t) + \dots \\ &+ a_1(t)u_2(t) + a_0(t)u_1(t)| = |y^{(n)}(t) \\ &+ a_{n-1}(t)y^{(n-1)}(t) + a_{n-2}(t)y^{(n-2)}(t) + \dots \\ &+ a_1(t)y'(t) + a_0(t)y(t)| \leq \varphi(t) \end{aligned} \quad (43)$$

for all $t \in I$. That is, we have

$$\|\vec{u}'(t) - \mathbf{A}(t)\vec{u}(t)\|_\infty \leq \varphi(t) \quad (44)$$

for all $t \in I$, where the coefficient matrix $\mathbf{A}(t)$ is defined in (38), the vector norm $\|\cdot\|_\infty$ is defined in (12), and the continuously differentiable vector function $\vec{u} : I \rightarrow \mathbb{C}^n$ is defined by

$$\vec{u}(t) := \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(n-1)}(t) \end{pmatrix}. \quad (45)$$

According to Theorem 1, there exists a continuously differentiable vector function $\vec{u}_c : I \rightarrow \mathbb{C}^n$ satisfying

$$\vec{u}'_c(t) = \mathbf{A}(t)\vec{u}_c(t), \quad (46)$$

$$\|\vec{u}(t) - \vec{u}_c(t)\|_\infty \leq \|U(t)\|_\infty \left| \int_c^t \|U^{-1}(\tau)\|_\infty \varphi(\tau) d\tau \right| \quad (47)$$

for all $t \in I$, where $U(t)$ is a fundamental matrix of the first-order linear homogeneous matrix differential equation (46).

We now set

$$\vec{u}_c(t) := \begin{pmatrix} u_{c,1}(t) \\ u_{c,2}(t) \\ \vdots \\ u_{c,n}(t) \end{pmatrix} \quad (48)$$

and then we consider the first $n-1$ elements of (46) to show that

$$\begin{aligned} u'_{c,1}(t) &= u_{c,2}(t), \\ u'_{c,2}(t) &= u_{c,3}(t) = u''_{c,1}(t), \\ &\vdots \\ u'_{c,n-1}(t) &= u_{c,n}(t) = u^{(n-1)}_{c,1}(t). \end{aligned} \quad (49)$$

Moreover, we use the last equality of (49) and then the last element of (46) to see

$$\begin{aligned} u'_{c,n}(t) &= u^{(n)}_{c,1}(t) \\ &= -a_0(t)u_{c,1}(t) - a_1(t)u_{c,2}(t) - \dots \\ &\quad - a_{n-2}(t)u_{c,n-1}(t) - a_{n-1}(t)u_{c,n}(t) \end{aligned} \quad (50)$$

for all $t \in I$. The continuous differentiability of $u_{c,n}$ implies that $u_{c,1}$ is n times continuously differentiable. Using (49) and (50), we further have

$$\begin{aligned} &u^{(n)}_{c,1}(t) + a_{n-1}(t)u^{(n-1)}_{c,1}(t) + \dots + a_1(t)u'_{c,1}(t) \\ &+ a_0(t)u_{c,1}(t) = 0 \end{aligned} \quad (51)$$

for every $t \in I$.

If we define a function $y_c : I \rightarrow \mathbb{C}$ by

$$y_c(t) := u_{c,1}(t), \quad (52)$$

then it follows from (51) that the n times continuously differentiable function y_c is a solution to the homogeneous matrix differential equation (37). Moreover, by (45), (47), and (48), we get

$$\begin{aligned} |y(t) - y_c(t)| &= |u_1(t) - u_{c,1}(t)| \leq \|\vec{u}(t) - \vec{u}_c(t)\|_\infty \\ &\leq \|U(t)\|_\infty \left| \int_c^t \|U^{-1}(\tau)\|_\infty \varphi(\tau) d\tau \right| \end{aligned} \quad (53)$$

for all $t \in I$. $\square$

Remark 4. Popa and Raşa [11] proved the Hyers-Ulam stability of the linear n th order differential equations with variable coefficients (37). For any continuous function $\lambda : I \rightarrow \mathbb{C}$, they introduced the differential operators D_λ and D^n by

$$\begin{aligned} D_\lambda(y)(t) &:= y'(t) + \lambda y(t), \\ D^n(y)(t) &:= y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots \\ &\quad + a_1(t)y'(t) + a_0(t)y(t) \end{aligned} \quad (54)$$

for any n times continuously differentiable function $y : I \rightarrow \mathbb{C}$. Under the assumptions that there exist continuous functions $r_1, r_2, \dots, r_n : I \rightarrow \mathbb{C}$ such that

$$D^n = D_{r_1} \circ D_{r_2} \circ \dots \circ D_{r_n} \quad (55)$$

and that $\inf_{t \in I} |r_k(t)| > 0$ for every $k \in \{1, 2, \dots, n\}$, they proved the Hyers-Ulam stability of the linear n th order differential equations (37) (see [11, Theorem 3.1]).

We proved the generalized Hyers-Ulam stability of the linear n th order differential equations (37) in Theorem 3 without additional assumptions. Based on this argument, Theorem 3 gains an advantage over [11, Theorem 3.1].

However, we remark that the hypotheses in [11, Theorem 3.1] are stronger, but the conclusions are also stronger: the Hyers-Ulam constant is described in a more explicit fashion.

Finally, we introduce an example to illustrate the usefulness of Theorem 3 for proving the generalized Hyers-Ulam stability of the second-order linear differential equation

$$y''(t) + \omega^2 y(t) = 0 \quad (56)$$

for a simple harmonic oscillator.

Example 5. Assume that $I = (a, b)$ is an arbitrary open interval with $-\infty \leq a < b \leq \infty$. Let ω be a positive real number, let c be an arbitrary real number included in I , and let $\varphi : I \rightarrow [0, \infty)$ be a continuous function such that

$$\int_c^t \varphi(\tau) d\tau \quad (57)$$

exists for each $t \in I$. If a twice continuously differentiable function $y : I \rightarrow \mathbb{C}$ satisfies the inequality

$$|y''(t) + \omega^2 y(t)| \leq \varphi(t) \quad (58)$$

for all $t \in I$, then there exists a solution $y_c : I \rightarrow \mathbb{C}$ to the second-order linear differential equation (56) such that

$$|y(t) - y_c(t)| \leq \max\{1, \omega\} \frac{\omega + 1}{\omega} \left| \int_c^t \varphi(\tau) d\tau \right| \quad (59)$$

for all $t \in I$.

Proof. If we set

$$\begin{aligned} \mathbf{A} &:= \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \\ \vec{u}(t) &:= \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}, \end{aligned} \quad (60)$$

then it follows from (58) that

$$\|\vec{u}'(t) - \mathbf{A}\vec{u}(t)\|_\infty \leq \varphi(t) \quad (61)$$

for all $t \in I$.

In view of Theorem 3, there exists a twice continuously differentiable solution $y_c : I \rightarrow \mathbb{C}$ to (56) such that

$$|y(t) - y_c(t)| \leq \|U(t)\|_\infty \left| \int_c^t \|U^{-1}(\tau)\|_\infty \varphi(\tau) d\tau \right| \quad (62)$$

for all $t \in I$, where $U(t)$ is a fundamental matrix of the matrix differential equation $\vec{u}'(t) = \mathbf{A}\vec{u}(t)$.

We know that the eigenvalues of the matrix $\mathbf{A}$ are $\lambda_1 = -i\omega$ and $\lambda_2 = i\omega$ and the eigenvectors of $\mathbf{A}$ corresponding to λ_1 and λ_2 are

$$\begin{aligned} &\begin{pmatrix} 1 \\ -i\omega \end{pmatrix}, \\ &\begin{pmatrix} 1 \\ i\omega \end{pmatrix}, \end{aligned} \quad (63)$$

respectively. If we set

$$\begin{aligned} \mathbf{P} &:= \begin{pmatrix} 1 & 1 \\ -i\omega & i\omega \end{pmatrix}, \\ \mathbf{D} &:= \begin{pmatrix} -i\omega & 0 \\ 0 & i\omega \end{pmatrix}, \end{aligned} \quad (64)$$

then $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ or $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

If we set $\vec{z}(t) := \mathbf{P}^{-1}\vec{u}(t)$, then it follows from (61) that

$$\begin{aligned} &\|\vec{z}'(t) - \mathbf{D}\vec{z}(t)\|_\infty \\ &= \|\mathbf{P}^{-1}\vec{u}'(t) - (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})\mathbf{P}^{-1}\vec{u}(t)\|_\infty \\ &\leq \|\mathbf{P}^{-1}\|_\infty \varphi(t) \end{aligned} \quad (65)$$

for all $t \in I$. It is not difficult to find a fundamental matrix $Z(t)$ of the homogeneous matrix differential equation $\vec{z}'(t) = \mathbf{D}\vec{z}(t)$:

$$Z(t) = \begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{i\omega t} \end{pmatrix}. \quad (66)$$

Using this fundamental matrix, we further get

$$U(t) = \mathbf{P}Z(t) = \begin{pmatrix} e^{-i\omega t} & e^{i\omega t} \\ -i\omega e^{-i\omega t} & i\omega e^{i\omega t} \end{pmatrix}, \quad (67)$$

$$U^{-1}(t) = \frac{1}{2i\omega} \begin{pmatrix} i\omega e^{i\omega t} & -e^{i\omega t} \\ i\omega e^{-i\omega t} & e^{-i\omega t} \end{pmatrix}.$$

Since

$$\begin{aligned} \|U(t)\|_\infty &= \max\{2, 2\omega\}, \\ \|U^{-1}(t)\|_\infty &= \frac{\omega + 1}{2\omega}, \end{aligned} \quad (68)$$

it follows from (62) that

$$|y(t) - y_c(t)| \leq \max\{1, \omega\} \frac{\omega + 1}{\omega} \left| \int_c^t \varphi(\tau) d\tau \right| \quad (69)$$

for all $t \in I$. $\square$

Conflict of Interests

The author declares that he has no competing interests.

Authors' Contribution

The author read and approved the final paper.

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