Research Article

Positive Solutions of Two-Point Boundary Value Problems for Monge-Ampère Equations

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This paper considers the following boundary value problem: $((-u'(t))^n)' = nt^{n-1}f(u(t))$, $0 < t < 1$, $u'(0) = 0$, $u(1) = 0$, where $n > 1$ is odd. We establish the method of lower and upper solutions for some boundary value problems which generalizes the above equations and using this method we present a necessary and sufficient condition for the existence of positive solutions to the above boundary value problem and some sufficient conditions for the existence of positive solutions.

1. Introduction

In [1], Kutev obtained the existence of a unique nontrivial solution of the following boundary value problem (BVP)

$$\begin{aligned} ((-u'(t))^n)' &= nt^{n-1}f(u(t)), \quad 0 < t < 1, \\ u'(0) &= 0, \\ u(1) &= 0, \end{aligned} \quad (1)$$

where $n > 1$ and $f(u) = u^p$, $\forall 0 < p \neq n$, which is related to the following Dirichlet problem of the Monge-Ampère equations in R^n

$$\begin{aligned} \det(D^2u) &= \lambda f(-u) \quad \text{in } B, \\ u &= 0 \quad \text{on } \partial B, \end{aligned} \quad (2)$$

where $B = \{x \in R^n : |x| < 1\}$ is the unit ball in R^n and $D^2u = (\partial^2 u / \partial x_i \partial x_j)$ is the Hessian of u .

After Kutev's works, the Monge-Ampère equation has attracted a growing attention in recent years because of its important role in several areas of applied mathematics. For instance, in [2], Hu and Wang obtained some sufficient conditions for the existence and multiplicity of the positive solutions for problem (1) and Dai [3] and Wang [4] discussed the unilateral global bifurcation results for the problem with $f(u) = u^n + g(u)$. And there are some results on the existence

and multiplicity and nonexistence of nontrivial radial convex solutions of systems of Monge-Ampère equations (see [5–8]) also. We notice that the nonlinearity $f(u)$ is continuous at $u = 0$ in the above works. For the case that $f(u)$ is singular at $u = 0$, there are fewer results for BVP (1). But some interesting results are presented for BVP (2) in [9–12] where $f(u)$ is singular at $u = 0$. We also refer to [7, 8, 13, 14] and references therein for further discussions regarding solutions of the Monge-Ampère equations.

One goal in this paper is to consider the existence of positive solutions under the conditions that $n > 1$ and $f(u)$ is singular at $u = 0$. Compared to the results in [10], we do not need the monotonicity of f .

Our paper is organized as follows. Section 2 lists some lemmas. In Section 3, we obtain a theorem on the upper and lower solutions which is an extension of this method to the class of problem we consider. Section 4 is devoted to our main results on some sufficient conditions for positive solutions of the BVP (1).

2. Preliminaries

Let X be the Banach space $C[0, 1]$ with $\|v\| = \max_{t \in [0, 1]} |v(t)|$. Define a set $K \subseteq X$ by

$$\begin{aligned} K &= \{v \in X : v(t) \text{ is nonincreasing in } t, v(1) \\ &= 0, v(t) \geq \|v\|(1-t) \quad \forall t \in [0, 1]\}. \end{aligned} \quad (3)$$

It can be easily verified that K is indeed a normal cone in X (see [2]).

Lemma 1 (see [15, 16]). *Let E be a Banach space, and let $D \subseteq E$ be a bounded, closed, and convex set. Assume that $F : D \rightarrow D$ is a continuous compact operator. Then, F has at least one fixed point in D .*

Remark 2. Obviously, assume that E is a Banach space, $D \subseteq E$ is a bounded, closed, and convex set and let $F : E \rightarrow D$ be a continuous compact operator. Then, F has at least one fixed point in D also.

Lemma 3. *Assume that function $v \in C[0, 1]$ with $v(t) \geq 0$, $v(0) = \|v\| = \max_{t \in [0, 1]} |v(t)|$, and $v'(t)$ decreasing in $[0, 1]$. Then,*

$$v(t) \geq (1-t)\|v\|, \quad t \in [0, 1]. \quad (4)$$

Proof. Since $v'(t)$ is decreasing in $[0, 1]$, we have

$$\begin{aligned} v(t) - v(0) &= \int_0^t v'(s) ds \geq tv'(t), \quad \forall t \in [0, 1], \\ v(1) - v(t) &= \int_t^1 v'(s) ds \leq (1-t)v'(t), \quad (5) \\ &\quad \forall t \in [0, 1], \end{aligned}$$

which implies

$$\begin{aligned} (1-t)(v(t) - v(0)) &= (1-t) \int_0^t v'(s) ds \\ &\geq (1-t)tv'(t), \quad \forall t \in [0, 1], \\ t(v(1) - v(t)) &= t \int_t^1 v'(s) ds \leq t(1-t)v'(t), \\ &\quad \forall t \in [0, 1]. \end{aligned} \quad (6)$$

Then, for $t \in [0, 1]$, we have

$$(1-t)(v(t) - v(0)) \geq t(v(1) - v(t)). \quad (7)$$

That is,

$$v(t) \geq (1-t)v(0) + tv(1). \quad (8)$$

Since $v(0) = \max_{t \in [0, 1]} |v(t)|$ and $v(1) \geq 0$, we have

$$v(t) \geq (1-t)\|v\|. \quad (9)$$

The proof is complete. $\square$

Remark 4. The idea of the proof of Lemma 3 comes from Lemma 2.2 in [2].

Now, we list some conditions for convenience:

$$(C_1)$$

$$f : (0, \infty) \rightarrow (-\infty, \infty) \text{ is continuous}; \quad (10)$$

$$(C_2) \text{ } n \text{ is odd.}$$

Throughout this paper, we always assume that condition (C_2) holds; that is to say, n is odd.

3. Upper and Lower Solutions

Consider the following two-boundary value problem:

$$\begin{aligned} \left((-v'(t))^n\right)' &= nt^{n-1}f(v(t)), \quad 0 < t < 1, \\ v'(0) &= a, \\ v(1) &= b, \end{aligned} \quad (11)$$

where $f : D \rightarrow R$ is continuous and $D \subseteq R$.

Now, we give following definitions.

Definition 5. If $\alpha \in C[0, 1] \cap C^1[0, 1) \cap C^2(0, 1)$ and

$$\begin{aligned} \left(-\alpha'(t)\right)' &\leq nt^{n-1}f(\alpha(t)), \quad 0 < t < 1, \\ \alpha'(0) &= a, \\ \alpha(1) &\leq b, \end{aligned} \quad (12)$$

then $\alpha(t)$ is called a lower solution of the BVP (11).

Similarly, we can define an upper solution of BVP (11).

Definition 6. If $\beta \in C[0, 1] \cap C^1[0, 1) \cap C^2(0, 1)$ and

$$\begin{aligned} \left(-\beta'(t)\right)' &\geq nt^{n-1}f(\beta(t)), \quad 0 < t < 1, \\ \beta'(0) &= a, \\ \beta(1) &\geq b, \end{aligned} \quad (13)$$

then $\beta(t)$ is called an upper solution of BVP (11).

If $\alpha(t) \leq \beta(t)$, define

$$D_\alpha^\beta(t) := \{x \in \mathbb{R} : \alpha(t) \leq x \leq \beta(t) \quad \forall t \in [0, 1]\}. \quad (14)$$

Theorem 7. *Let $\alpha, \beta \in C[0, 1] \cap C^1[0, 1) \cap C^2(0, 1)$, respectively, be lower and upper solutions of BVP (11) with $\alpha(t) \leq \beta(t)$ on $[0, 1]$ and $D_\alpha^\beta(t) \subseteq D$ for all $t \in (0, 1)$ and*

$$\text{there exists } h \in L^1(0, 1)$$

$$\text{with } \int_0^1 nt^{n-1}h(t) dt < \infty, \quad (15)$$

$$\text{such that } \sup_{x \in D_\alpha^\beta(t)} |f(x)| \leq h(t) \quad \text{for a.e. } t \in (0, 1)$$

holds. Then, BVP (11) has at least one solution $v \in C[0, 1] \cap C^1[0, 1) \cap C^2(0, 1)$ and

$$\alpha(t) \leq v(t) \leq \beta(t), \quad t \in [0, 1]. \quad (16)$$

Proof. Consider the modified boundary value problem of the form

$$\begin{aligned} \left((-v'(t))^n\right)' &= nt^{n-1}F(v(t)), \quad 0 < t < 1, \\ v'(0) &= a, \\ v(1) &= b, \end{aligned} \quad (17)$$

where the function

$$F(v(t)) = \begin{cases} f(\alpha(t)) + \min\{1, \alpha(t) - v(t)\}, & v(t) < \alpha(t), \\ f(v(t)), & \alpha(t) \leq v(t) \leq \beta(t), \\ f(\beta(t)) + \min\{1, v(t) - \beta(t)\}, & v(t) > \beta(t). \end{cases} \quad (18)$$

Obviously, if the problem (17) has a solution $v(t)$ with $\alpha(t) \leq v(t) \leq \beta(t)$, $t \in [0, 1]$, then problem (11) has a solution $v(t)$, $t \in [0, 1]$.

For $v \in X = C[0, 1]$, define an operator

$$(Av)(t) = b + (t-1)a + \int_t^1 \left(\int_0^s n\tau^{n-1} F(v(\tau)) d\tau \right)^{1/n} ds, \quad 0 \leq t \leq 1. \quad (19)$$

A standard argument shows that $A : X \rightarrow \{v : \|v\| \leq |a| + |b| + (\int_0^1 n\tau^{n-1} [h(\tau) + 1] d\tau)^{1/n}\}$ is continuous and compact. Lemma 1 guarantees that A has a fixed point $v \in C[0, 1] \cap C^1[0, 1] \cap C^2(0, 1)$; that is BVP (17) has a solution $v(t)$.

Now, we claim that

$$\alpha(t) \leq v(t) \leq \beta(t), \quad t \in [0, 1], \quad (20)$$

which guarantees that $v \in C[0, 1] \cap C^1[0, 1] \cap C^2(0, 1)$ is a solution of BVP (11).

Let $x(t) = \alpha(t) - v(t)$, $t \in [0, 1]$.

First, we show that

$$x(0) \leq 0, \quad \text{that is, } \alpha(0) \leq v(0). \quad (21)$$

Suppose that $x(0) > 0$. Then, $\exists t_0 > 0$ such that $x(t) > 0$, $t \in [0, t_0]$. Let $t_* = \sup\{t : x(s) > 0, s \in [0, t]\}$. Obviously, $t_* \leq 1$ and

$$\alpha(t) > v(t), \quad t \in [0, t_*), \quad x(t_*) = 0. \quad (22)$$

Let $y(t) = (-\alpha'(t))^n - (-v'(t))^n$, $t \in (0, t_*)$. Then

$$\begin{aligned} y'(t) &= \left((-\alpha'(t))^n \right)' - \left((-v'(t))^n \right)' \\ &\leq nt^{n-1} f(\alpha(t)) \\ &\quad - \left(nt^{n-1} f(\alpha(t)) + \min\{1, \alpha(t) - v(t)\} \right) \\ &< 0, \quad t \in (0, t_*), \end{aligned} \quad (23)$$

which implies that $y(t)$ is decreasing on $[0, t_*)$. And so $y(t) < y(0)$ for all $t \in (0, t_*)$; that is,

$$(-\alpha'(t))^n - (-v'(t))^n < (-\alpha'(0))^n - (-v'(0))^n = 0. \quad (24)$$

That is,

$$(-\alpha'(t))^n < (-v'(t))^n, \quad t \in (0, t_*). \quad (25)$$

From (C_2) , we have

$$-\alpha'(t) < -v'(t). \quad (26)$$

That is,

$$x'(t) = \alpha'(t) - v'(t) > 0, \quad (27)$$

which means that $x(t)$ is increasing on $[0, t_*)$. Then, $x(t) > x(0) > 0$, $t \in (0, t_*)$. The continuity of $x(t)$ at $t = t_*$ means that $x(t_*) \geq x(0) > 0$, which contradicts (22).

Secondly, we prove that

$$\alpha(t) \leq v(t), \quad \forall t \in [0, 1]. \quad (28)$$

Assume that $v(t) < \alpha(t)$ for some $t \in (0, 1)$. Since $\alpha(0) \leq v(0)$ and $\alpha(1) \leq v(1)$, there exists a $\tilde{t} \in (0, 1)$ such that $x(\tilde{t}) = \max_{t \in [0, 1]} x(t) = \max_{t \in [0, 1]} (\alpha(t) - v(t)) > 0$ with $x'(\tilde{t}) = 0$ (note $\alpha'(\tilde{t}) = v'(\tilde{t})$). Then,

$$\begin{aligned} \left((-v'(\tilde{t}))^n \right)' &= n\tilde{t}^{n-1} F(v(\tilde{t})) \\ &= n\tilde{t}^{n-1} f(\alpha(\tilde{t})) \\ &\quad + n\tilde{t}^{n-1} \min\{1, \alpha(\tilde{t}) - v(\tilde{t})\} \\ &> n\tilde{t}^{n-1} f(\alpha(\tilde{t})) \geq \left((-\alpha'(\tilde{t}))^n \right)'. \end{aligned} \quad (29)$$

From the continuity of $((-v'(t))^n)'$ and $((-\alpha'(t))^n)'$ in $(0, 1)$, there exists a $\delta > 0$ such that

$$\begin{aligned} \left((-v'(t))^n \right)' &> \left((-\alpha'(t))^n \right)', \\ t \in (\tilde{t} - \delta, \tilde{t} + \delta) &\subseteq (0, 1). \end{aligned} \quad (30)$$

Integrating both sides of the above inequality from $\tilde{t}$ to $t \in (\tilde{t}, \tilde{t} + \delta)$, we get

$$\begin{aligned} (-v'(t))^n - (-v'(\tilde{t}))^n &> (-\alpha'(t))^n - (-\alpha'(\tilde{t}))^n, \\ t \in (\tilde{t}, \tilde{t} + \delta). \end{aligned} \quad (31)$$

That is,

$$(-v'(t))^n > (-\alpha'(t))^n, \quad t \in (\tilde{t}, \tilde{t} + \delta). \quad (32)$$

From (C_2) , we have

$$-v'(t) > -\alpha'(t), \quad t \in (\tilde{t}, \tilde{t} + \delta). \quad (33)$$

And so

$$\alpha'(t) - v'(t) > 0, \quad t \in (\tilde{t}, \tilde{t} + \delta). \quad (34)$$

Then,

$$x'(t) > 0, \quad t \in (\tilde{t}, \tilde{t} + \delta), \quad (35)$$

which contradicts the fact that $x(\tilde{t}) = \max_{t \in [0, 1]} x(t)$. Hence, (28) is true.

The same argument shows that

$$v(t) \leq \beta(t), \quad \forall t \in [0, 1]. \quad (36)$$

Consequently, (28) and (36) show that (20) holds. Then, $v(t)$ is a solution to BVP (11). $\square$

Remark 8. The idea of the proof comes from [17, 18].

4. Main Results

Suppose that K is defined in Section 2. Now, we list the following condition for convenience:

(H_1) $f : (0, +\infty) \rightarrow (0, +\infty)$, continuous and $\exists \lambda, \mu, \delta, (-\infty < \lambda < 0 < \mu < 1, 0 < \delta \leq 1)$, $\forall v \in (0, +\infty)$ such that

$$\begin{aligned} c_0^\mu f(v) &\leq f(c_0 v) \leq c_0^\lambda f(v), \quad 0 \leq c_0 \leq \delta, \\ c_0^\lambda f(v) &\leq f(c_0 v) \leq c_0^\mu f(v), \quad c_0 \geq \frac{1}{\delta}. \end{aligned} \quad (37)$$

Remark 9. Assume that

$$f(v) = v^{-\alpha_1} + v^{-\alpha_2} + v^{\beta_1} + v^{\beta_2}, \quad v \in (0, +\infty), \quad (38)$$

where $\alpha_1, \alpha_2 \in (0, +\infty)$, $\beta_1, \beta_2 \in (0, 1)$ are constants. Let $\delta = 1$, $\lambda = \min\{-\alpha_1, -\alpha_2\}$, $\mu = \max\{\beta_1, \beta_2\}$. Then, if $0 \leq c_0 \leq \delta$, we have

$$\begin{aligned} f(c_0 v) &= (c_0 v)^{-\alpha_1} + (c_0 v)^{-\alpha_2} + (c_0 v)^{\beta_1} + (c_0 v)^{\beta_2} \\ &= (c_0)^{-\alpha_1} v^{-\alpha_1} + (c_0)^{-\alpha_2} v^{-\alpha_2} + (c_0)^{\beta_1} v^{\beta_1} \\ &\quad + (c_0)^{\beta_2} v^{\beta_2} \\ &\leq (c_0)^\lambda (v^{-\alpha_1} + v^{-\alpha_2} + v^{\beta_1} + v^{\beta_2}) \\ &= c^\lambda f(v), \\ f(c_0 v) &= (c_0 v)^{-\alpha_1} + (c_0 v)^{-\alpha_2} + (c_0 v)^{\beta_1} + (c_0 v)^{\beta_2} \\ &= (c_0)^{-\alpha_1} v^{-\alpha_1} + (c_0)^{-\alpha_2} v^{-\alpha_2} + (c_0)^{\beta_1} v^{\beta_1} \\ &\quad + (c_0)^{\beta_2} v^{\beta_2} \\ &\geq (c_0)^\mu (v^{-\alpha_1} + v^{-\alpha_2} + v^{\beta_1} + v^{\beta_2}) \\ &= c^\mu f(v). \end{aligned} \quad (39)$$

A similar argument shows that if $c_0 \geq 1/\delta$, we have

$$c_0^\lambda f(v) \leq f(c_0 v) \leq c_0^\mu f(v). \quad (40)$$

We have the following theorems.

Theorem 10. Suppose that (H_1) holds and $2(\mu - \lambda)/(1 - \mu) < 1$. Then, the necessary and sufficient condition of the existence of the positive solution in $C^1[0, 1] \cap C^2(0, 1)$ for BVP (1) is

$$0 < \int_0^1 t^{n-1} f(1-t) dt < \infty. \quad (41)$$

Proof.

Necessity. Suppose that $v \in C^1[0, 1] \cap C^2(0, 1)$ is a positive solution. We have

$$\begin{aligned} n(-v'(t))^{n-1}(-v''(t)) &= nt^{n-1}f(v(t)) > 0, \\ t &\in (0, 1), \\ -\infty < v'(t) &= -\left(\int_0^t n\tau^{n-1}f(v(\tau))d\tau\right)^{1/n} < 0, \\ t &\in (0, 1], \end{aligned} \quad (42)$$

$$\begin{aligned} v(t) &= \int_t^1 \left(\int_0^s n\tau^{n-1}f(v(\tau))d\tau\right)^{1/n} ds \\ &\leq (1-t) \left(\int_0^1 n\tau^{n-1}f(v(\tau))d\tau\right)^{1/n}. \end{aligned}$$

Let

$$M_v = \left(\int_0^1 n\tau^{n-1}f(v(\tau))d\tau\right)^{1/n}. \quad (43)$$

From $v(t) > 0$ and $v(1) = 0$, Lemma 3 implies that $v \in K$ and

$$\|v\|(1-t) \leq v(t) \leq M_v(1-t). \quad (44)$$

Let $c_1 > 0$ be a constant such that $c_1 M_v \leq \delta$, $c_1 \leq \delta$. By (37) and (44), we have

$$\begin{aligned} f(v(t)) &= f\left(\frac{1}{c_1} \frac{c_1 v(t)}{1-t} (1-t)\right) \\ &\geq \left(\frac{1}{c_1}\right)^\lambda f\left(\frac{c_1 v(t)}{1-t} (1-t)\right) \\ &\geq c_1^{\mu-\lambda} \left(\frac{v(t)}{1-t}\right)^\mu f(1-t) \\ &\geq c_1^{\mu-\lambda} \|v\|^\mu f(1-t), \quad t \in (0, 1). \end{aligned} \quad (45)$$

That is,

$$nt^{n-1}f(v(t)) \geq nc_1^{\mu-\lambda} \|v\|^\mu t^{n-1}f(1-t), \quad t \in (0, 1). \quad (46)$$

Since

$$(-v'(1))^n = \int_0^1 ns^{n-1}f(v(s))ds < +\infty \quad (47)$$

we have from (46)

$$\begin{aligned} 0 &< \int_0^1 nt^{n-1}f(1-t)dt \\ &\leq \int_0^1 nc_1^{\lambda-\mu} \|v\|^{-\mu} t^{n-1}f(v(t))dt \\ &= c_1^{\lambda-\mu} \|v\|^{-\mu} (-v'(1))^n < +\infty. \end{aligned} \quad (48)$$

Hence, (41) holds.

Sufficiency. Since (41) holds, set

$$h(t) := \int_t^1 \left(\int_0^s n\tau^{n-1} f(1-\tau) d\tau \right)^{1/n} ds, \quad 0 \leq t \leq 1. \quad (49)$$

Obviously $h(t) \geq 0$ and $h'(t)$ are decreasing in $[0, 1]$. Then, Lemma 3 means that $h(t) \geq \|h\|(1-t)$. From (41), we know that there exists $N > 0$ such that $\int_0^1 n\tau^{n-1} f(1-t) dt \leq N$. Consequently,

$$a_1(1-t) \leq h(t) \leq a_2(1-t), \quad (50)$$

where $a_1 = \min\{1, \|h\|\}$, $a_2 = \max\{1, N^{1/n}\}$.

Let $\alpha(t) := k_1 h(t)$ and $\beta(t) := k_2 h(t)$, where $c_2 \geq 1/a_1 \delta$ is big enough such that $k_1 = (1/2)(c_2^{\lambda-\mu} a_1^\lambda)^{1/(1-\mu)} \leq 1/2$, $k_2 = 2(c_2^{\mu-\lambda} a_1^\mu)^{1/(1-\mu)} \geq 2$. Obviously,

$$0 \leq \alpha(t) \leq \beta(t), \quad t \in [0, 1]. \quad (51)$$

Since (note (50))

$$\begin{aligned} f(c_2 h(t)) &= f\left(\frac{c_2 h(t)}{1-t}(1-t)\right) \\ &\geq c_2^\lambda \left(\frac{h(t)}{1-t}\right)^\lambda f(1-t) \geq c_2^\lambda a_1^\lambda f(1-t), \end{aligned} \quad (52)$$

$t \in (0, 1),$

we have (note $k_1/c_2 \leq \delta$)

$$\begin{aligned} f(\alpha(t)) &= f(k_1 h(t)) \geq \left(\frac{k_1}{c_2}\right)^\mu f(c_2 h(t)) \\ &\geq k_1^\mu c_2^{\lambda-\mu} a_1^\lambda f(1-t). \end{aligned} \quad (53)$$

From

$$k_1 = \frac{1}{2} (c_2^{\lambda-\mu} a_1^\lambda)^{1/(1-\mu)}, \quad (54)$$

we can obtain

$$k_1^{1-\mu} = \left(\frac{1}{2}\right)^{1-\mu} c_2^{\lambda-\mu} a_1^\lambda. \quad (55)$$

That is,

$$c_2^{\lambda-\mu} a_1^\lambda = k_1^{1-\mu} \left(\frac{1}{2}\right)^{\mu-1}. \quad (56)$$

And so (53) is

$$\begin{aligned} f(\alpha(t)) &\geq \left(\frac{1}{2}\right)^{\mu-1} k_1^{1-\mu} k_1^\mu f(1-t) \geq k_1^{1-\mu} k_1^\mu f(1-t) \\ &= k_1 f(1-t), \quad t \in (0, 1). \end{aligned} \quad (57)$$

Similarly, we have

$$\begin{aligned} f(\beta(t)) &= f(k_2 h(t)) \leq \left(\frac{1}{c_2}\right)^\lambda f(c_2 k_2 h(t)) \\ &\leq k_2 f(1-t). \end{aligned} \quad (58)$$

Differentiating $\alpha(t)$ on $(0, 1)$, we have

$$\begin{aligned} \alpha'(t) &= -k_1 \left(\int_0^t n\tau^{n-1} f(1-\tau) d\tau \right)^{1/n}, \\ (-\alpha'(t))^n &= k_1^n \int_0^t n\tau^{n-1} f(1-\tau) d\tau, \\ ((-\alpha'(t))^n)' &= k_1^n n t^{n-1} f(1-t) \leq k_1 n t^{n-1} f(1-t) \\ &\leq n t^{n-1} f(\alpha(t)), \end{aligned} \quad (59)$$

with $\alpha'(0) = 0$, $\alpha(1) = 0$, which implies that $\alpha(t)$ is a lower solution to BVP (1).

Similarly, we have

$$((-\beta'(t))^n)' \geq n t^{n-1} f(\beta(t)), \quad (60)$$

with $\beta'(0) = 0$, $\beta(1) = 0$. Hence, $\beta(t)$ is an upper solution to BVP (1).

Theorem 7 guarantees that BVP (1) has at least one positive solution in $C^1[0, 1] \cap C^2(0, 1)$. $\square$

Remark 11. Idea of the proof comes from [19].

Theorem 12. Assume that (C_1) holds and the following conditions are true:

(1) there exist $L > 0$ and $\varepsilon > 0$ such that

$$f(x) > L, \quad \forall x \in (0, \varepsilon]; \quad (61)$$

(2) there exists $M > 0$ such that

$$\sup_{x \geq M} |f(x)| < +\infty. \quad (62)$$

Then, BVP (1) has at least one positive solution $v \in C[0, 1] \cap C^1[0, 1] \cap C^2(0, 1)$. If, moreover, f is decreasing, then the solution is unique.

Proof. By assumption (1), there is $\varepsilon_m > 0$ such that

$$f(x) > L, \quad \forall x \in (0, \varepsilon_m]. \quad (63)$$

Without restriction, we can assume that $\{\varepsilon_m\}$ is a decreasing sequence and $\lim_{m \rightarrow +\infty} \varepsilon_m = 0$ with $\varepsilon_m < \min\{1, L\}$ for all $m \in \mathbb{N}$.

Let

$$\gamma(t) := \int_t^1 \left(\int_0^s n\tau^{n-1} (1-\tau) d\tau \right)^{1/n} ds, \quad t \in [0, 1]. \quad (64)$$

It is easy to see that

$$0 < \gamma(t), \quad t \in [0, 1), \quad \gamma \in C^2[0, 1]. \quad (65)$$

Let

$$k_0 := \frac{1}{2 + \|\gamma\|} \varepsilon_1, \quad (66)$$

which guarantees that

$$0 \leq k_0 \gamma(t) < \varepsilon_1, \quad t \in [0, 1]. \quad (67)$$

Set

$$\alpha_1(t) := \varepsilon_1, \quad t \in [0, 1]. \quad (68)$$

From (63), we have

$$\begin{aligned} \left((-\alpha_1'(t))^n \right)' &= 0 < nt^{n-1}L < nt^{n-1}f(\varepsilon_1) \\ &= nt^{n-1}f(\alpha_1(t)), \quad t \in (0, 1), \end{aligned} \quad (69)$$

$$\alpha_1'(0) = 0,$$

$$\alpha_1(1) = \varepsilon_1.$$

From (C₁) and condition (2), we choose an $M' > \max\{\varepsilon_1, M\}$ such that

$$\sup_{x \in [M', +\infty)} |f(x)| = R < +\infty. \quad (70)$$

Let

$$\beta_1(t) := M' + \int_t^1 \left(\int_0^s nt^{n-1} (M' + R + 1) d\tau \right)^{1/n} ds, \quad t \in [0, 1]. \quad (71)$$

Equation (70) guarantees that

$$\begin{aligned} f(\beta_1(t)) &\leq R, \quad \forall t \in [0, 1], \\ \left((-\beta_1'(t))^n \right)' &= nt^{n-1} (M' + R + 1) \\ &\geq nt^{n-1} f(\beta_1(t)), \quad t \in (0, 1), \end{aligned} \quad (72)$$

$$\beta_1'(0) = 0,$$

$$\beta_1(1) = M' \geq \varepsilon_1.$$

Now, Theorem 7 implies that there exists $u_1 \in C[0, 1] \cap C^1[0, 1] \cap C^2(0, 1)$ such that

$$\begin{aligned} \left((-u_1'(t))^n \right)' &= nt^{n-1} f(u_1(t)), \quad t \in (0, 1), \\ u_1'(0) &= 0, \\ u_1(1) &= \varepsilon_1. \end{aligned} \quad (73)$$

Let $\alpha_2(t) := \varepsilon_2$, $t \in [0, 1]$, and $\beta_2(t) := u_1(t)$, $t \in [0, 1]$. It is easy to prove that

$$\begin{aligned} \left((-\alpha_2'(t))^n \right)' &= 0 < nt^{n-1}L < nt^{n-1}f(\varepsilon_2) \\ &= nt^{n-1}f(\alpha_2(t)), \quad t \in (0, 1), \\ \alpha_2'(0) &= 0, \\ \alpha_2(1) &= \varepsilon_2, \end{aligned} \quad (74)$$

$$\begin{aligned} \left((-\beta_2'(t))^n \right)' &= nt^{n-1}f(\beta_2(t)), \quad t \in (0, 1), \\ \beta_2'(0) &= 0, \\ \beta_2(1) &= \varepsilon_1 > \varepsilon_2. \end{aligned}$$

Theorem 7 guarantees that there exists $u_2 \in C[0, 1] \cap C^1[0, 1] \cap C^2(0, 1)$ such that

$$\begin{aligned} \left((-u_2'(t))^n \right)' &= nt^{n-1}f(u_2(t)), \quad t \in (0, 1), \\ u_2'(0) &= 0, \\ u_2(1) &= \varepsilon_2. \end{aligned} \quad (75)$$

Proceeding the above proof, we can get a sequence $\{u_m(\cdot)\}$ such that

$$\begin{aligned} \varepsilon_m &\leq u_m(t) \leq u_{m-1}(t), \quad t \in [0, 1], \\ \left((-u_m'(t))^n \right)' &= nt^{n-1}f(u_m(t)), \quad t \in (0, 1), \\ u_m'(0) &= 0, \\ u_m(1) &= \varepsilon_m. \end{aligned} \quad (76)$$

Now, we claim that

$$u_m(t) \geq k_0 \gamma(t), \quad t \in [0, 1], \quad m \in \mathbb{N}. \quad (77)$$

First, we show that

$$\begin{aligned} u_m(0) &\geq k_0 \gamma(0), \\ u_m(1) &\geq k_0 \gamma(1), \\ m &\in \mathbb{N}. \end{aligned} \quad (78)$$

Obviously, $u_m(1) = \varepsilon_m > 0 = k_0 \gamma(1)$. Suppose $u_m(0) < k_0 \gamma(0)$. Let $h(t) := u_m(t) - k_0 \gamma(t)$, $t \in [0, 1]$, and $t' = \sup\{t : h(s) < 0, s \in [0, t]\}$. Obviously, $t' < 1$ (note $u_m(1) \geq k_0 \gamma(1)$) and $h(t') = 0$, $h(t) < 0$, for all $t \in [0, t']$, which implies that

$$\varepsilon_m \leq u_m(t) < k_0 \gamma(t) < \varepsilon_1, \quad t \in [0, t']. \quad (79)$$

For $t \in (0, t')$, condition (1) guarantees that

$$\begin{aligned} \left((-u_m'(t))^n \right)' &= nt^{n-1}f(u_m(t)) > nt^{n-1}L \\ &> nt^{n-1}k_0(1-t) = \left((-k_0 \gamma'(t))^n \right)'. \end{aligned} \quad (80)$$

Integrate (80) from 0 to $t \in (0, t')$ to obtain

$$\left((-u_m'(t))^n \right) > \left((-k_0 \gamma'(t))^n \right), \quad t \in (0, t'). \quad (81)$$

From (C₂), we have

$$-u_m'(t) > -k_0 \gamma'(t), \quad t \in (0, t'). \quad (82)$$

That is,

$$h'(t) = u_m'(t) - k_0 \gamma'(t) < 0, \quad t \in (0, t'), \quad (83)$$

which means that $h(t)$ is decreasing on $(0, t']$. The continuity of $h(t)$ at t' means that $h(t') < 0$. This contradicts that $h(t') = 0$. Hence, (78) is true.

Secondly, we show that

$$u_m(t) \geq k_0 \gamma(t), \quad \forall t \in [0, 1], \quad m \in \mathbb{N}. \quad (84)$$

Suppose that there is a $t \in (0, 1)$ such that $h(t) < 0$. There exists a $t_0 \in (0, 1)$ with $h(t_0) = \min_{t \in [0, 1]} h(t) < 0$ and $h'(t_0) = 0$; that is, $u'_m(t_0) = k_0 \gamma'(t_0)$. Now, $h(t_0) < 0$ implies that $\varepsilon_m \leq u_m(t_0) < k_0 \gamma(t_0) < \varepsilon_1$, which together with condition (1) guarantees that

$$\begin{aligned} \left((-u'_m(t_0))^n \right)' &= nt_0^{n-1} f(u_m(t_0)) > nt_0^{n-1} L \\ &> nt_0^{n-1} k_0 (1 - t_0) \\ &= \left((-k_0 \gamma'(t_0))^n \right)'. \end{aligned} \quad (85)$$

The continuity of $((-u'_m(t))^n)'$ and $((-k_0 \gamma'(t))^n)'$ at $t = t_0$ guarantees that there exists a $\delta > 0$ such that

$$\begin{aligned} \left((-u'_m(t))^n \right)' &> \left((-k_0 \gamma'(t))^n \right)', \\ t &\in (t_0 - \delta, t_0 + \delta). \end{aligned} \quad (86)$$

Integrate (86) from t_0 to $t \in (t_0, t_0 + \delta)$ to obtain

$$\left(-u'_m(t) \right)^n > \left(-k_0 \gamma'(t) \right)^n, \quad t \in (t_0, t_0 + \delta). \quad (87)$$

From (C_2) , we have

$$-u'_m(t) > -k_0 \gamma'(t), \quad t \in (t_0, t_0 + \delta). \quad (88)$$

That is,

$$h'(t) = u'_m(t) - k_0 \gamma'(t) < 0, \quad t \in (t_0, t_0 + \delta), \quad (89)$$

which means that $h(t)$ is decreasing on $(t_0, t_0 + \delta)$. This contradicts that $h(t_0) = \min_{t \in [0, 1]} h(t)$. Consequently, (84) is true.

Consider now the pointwise limit as follows:

$$u_*(t) := \lim_{m \rightarrow +\infty} u_m(t), \quad t \in [0, 1]. \quad (90)$$

It is clear that

$$k_0 \gamma(t) \leq u_*(t) \leq u_m(t), \quad \forall t \in [0, 1], \quad (91)$$

which means that

$$u_*(1) = 0. \quad (92)$$

For any $1 > \epsilon > 0$, from (91) and

$$\left((-u'_m(t))^n \right)' = nt^{n-1} f(u_m(t)), \quad \forall t \in (0, 1), \quad (93)$$

we have

$$\begin{aligned} \sup \{ nt^{n-1} f(x) : t \in [0, 1 - \epsilon], k_0 \gamma(t) \leq x \leq u_1(t) \} \\ < +\infty. \end{aligned} \quad (94)$$

That is, $\forall t \in [0, 1 - \epsilon]$

$$\begin{aligned} \left| \left((-u'_m(t))^n \right)' \right| &\leq \sup \{ nt^{n-1} f(x) : t \\ &\in [0, 1 - \epsilon], k_0 \gamma(t) \leq x \leq u_1(t) \} < +\infty. \end{aligned} \quad (95)$$

The Ascoli-Arzelà theorem guarantees that $\{u_m(t)\}$ converges uniformly $u_*(t)$ on $[0, 1 - \epsilon]$ and $\{u'_m(t)\}$ converges uniformly $u'_*(t)$ on $[0, 1 - \epsilon]$. Since $\epsilon > 0$ is arbitrary, we find that

$$\begin{aligned} u_* &\in C^1[0, 1], \\ \left((-u'_*(t))^n \right)' &= nt^{n-1} f(u_*(t)), \quad t \in (0, 1), \\ u'_*(0) &= 0. \end{aligned} \quad (96)$$

It remains only to prove the continuity of $u_*(t)$ at $t = 1$.

Let $\varepsilon > 0$ be given. Take $u_{m_\varepsilon}(t)$ with $u_{m_\varepsilon}(0) < \varepsilon$. Since $u_{m_\varepsilon}(t)$ is continuous at $t = 1$, we can find $\delta > 0$ such that

$$0 < u_{m_\varepsilon}(t) < \varepsilon, \quad \forall t \in [1 - \delta, 1]. \quad (97)$$

From (91), we obtain that

$$0 < u_*(t) < \varepsilon, \quad t \in [1 - \delta, 1], \quad (98)$$

which together with (92) means that $u_*(t)$ is continuous at $t = 1$. From (92) and (96), u_* is positive solution to BVP (1) in $C^1[0, 1] \cap C^2(0, 1)$.

Now, we show that the uniqueness of u_* follows when f is decreasing on $(0, +\infty)$.

Suppose that, by contradiction, v_1 and v_2 are two different solutions. Obviously

$$v_1(1) = v_2(1) = 0. \quad (99)$$

First, we show that

$$v_1(0) = v_2(0). \quad (100)$$

In fact, if $v_1(0) < v_2(0)$, let $t^* = \sup\{t : v_1(s) < v_2(s), s \in (0, t)\}$. Obviously $1 \geq t^* > 0$, $v_1(t) < v_2(t)$ for all $t \in [0, t^*]$ and $v_1(t^*) = v_2(t^*)$ (notice $v_1(1) = v_2(1) = 0$). The monotonicity of f implies that

$$\begin{aligned} \left((-v'_1(t))^n \right)' &= nt^{n-1} f(v_1(t)) > nt^{n-1} f(v_2(t)) \\ &= \left((-v'_2(t))^n \right)', \quad \forall t \in (0, t^*). \end{aligned} \quad (101)$$

Integrate (101) from 0 to t to obtain

$$\left(-v'_1(t) \right)^n > \left(-v'_2(t) \right)^n, \quad \forall t \in (0, t^*). \quad (102)$$

From (C_2) , we have

$$-v'_1(t) > -v'_2(t), \quad \forall t \in (0, t^*). \quad (103)$$

That is,

$$(v_1(t) - v_2(t))' < 0, \quad \forall t \in (0, t^*). \quad (104)$$

Then, $v_1(t) - v_2(t)$ is decreasing on $(0, t^*)$, which together with $v_1(0) - v_2(0) < 0$ means that $v_1(t^*) - v_2(t^*) < 0$. This is a contradiction. Hence, (100) is true.

Secondly, we show that

$$v_1(t) \equiv v_2(t), \quad t \in (0, 1). \quad (105)$$

Suppose that there exists $t_0 \in (0, 1)$ with $v_1(t_0) \neq v_2(t_0)$. Without loss of generality, we suppose that $v_1(t_0) < v_2(t_0)$. Let $t' = \min_{t \in [0, 1]} (v_1(t) - v_2(t))$. Equation (100) guarantees that $t' \in (0, 1)$, $v_1(t') < v_2(t')$, and $v_1'(t') = v_2'(t')$. The monotonicity of f implies that

$$\begin{aligned} ((-v_1'(t'))^n)' &= nt^{n-1} f(v_1(t')) \\ &> nt^{n-1} f(v_2(t')) = ((-v_2'(t'))^n)'. \end{aligned} \quad (106)$$

The continuity of $((-v_1'(t))^n)'$ and $((-v_2'(t))^n)'$ at $t = t'$ implies that there exists a $\delta > 0$ such that

$$\begin{aligned} ((-v_1'(t))^n)' &> ((-v_2'(t))^n)', \\ \forall t \in (t' - \delta, t' + \delta). \end{aligned} \quad (107)$$

Integrate (107) from t' to $t \in (t', t' + \delta)$ to obtain

$$(-v_1'(t))^n > (-v_2'(t))^n, \quad \forall t \in (t', t' + \delta). \quad (108)$$

Now, (C_2) implies that

$$-v_1'(t) > -v_2'(t), \quad \forall t \in (t', t' + \delta). \quad (109)$$

That is,

$$(v_1(t) - v_2(t))' < 0, \quad \forall t \in (t', t' + \delta). \quad (110)$$

Then, $v_1(t) - v_2(t)$ is decreasing on $t \in (t', t' + \delta)$, which contradicts $t' = \min_{t \in [0, 1]} (v_1(t) - v_2(t))$. Hence, (105) is true. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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