Properties of Functions in the Wiener Class $BV_p[a, b]$ for $0 < p < 1$

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We will investigate properties of functions in the Wiener class $BV_p[a, b]$ with $0 < p < 1$. We prove that any function in $BV_p[a, b]$ $(0 < p < 1)$ can be expressed as the difference of two increasing functions in $BV_p[a, b]$. We also obtain the explicit form of functions in $BV_p[a, b]$ and show that their derivatives are equal to zero a.e. on $[a, b]$.

1. Introduction

Let $0 < p < \infty$. We say that a real valued function $f$ on $[a, b]$ is of bounded $p$-variation and is denoted by $f \in BV_p[a, b]$, if

$$V_p f = \sup_T \left( \frac{1}{p} \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|^p \right)^{1/p} < \infty,$$

where the supremum is taken over all partitions $T: a = x_0 < x_1 < \cdots < x_n = b$. When $p = 1$, we get the well-known Jordan bounded variation $BV[a, b]$; and when $1 < p < \infty$, we get Wiener’s definition of bounded $p$-variation. There are many other generalizations of $BV$, such as bounded $\Phi$-variation in the sense of Young (see [1]) and Waterman’s $\Lambda$-bounded variation (see [2]). The class $BV_p$ and generalizations of $BV$ have been studied mainly because of their applicability to the theory of Fourier series and some good approximative properties (see, e.g., [1–7]).

However, it should be mentioned that results of most papers deal mostly with the case $p \geq 1$. This is because that in this case $BV_p[a, b]$ is a Banach space with the norm $\|f\|_{BV_p} = |f(a)| + V_p f$ (see, e.g., [3]). In the case $0 < p < 1$, $BV_p[a, b]$ is no longer a Banach space and has not been studied as far as we know. Nevertheless, functions in $BV_p[a, b]$ $(0 < p < 1)$ have many interesting properties; for example, their derivatives are equal to zero a.e. on $[a, b]$.

In this paper, we will investigate properties of functions in the class $BV_p[a, b]$ with $0 < p < 1$. We will show that $BV_p[a, b]$ is a Frechet space with the quasinorm

$$q(f) = |f(a)|^p + (V_p f)^p.$$

We will get the Jordan type decomposition theorem which says that any function in $BV_p[a, b]$ $(0 < p < 1)$ can be expressed as the difference of two increasing functions in $BV_p[a, b]$. We also get the representation theorem which gives the explicit form of functions in $BV_p[a, b]$ $(0 < p < 1)$.

2. Statement of Main Results

Clearly, for any fixed $p \in (0, 1)$, the Wiener class $BV_p[a, b]$ is a linear space. We define the functional $q$ on $BV_p[a, b]$ by

$$q(f) = |f(a)|^p + (V_p f)^p = |f(a)|^p + \sup_T \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|^p,$$

where $f \in BV_p[a, b]$.

From the inequality $(a + b)^p \leq a^p + b^p$ $(a, b \geq 0, 0 < p < 1)$, we get that $q(f + g) \leq q(f) + q(g)$. It then follows that $q$ is a quasinorm on $BV_p[a, b]$.

Our first result claims that $BV_p[a, b]$ $(0 < p < 1)$ equipped with the quasinorm $q$ is a Frechet space.
Theorem 1. The Wiener class $BV_p [a, b]$ $(0 < p < 1)$ equipped with the quasinorm $q$ is a Frechet space.

From the inequality
\[
\left( \sum_{i=1}^{\infty} a_i^p \right)^{1/p} \leq \left( \sum_{i=1}^{\infty} a_i^{p_1} \right)^{1/p_1}, \quad a_i \geq 0, \quad 0 < p_1 \leq p < \infty,
\]
we get that, for any $f \in BV_p [a, b],
\[
V_{p_1} f \leq V_{p_1} f,
\]
which means that $BV_{p_1} [a, b] \subseteq BV_p [a, b]$. Specially, for $0 < p < 1$, $BV_p [a, b] \subseteq BV_1 [a, b] \equiv BV [a, b]$. This implies that $BV_p [a, b]$ functions are bounded, and the discontinuities of a $BV_p [a, b]$ function are simple and, therefore, at most denumerable (see [8, Theorem 13.7 and Lemma 13.2]). By the Jordan decomposition theorem, we know that every function $f$ in $BV_p [a, b]$ can be expressed as the difference of two increasing functions $g$ and $h$ defined on $[a, b]$ (see [8, Corollary 13.6]). If $f \in BV_p [a, b] \subseteq BV [a, b]$, we can require that the above increasing functions $g$ and $h$ are still in $BV_{p_1} [a, b]$. This is our next theorem.

Theorem 2 (Jordan type decomposition theorem). Any function in $BV_p [a, b]$ $(0 < p < 1)$ can be expressed as the difference of two increasing functions in $BV_p [a, b]$.

Let $t$ be an atom, and $0 \leq d' \leq d$. Set
\[
h_{t,d,d'} (x) = \begin{cases} 
0, & x < t, \\
d', & x = t, \\
d, & x > t.
\end{cases}
\]
Then $h_{t,d,d'} (x)$ is increasing on $[a, b]$ with only one discontinuity point $t$. Also, $h_{t,d,d'} (x)' = 0$ for $x \neq t$.

Let $f$ be an increasing function in $BV_p [a, b]$ $(0 < p < 1)$. Denote by $A \equiv A(f)$ the set of points of discontinuity of $f$. Then $A$ is at most countable (see [8, Theorem 2.17]). Since $f$ is increasing, we get that, for any $t \in A$, the right and left limits $f(t+0)$ and $f(t-0)$ of the function $f$ at $t$ exist, $f(t+0) - f(t-0) > 0$, and $0 \leq f(t) - f(t-0) \leq f(t+0) - f(t-0)$. For $t \in A$, we define
\[
\tilde{h}_t (x) = \sup_{x \neq t} h_{t,d,d'} (x).
\]

Our next theorem characterizes the form of an increasing function in $BV_p [a, b]$. Any increasing function $f$ in $BV_p [a, b]$ must be as follows:
\[
f(x) = \sum_{n=1}^{N} h_{t_n,d_n,d_n'} (x) + \epsilon,
\]
where $N \leq \infty$, $t_n \in [a, b]$, $d_n > 0$, $d_n' \in [0, d_n]$, and $\sum_{n=1}^{N} d_n < \infty$.

Theorem 3. (1) If $f(x) = c + \sum_{n=1}^{N} h_{t_n,d_n,d_n'} (x)$, where $N \leq \infty$, $t_n \in [a, b]$, $d_n > 0$, and $d_n' \in [0, d_n]$, then $f \in BV_p [a, b]$ $(0 < p < 1)$ if and only if $\sum_{n=1}^{N} d_n^p < \infty$. In this case,
\[
\left( \sum_{n=1}^{N} d_n^p \right)^{1/p} \leq V_p (f) \leq \left( 2 \sum_{n=1}^{N} d_n^p \right)^{1/p} .
\]

(2) Let $f$ be an increasing function in $BV_p [a, b]$ $(0 < p < 1)$, by Theorem 3 we have $f(x) = \sum_{x \in A_1} \tilde{h}_i (x) + c$, where $A$ is the set of points of discontinuity of $f$, and $\tilde{h}_i (x)$ is defined by (7).

Finally, for an increasing function $f$ in $BV_p [a, b]$ $(0 < p < 1)$, by Theorem 3 we have $f(x) = \sum_{x \in A_1} \tilde{h}_i (x) + c$, where $A$ is the set of points of discontinuity of $f$ and at most countable. Since $(h_i (x))' = 0$, a.e. $x \in [a, b]$, by the Fubini term by term differentiation theorem (see [9, Proposition 4.6]), we get $f' (x) = 0$, a.e. $x \in [a, b]$. By Theorem 2, any function $f$ in $BV_p [a, b]$ can be expressed as the difference of two increasing functions $g(x)$ and $r(x)$ in $BV_p [a, b]$. Applying Theorem 3, we get the representation theorem of functions in $BV_p [a, b]$ $(0 < p < 1)$ as follows.

Corollary 4. Let $f \in BV_p [a, b]$ $(0 < p < 1)$. Then $f$ can be expressed in the following form:
\[
f(x) = g(x) - r(x) = \sum_{x \in A_1} \tilde{h}_g (x) - \sum_{x \in A_2} \tilde{h}_r (x) + c,
\]
where $c$ is a constant, $g(x)$, $r(x)$ are increasing functions in $BV_p [a, b]$, $\tilde{h}_g (x)$ and $\tilde{h}_r (x)$ are defined by (7), $A_1, A_2 \subseteq A$, and $A_1, A_2, A$ are the sets of points of discontinuity of $g$, $r$, and $f$, respectively. Furthermore, $f' (x) = 0$, a.e. $x \in [a, b]$.

3. Proofs of Theorems 1–3

Proof of Theorem 1. It suffices to prove that $BV_p [a, b]$ is complete. Let $\{f_n\}$ be a Cauchy sequence in $BV_p [a, b]$; that is, $q(f_n - f_m) = |f_n(a) - f_m(a)|^p + (V_p (f_n - f_m))^p \to 0$ as $n, m \to \infty$. For any $\xi \in [a, b]$, using the partition $T: a \leq \xi \leq b$ and the definition of $V_p f$, we get that $\{ f_n (\xi) \}$ is a Cauchy sequence in $\mathbb{R}$ and converges to a number denoted by $f(\xi)$. For any $\varepsilon > 0$, there exists an integer $N$ such that $q(f_n - f_m) \leq \varepsilon$ for $m, n > N$. Let $T: a = x_0 < x_1 < \cdots < x_k = b$ be an arbitrary partition of $[a, b]$. Then
\[
|f_n (a) - f_m (a)|^p
\]
\[
+ \sum_{i=1}^{k} \left| (f_n - f_m) (x_i) - (f_m - f_m) (x_{i-1}) \right|^p \leq q(f_n - f_m) \leq \varepsilon.
\]
Letting $m \to \infty$, we get that
\[
\left| f_n (a) - f_m (a) \right|^p + \sum_{i=1}^{k} \left| (f_n - f_m) (x_i) - (f_m - f_m) (x_{i-1}) \right|^p \leq \varepsilon.
\]
Taking the supremum over all partitions $T$, we have $q(f-f_n) \leq \varepsilon$ for $n > N$. This means that $f = (f-f_n) + f_n \in BV_p[a,b]$, and $q(f-f_n) \to 0$ as $n \to \infty$. Hence, $BV_p[a,b] \subset (0 < p < 1)$ is complete. Theorem 1 is proved.

**Proof of Theorem 2.** Suppose that $f \in BV_p[a,b]$ $(0 < p < 1)$. Since $f \in BV_p[a,b] \subset BV[a,b]$, by the Jordan decomposition theorem (see [8, Corollary 13.6]), we have $f(x) = g(x) - r(x)$, where $g(x), r(x)$ are increasing functions on $[a,b]$. Indeed, we can choose $g(x)$ to be $V_a^x(f)$, the total variation function of $f$ defined by

$$V_a^x(f) = \sup_T \left\{ \frac{1}{n} \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \right\},$$

where the supremum is taken over all partitions $T : a = x_0 < x_1 < \cdots < x_n = x$ of $[a,x]$, $r(x) = V_a^x(f) - f(x)$. It suffices to show that $g(x) = V_a^x(f) \in BV_p[a,b]$. For any fixed partition $T : a = x_0 < x_1 < \cdots < x_n = b$, we note that

$$g(x_i) - g(x_{i-1})^p = |V_{x_{i-1}}^{x_i} f|^p,$$

$$= \sup_{T_i} \left\{ \frac{1}{n} \sum_{j=1}^m |f(\xi_{i,j}) - f(\xi_{i,j-1})| \right\}^p,$$

$$\leq \sup_{T_i} \frac{1}{n} \sum_{j=1}^m |f(\xi_{i,j}) - f(\xi_{i,j-1})|^p,$$

where the supremum is taken over all partitions $T_i : x_{i-1} = \xi_{i,1} < \xi_{i,2} < \cdots < \xi_{i,m} = x_i$ of $[x_{i-1}, x_i]$. It follows that

$$\sum_{i=1}^n |g(x_i) - g(x_{i-1})|^p \leq \sum_{i=1}^n \sup_{T_i} \frac{1}{n} \sum_{j=1}^m |f(\xi_{i,j}) - f(\xi_{i,j-1})|^p,$$

$$= \sup_{T_i, 1 \leq i \leq m} \sum_{j=1}^n \frac{1}{n} \sum_{j=1}^m |f(\xi_{i,j}) - f(\xi_{i,j-1})|^p,$$

$$\leq (V_p f)^p,$$

(14)

which implies $g \in BV_p[a,b]$. This completes the proof of Theorem 2.

To prove Theorem 3, we introduce the next lemma.

**Lemma 5.** If $f \in BV_p[a,b] \cap C[a,b]$ $(0 < p < 1)$, then $f$ is a constant function.

**Proof.** It suffices to show that, for any $d \in [a,b]$, $f(d) = f(a)$. Assume that there exists $e \in (a,b)$ such that $f(d) \neq f(a)$. Without loss of generality, we assume that $f(d) < f(a)$. Since $f \in C[a,b]$, there exist $n-1$ points $\xi_1, \xi_2, \cdots, \xi_{n-1}$ such that $a = \xi_0 < \xi_1 < \cdots < \xi_{n-1} < \xi_n = d$ and $f(\xi_i) = f(a) + ((f(d) - f(a))/n)i$. Hence,

$$\left( V_p f \right)^p \geq \sum_{i=1}^n |f(\xi_i) - f(\xi_{i-1})|^p,$$

$$= n^{1-p} \left| f(d) - f(a) \right|^p \to \infty,$$

as $n \to \infty$, which implies that $f \notin BV_p[a,b]$. This leads to a contradiction. Lemma 5 is proved.

**Proof of Theorem 3.** (1) Without loss of generality, we may assume that $N = \infty$. Let $T : a = y_0 < y_1 < \cdots < y_m = b$ be a partition of $[a,b]$. For $j, 1 \leq j \leq m$, we note that

$$\left| f(y_j) - f(y_{j-1}) \right|^p \leq \sum_{i=1}^{\infty} (h_{i-1}d_{i,n}^p (y_j) - h_{i-1}d_{i-1,n}^p (y_{j-1}))^p,$$

$$= \sum_{i=1}^{\infty} d_n^p + \sum_{t_i = y_{j-1}}^{y_j} (d_n^p - d_{n-1}^p) + \sum_{t_{n-1} = y_{j-1}}^{y_j} d_n^p,$$

(17)

$$\leq \sum_{y_{j-1} < t_n \leq y_j} d_n^p,$$

where an empty sum denotes 0. It follows that

$$\sum_{j=1}^m \left| f(y_j) - f(y_{j-1}) \right|^p \leq \sum_{j=1}^m \left( \sum_{y_{j-1} < t_n \leq y_j} d_n^p \right) \leq 2 \sum_{n=1}^m d_n^p.$$

(18)

Taking the supremum over all partitions of $[a,b]$, we obtain that

$$\left( V_p f \right)^p \leq 2 \sum_{n=1}^m d_n^p.$$

(19)

On the other hand, for any fixed $m$, by renumbering $\{t_n\}_{n=1}^m$ if necessary, we may assume that $a = t_1 < t_2 < \cdots < t_m = b$. We set $y_i = (t_i + t_{i-1})/2$ $(1 \leq i \leq m-1)$. Then $T : a = y_0 < y_1 < \cdots < y_{m-1} < y_m = b$ is a partition of $[a,b]$. It follows that

$$\left( V_p f \right)^p \geq \sum_{j=1}^m \left| f(y_j) - f(y_{j-1}) \right|^p \geq \sum_{j=1}^m \left( \sum_{y_{j-1} < t_n \leq y_j} d_n^p \right)^p,$$

$$\geq \sum_{j=1}^m d_n^p.$$

(20)

Letting $m \to \infty$, we get

$$V_p f \geq \left( \sum_{n=1}^\infty d_n^p \right)^{1/p}.\tag{21}$$

Combining (19) with (21), we get (9). Hence, $f \in BV_p[a,b]$ $(0 < p < 1)$ and $A$ the set of points of discontinuity of $f$ on $[a,b]$. We set $h_l(x) = \sum_{t \in A} h_t(x)$, where $h_t(x)$ is defined by (7). Similar to the proof of (21), we have

$$\sum_{t \in A} \left| f(t+0) - f(t-0) \right|^p \leq \left( V_p f \right)^p \to \infty.$$

(22)
Applying the above proved result, we obtain that $h_f(x) \in BV_p[a,b]$. We set $g(x) = f(x) - h_f(x)$; then $g \in BV_p[a,b]$. We will show that $g(x)$ is continuous on $[a,b]$.

Indeed, for $x \in [a,b]$, we have

$$\sum_{t \in A} \tilde{h}_t(x) \leq \sum_{t \in A} (f(t+0) - f(t-0)) \leq \left( \sum_{t \in A} (f(t+0) - f(t-0))^p \right)^{1/p} \leq V_p f < \infty.$$  

(23)

By Weierstrass $M$-test (see [10, Theorem 7.10]), we get that the series $\sum_{t \in A} \tilde{h}_t(x)$ converges uniformly on $[a,b]$. For $x_0 \in [a,b] \setminus A$, $\tilde{h}_t(x)$ ($t \in A$) is continuous at $x_0$, so $h_f(x) = \sum_{t \in A} \tilde{h}_t(x)$ is also continuous at $x_0$. It follows that $g(x)$ is continuous at $x_0$.

For $x_0 \in A$, we set $u(x) = \sum_{t \in A \setminus \{x_0\}} \tilde{h}_t(x)$. Then $u(x)$ is continuous at $x_0$ and $h_f(x) = u(x) + \tilde{h}_{x_0}(x)$. Hence,

$$h_f(x_0 + 0) = u(x_0) + (f(x_0 + 0) - f(x_0 - 0)),$$

$$h_f(x_0 - 0) = u(x_0),$$

(24)

$$h_f(x_0) = u(x_0) + (f(x_0) - f(x_0 - 0)).$$

Thus,

$$g(x_0 + 0) = g(x_0) = g(x_0 - 0) = f(x_0 - 0) - u(x_0),$$

(25)

from which we can deduce that $g$ is continuous at $x_0$. Hence, $g(x) \in C[a,b]$.

Since $g(x) \in C[a,b] \cap BV_p[a,b]$, it follows from Lemma 5 that $g(x)$ is a constant $c$. Thus $f(x) = h_f(x) + c = \sum_{t \in A} \tilde{h}_t(x) + c$. The proof of Theorem 3 is complete.

\[
\square
\]

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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