Research Article

Estimates for Multilinear Commutators of Generalized Fractional Integral Operators on Weighted Morrey Spaces

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Received 28 July 2014; Accepted 29 October 2014

Academic Editor: Nelson José Merentes Díaz

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Let L be the infinitesimal generator of an analytic semigroup on $L^2(\mathbb{R}^n)$ with Gaussian kernel bounds, and let $L^{-\alpha/2}$ be the fractional integrals of L for $0 < \alpha < n$. Assume that $\vec{b} = (b_1, b_2, \dots, b_m)$ is a finite family of locally integrable functions; then the multilinear commutators generated by $\vec{b}$ and $L^{-\alpha/2}$ are defined by $L_{\vec{b}}^{-\alpha/2} f = [b_m, \dots, [b_2, [b_1, L^{-\alpha/2}]], \dots] f$. Assume that b_j belongs to weighted BMO space, $j = 1, 2, \dots, m$; the authors obtain the boundedness of $L_{\vec{b}}^{-\alpha/2}$ on weighted Morrey spaces. As a special case, when $L = -\Delta$ is the Laplacian operator, the authors also obtain the boundedness of the multilinear fractional commutator $I_{\alpha}^{\vec{b}}$ on weighted Morrey spaces. The main results in this paper are substantial improvements and extensions of some known results.

1. Introduction and Main Results

Assume that L is a linear operator on $L^2(\mathbb{R}^n)$, which generates an analytic semigroup e^{-tL} with a kernel $p_t(x, y)$ satisfying a Gaussian upper bound; that is,

$$|p_t(x, y)| \leq \frac{C}{t^{n/2}} e^{-c(|x-y|^2/t)}, \quad (1)$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$.

The property (1) is satisfied by a large amount of differential operators. One can see [1] for details and examples.

For $0 < \alpha < n$, the fractional integral $L^{-\alpha/2}$ generated by the operator L is defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL}(f) \frac{dt}{t^{-\alpha/2+1}}(x). \quad (2)$$

Let $\vec{b} = (b_1, b_2, \dots, b_m)$ be a finite family of locally integrable functions; then the multilinear commutators generated by $L^{-\alpha/2}$ and $\vec{b}$ are defined by

$$L_{\vec{b}}^{-\alpha/2} f = [b_m, \dots, [b_2, [b_1, L^{-\alpha/2}]], \dots] f, \quad (3)$$

where $m \in \mathbb{Z}^+$.

Note that if $L = -\Delta$, which is Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the classical fractional integral I_α :

$$I_\alpha f(x) = \frac{\Gamma((n-\alpha)/2)}{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad (4)$$

while $L_{\vec{b}}^{-\alpha/2}$ is the iterated commutator generated by $\vec{b}$ and I_α :

$$I_{\alpha}^{\vec{b}} f = [b_m, \dots, [b_2, [b_1, I_\alpha]], \dots] f, \quad (5)$$

where $m \in \mathbb{Z}^+$.

When $m = 1$, it is easy to see that $L_b^{-\alpha/2} f = [b, L^{-\alpha/2}] f$ is the commutator generated by $L^{-\alpha/2}$ and b , and when $b_1 = b_2 = \dots = b_m$, $L_{\vec{b}}^{-\alpha/2}$ is the higher commutator.

As we all know, if $b \in \text{BMO}$, the commutator of fractional integral operator $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1 < p < n/\alpha$ and $1/q = 1/p -$

α/n (see [2]). In 2004, Duong and Yan [1] generalized the above classical result and obtained the (L^p, L^q) boundedness of the commutator $[b, L^{-\alpha/2}]$ under the same conditions. Simultaneously, the theory on multilinear integral operators and multilinear commutators has attracted much attention as a rapid developing field in harmonic analysis. Mo and Lu [3] studied the (L^p, L^q) boundedness of the multilinear commutators $L_{\vec{b}}^{-\alpha/2}$, where $\vec{b} = (b_1, \dots, b_m)$, $b_j \in \text{BMO}$, and $j = 1, 2, \dots, m$.

On the other hand, Muckenhoupt and Wheeden [4] gave some definitions of weighted bounded mean oscillation and obtained some equivalent conditions for them.

Definition 1 (see [4]). Let $1 \leq p < \infty$ and w be locally integral in $\mathbb{R}^n$ and let $w \geq 0$. A locally integrable function b is said to be in $\text{BMO}_p(w)$ if

$$\|b\|_{\text{BMO}_p(w)} = \sup_Q \left(\frac{1}{w(Q)} \int_Q |b(x) - b_Q|^p w(x)^{1-p} dx \right)^{1/p} \leq C, \quad (6)$$

where $b_Q = (1/|Q|) \int_Q b(y) dy$ and the supremum is taken over all balls $Q \in \mathbb{R}^n$.

We may note that other weighted definitions for the bounded mean oscillation also have been given by Muckenhoupt and Wheeden in [4].

Definition 2 (see [4]). Let w be locally integral in $\mathbb{R}^n$ and $w \geq 0$. A locally integrable function b is said to be in $\text{BMO}(w)$ if the norm of $\text{BMO}(w)$: $\|\cdot\|_{*,w}$ satisfies

$$\|b\|_{*,w} = \sup_Q \frac{1}{w(Q)} \int_Q |b(x) - b_{Q,w}| w(x) dx \leq C, \quad (7)$$

where $b_{Q,w} = (1/w(Q)) \int_Q b(z)w(z)dz$ and the supremum is taken over all balls $Q \in \mathbb{R}^n$.

The above two definitions cannot contain each other; throughout this paper, we will make some investigations on the basis of Definition 2.

Recently, Wang [5] obtained some estimates for the commutator $[b, I_\alpha]$ on weighted Morrey space (see Definitions 3 and 4), where $b \in \text{BMO}_1(w)$. Furthermore, Wang and Si [6] obtained the necessary and sufficient conditions for the boundedness of $[b, L^{-\alpha/2}]$ on weighted Morrey spaces when $b \in \text{BMO}_1(w)$.

Motivated by [1, 3, 5, 6], it is natural to raise the following question: how to establish corresponding boundedness of the multilinear commutator $L_{\vec{b}}^{-\alpha/2} f$ on the weighted Morrey space, where $\vec{b} = (b_1, \dots, b_m)$, $b_j \in \text{BMO}(w)$?

The question is not motivated only by a mere quest to extend the multilinear commutator $L_{\vec{b}}^{-\alpha/2} f$ from the classical commutator $[b, I_\alpha]$ but rather by their natural appearance in analysis (see [3]).

To state the main results, we now give some definitions and notations.

A weight is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, \infty)$ almost everywhere. For a weight w and a measurable set E , we define $w(E) = \int_E w(x) dx$, the Lebesgue measure of E , by $|E|$ and the characteristic function of E by χ_E . For a real number p , $1 < p < \infty$; p' is the conjugate of p ; that is, $1/p + 1/p' = 1$. The letter C denotes a positive constant that may vary at each occurrence but is independent of the essential variable.

Definition 3 (see [7]). Let $1 \leq p < \infty$, $0 < \kappa < 1$; let w be a weight; then weighted Morrey space is defined by

$$L^{p,\kappa}(w) := \{f \in L^p_{\text{loc}}(w) : \|f\|_{L^{p,\kappa}(w)} < \infty\}, \quad (8)$$

where

$$\|f\|_{L^{p,\kappa}(w)} = \sup_B \left(\frac{1}{w(B)^\kappa} \int_B |f(x)|^p w(x) dx \right)^{1/p}, \quad (9)$$

and the supremum is taken over all balls B in $\mathbb{R}^n$.

Definition 4 (see [7]). Let $1 \leq p < \infty$, $0 < \kappa < 1$; let u, v be weight; then two weights weighted Morrey space are defined by

$$L^{p,\kappa}(u, v) := \{f : \|f\|_{L^{p,\kappa}(u,v)} < \infty\}, \quad (10)$$

where

$$\|f\|_{L^{p,\kappa}(u,v)} = \sup_B \left(\frac{1}{v(B)^\kappa} \int_B |f(x)|^p u(x) dx \right)^{1/p}, \quad (11)$$

and the supremum is taken over all balls B in $\mathbb{R}^n$. If $u = v$, then we denote $L^{p,\kappa}(u)$ for short.

Remark 5. (1) If $w = 1$, $\kappa = \lambda/n$, and $0 < \lambda < n$, then $L^{p,\kappa}(w) = L^{p,\lambda}(\mathbb{R}^n)$, the classical Morrey space.

(2) If $\kappa = 0$, then $L^{p,0}(w) = L^p(w)$, the weighted Lebesgue space; if $w = 1$, $\kappa = 0$, then $L^{p,\kappa}(w) = L^p(\mathbb{R}^n)$, the Lebesgue space.

Definition 6 (see [8]). A weight function w is in the Muckenhoupt class A_p with $1 < p < \infty$ if for every ball B in $\mathbb{R}^n$, there exists a positive constant C which is independent of B such that

$$\left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C. \quad (12)$$

When $p = 1$, $w \in A_1$, if

$$\frac{1}{|B|} \int_B w(x) dx \leq C \operatorname{ess\,inf}_{x \in B} w(x). \quad (13)$$

When $p = \infty$, $w \in A_\infty$, if there exist positive constants δ and C such that given a ball B and E is a measurable subset of B , then

$$\frac{w(E)}{w(B)} \leq C \left(\frac{|E|}{|B|} \right)^\delta. \quad (14)$$

Definition 7 (see [9]). A weight function w belongs to the reverse Hölder class RH_r , if there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality

$$\left(\frac{1}{|B|} \int_B w(x)^r dx \right)^{1/r} \leq C \frac{1}{|B|} \int_B w(x) dx, \quad (15)$$

holds for every ball B in $\mathbb{R}^n$.

It is well known that if $w \in A_p$ with $1 \leq p < \infty$, then there exists $r > 1$ such that $w \in \text{RH}_r$. It follows from Hölder's inequality that $w \in \text{RH}_r$ implies $w \in \text{RH}_s$ for all $1 < s < r$. Moreover, if $w \in \text{RH}_r$, $r > 1$, then we have $w \in \text{RH}_{r+\varepsilon}$ for some $\varepsilon > 0$. We thus write $r_w \equiv \sup\{r > 1 : w \in \text{RH}_r\}$ to denote the critical index of w for the reverse Hölder condition.

Definition 8. The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy. \quad (16)$$

Let w be a weight. The weighted maximal operator M_w is defined by

$$M_w f(x) = \sup_{x \in B} \frac{1}{w(B)} \int_B |f(y)| w(y) dy. \quad (17)$$

For $0 < \alpha < n$, $r \geq 1$, the fractional maximal operator $M_{\alpha,r}$ is defined by

$$M_{\alpha,r} f(x) = \sup_{x \in B} \left(\frac{1}{|B|^{1-\alpha r/n}} \int_B |f(y)|^r dy \right)^{1/r}. \quad (18)$$

And the fractional weighted maximal operator $M_{\alpha,r,w}$ is defined by

$$M_{\alpha,r,w} f(x) = \sup_{x \in B} \left(\frac{1}{w(B)^{1-\alpha r/n}} \int_B |f(y)|^r w(y) dy \right)^{1/r}. \quad (19)$$

If $\alpha = 0$, we denote $M_{r,w}$ for short.

Definition 9. A family of operators $\{A_t : t > 0\}$ is said to be an “approximation to identity” if, for every $t > 0$, A_t is represented by the kernel $a_t(x, y)$, which is a measurable function defined on $\mathbb{R}^n \times \mathbb{R}^n$, in the following sense: for every $f \in L^p(\mathbb{R}^n)$, $p \geq 1$,

$$A_t f(x) = \int_{\mathbb{R}^n} a_t(x, y) f(y) dy, \quad (20)$$

$$|a_t(x, y)| \leq h_t(x, y) = t^{-n/2} g\left(\frac{|x-y|^2}{t}\right),$$

for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $t > 0$. Here, g is a positive, bounded, decreasing function satisfying

$$\lim_{r \rightarrow \infty} r^{n+\varepsilon} g(r^2) = 0, \quad (21)$$

for some $\varepsilon > 0$.

Associated with an “approximation to identity” $\{A_t : t > 0\}$, Martell [10] introduced the sharp maximal function as follows:

$$M_A^\sharp f(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y) - A_{t_B} f(y)| dy, \quad (22)$$

where $t_B = r_B^2$, r_B is the radius of the ball B , and $f \in L^p(\mathbb{R}^n)$ for some $p \geq 1$.

Notice that our analytic semigroup $\{e^{-tL} : t > 0\}$ is an “approximation to identity.” In particular, denote

$$M_L^\sharp f(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y) - e^{-t_B L} f(y)| dy. \quad (23)$$

Next, we make some conventions on notation. Given any positive integer m , for all $0 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma_1, \sigma_2, \dots, \sigma_j\}$ of $\{1, 2, \dots, m\}$ of different elements, and, for any $\sigma \in C_j^m$, let $\sigma' = \{1, 2, \dots, m\} \setminus \sigma$. Let $\vec{b} = (b_1, b_2, \dots, b_m)$; then, for any $\sigma = \{\sigma_1, \sigma_2, \dots, \sigma_j\} \in C_j^m$, we denote $\vec{b}_\sigma = (b_{\sigma_1}, b_{\sigma_2}, \dots, b_{\sigma_j})$, $b_\sigma(x) = \prod_{\sigma_j \in \sigma} b_{\sigma_j}(x)$ and $\|\vec{b}_\sigma\|_{\text{BMO}(w)} = \prod_{\sigma_j \in \sigma} \|b_{\sigma_j}\|_{\text{BMO}(w)}$, and $\|\vec{b}\|_{\text{BMO}(w)} = \prod_{\sigma_j \in \{1, 2, \dots, m\}} \|b_{\sigma_j}\|_{\text{BMO}(w)} = \prod_{j=1}^m \|b_j\|_{\text{BMO}(w)}$.

In this paper, our main results are stated as follows.

Theorem 10. Assume the condition (1) holds. Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 \leq \kappa < p/q$, $w^{q/p} \in A_1$, and $r_w > (1 - \kappa)/(p/q - \kappa)$, where r_w denotes the critical index of w for the reverse Hölder condition. If $b_j \in \text{BMO}(w)$, $j = 1, 2, \dots, m$, then

$$\|L_b^{-\alpha/2} f\|_{L^{q, \kappa q/p}(w^{q/p}, w)} \leq C \|\vec{b}\|_{\text{BMO}(w)} \|f\|_{L^{p, \kappa}(w)}. \quad (24)$$

Theorem 11. Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $w^{q/p} \in A_1$, and $r_w > q/p$, where r_w denotes the critical index of w for the reverse Hölder condition. If $b_j \in \text{BMO}(w)$, $j = 1, 2, \dots, m$, then

$$\|L_b^{-\alpha/2} f\|_{L^q(w^{q/p})} \leq C \|\vec{b}\|_{\text{BMO}(w)} \|f\|_{L^p(w)}. \quad (25)$$

Moreover, if $L = -\Delta$ is the Laplacian, then

$$\|I_{\alpha}^{\vec{b}} f\|_{L^q(w^{q/p})} \leq C \|\vec{b}\|_{\text{BMO}(w)} \|f\|_{L^p(w)}. \quad (26)$$

Remark 12. We note that our results extend some results in [1, 3]. To be specific, if we take $m = 1$, $w = 1$, and $\kappa = 0$ in Theorem 10, it is easy to see that our conclusion is the main result of Duong and Yan [1]. If we only take $w = 1$ and $\kappa = 0$ in Theorem 10, our result contains the corresponding conclusion of [3].

The remaining part of this paper will be organized as follows. In Section 2, we will give some known results and prove some requisite lemmas. Section 3 is devoted to proving the theorems of this paper.

2. Requisite Lemmas

In this section, we will prove some lemmas and state some known results about weights and weighted Morrey space.

Lemma 13 (see [11]). *Let $s > 1$, $1 \leq p < \infty$, and $A_p^s = \{w : w^s \in A_p\}$. Then*

$$A_p^s = A_{1+(p-1)/s} \cap RH_s. \quad (27)$$

In particular, $A_1^s = A_1 \cap RH_s$.

Lemma 14 (see [10]). *Assume that the semigroup e^{-tL} has a kernel $p_t(x, y)$ which satisfies the upper bound (1). Take $\lambda > 0$, $f \in L_0^1(\mathbb{R}^n)$ (the set of functions in $L^1(\mathbb{R}^n)$ with bounded support) and a ball B_0 such that there exists $x_0 \in B_0$ with $Mf(x_0) \leq \lambda$. Then, for every $w \in A_\infty$, $0 < \eta < 1$, we can find $\gamma > 0$ (independent of λ, B_0, f, x_0) and constants $C, r > 0$ (which only depend on w) such that*

$$w\left(\left\{x \in B_0 : Mf(x) > A\lambda, M_L^\# f(x) \leq \gamma\lambda\right\}\right) \leq C\eta^r w(B_0), \quad (28)$$

where $A > 1$ is a fixed constant which depends only on n .

As a result, using the above good- λ inequality together with the standard arguments, we have the following estimates:

For every $f \in L^{p,\kappa}(u, v)$, $1 < p < \infty$, $0 \leq \kappa < 1$; if $u, v \in A_\infty$, then

$$\|f\|_{L^{p,\kappa}(u,v)} \leq \|Mf\|_{L^{p,\kappa}(u,v)} \leq \|M_L^\# f\|_{L^{p,\kappa}(u,v)}. \quad (29)$$

In particular, when $u = v = w$, $w \in A_\infty$, we have

$$\|f\|_{L^{p,\kappa}(w)} \leq \|Mf\|_{L^{p,\kappa}(w)} \leq \|M_L^\# f\|_{L^{p,\kappa}(w)}. \quad (30)$$

Lemma 15 (Kolmogorov's inequality; see [9, page 455]). *Let $0 < r < l < \infty$, for $f \geq 0$; define $\|f\|_{L^{l,\infty}} = \sup_{t>0} t|\{x \in \mathbb{R}^n : |f(x)| > t\}|^{1/l}$, $N_{l,r}(f) = \sup_E (\|f\chi_E\|_r / \|\chi_E\|_h)$, and $1/h = 1/r - 1/l$; then*

$$\|f\|_{L^{l,\infty}} \leq N_{l,r}(f) \leq \left(\frac{l}{l-r}\right)^{1/r} \|f\|_{L^{l,\infty}}. \quad (31)$$

Lemma 16 (see [4]). *Let $w \in A_\infty$. Then the norm of $BMO(w)$: $\|\cdot\|_{*,w}$, is equivalent to the norm of $BMO(\mathbb{R}^n)$: $\|\cdot\|_*$; that is, if b is a locally integrable function, then*

$$\|b\|_{*,w} = \sup_Q \frac{1}{w(Q)} \int_Q |b(x) - b_{Q,w}| w(x) dx \leq C \quad (32)$$

is equivalent to

$$\|b\|_* = \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q| dx \leq C. \quad (33)$$

Lemma 17 (see [5]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $w^{q/p} \in A_1$; if $0 < \kappa < p/q$, $r_w > (1 - \kappa)/(p/q - \kappa)$, then*

$$\|M_{\alpha,1} f\|_{L^{q,\kappa q/p}(w^{q/p},w)} \leq C \|f\|_{L^{p,\kappa}(w)}. \quad (34)$$

It also holds for I_α .

Lemma 18 (see [5]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $w^{q/p} \in A_1$; if $0 < \kappa < p/q$, $1 < r < p$, $r_w > (1 - \kappa)/(p/q - \kappa)$, then*

$$\|M_{r,w} f\|_{L^{q,\kappa q/p}(w^{q/p},w)} \leq C \|f\|_{L^{q,\kappa q/p}(w^{q/p},w)}. \quad (35)$$

Lemma 19 (see [5]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \kappa < p/q$, and $w \in A_\infty$. Then, for $1 < r < p$,*

$$\|M_{\alpha,r,w} f\|_{L^{q,\kappa q/p}(w)} \leq C \|f\|_{L^{p,\kappa}(w)}. \quad (36)$$

Remark 20. It is easy to see that Lemmas 17, 18, and 19 still hold for $\kappa = 0$.

Lemma 21. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $w^{q/p} \in A_1$; if $0 \leq \kappa < p/q$, $r_w > (1 - \kappa)/(p/q - \kappa)$, then*

$$\|L^{-\alpha/2} f\|_{L^{q,\kappa q/p}(w^{q/p},w)} \leq C \|f\|_{L^{p,\kappa}(w)}. \quad (37)$$

Proof. Since semigroup e^{-tL} has a kernel $p_t(x, y)$ that satisfies the upper bound (1), it is easy to see that, for $x \in \mathbb{R}^n$, $L^{-\alpha/2} f(x) \leq I_\alpha(|f|)(x)$. From the boundedness of I_α on weighted Morrey space (see Lemma 17), we get

$$\|L^{-\alpha/2} f\|_{L^{q,\kappa q/p}(w^{q/p},w)} \leq \|I_\alpha f\|_{L^{q,\kappa q/p}(w^{q/p},w)} \leq C \|f\|_{L^{p,\kappa}(w)}. \quad (38)$$

□

Remark 22. Since I_α is of weak $(1, n/(n - \alpha))$ type, from the above proof, we can obtain that $L^{-\alpha/2}$ is of weak $(1, n/(n - \alpha))$ type.

Lemma 23 (see [1]). *Assume the semigroup e^{-tL} has a kernel $p_t(x, y)$ which satisfies the upper bound (1). Then, for $0 < \alpha < n$, the differential operator $L^{-\alpha/2} - e^{-tL} L^{-\alpha/2}$ has an associated kernel $\tilde{K}_{\alpha,t}(x, y)$ which satisfies*

$$\tilde{K}_{\alpha,t}(x, y) \leq \frac{C}{|x - y|^{n-\alpha}} \frac{t}{|x - y|^2}. \quad (39)$$

Lemma 24. *Assume the semigroup e^{-tL} has a kernel $p_t(x, y)$ which satisfies the upper bound (1), $b \in BMO(w)$, and $w \in A_1$. Then, for $f \in L^p(\mathbb{R}^n)$, $p > 1$, $\sigma \in C_j^m$ ($j = 1, 2, \dots, m$), $1 < \tau < \infty$, and*

$$\begin{aligned} & \sup_{x \in B} \frac{1}{|B|} \int_B |e^{-t_B L} ((b - b_B)_\sigma f)(y)| dy \\ & \leq C \|\tilde{b}_\sigma\|_{BMO(w)} M_{\tau,w} f(x), \end{aligned} \quad (40)$$

where $t_B = r_B^2$ and r_B is the radius of B .

Proof. For any $f \in L^p(\mathbb{R}^n)$, $x \in B$, we have

$$\begin{aligned} & \frac{1}{|B|} \int_B |e^{-t_B L}((b - b_B)_\sigma f)(y)| dy \\ & \leq \frac{1}{|B|} \int_B \int_{\mathbb{R}^n} |p_{t_B}(y, z)| |(b(z) - b_B)_\sigma f(z)| dz dy \\ & \leq \frac{1}{|B|} \int_B \int_{2B} |p_{t_B}(y, z)| |(b(z) - b_B)_\sigma f(z)| dz dy \quad (41) \\ & \quad + \frac{1}{|B|} \int_B \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |p_{t_B}(y, z)| \\ & \quad \times |(b(z) - b_B)_\sigma f(z)| dz dy \end{aligned}$$

$$:= I + II.$$

Noticing that $y \in B, z \in 2B$, from (1), we get

$$|p_{t_B}(y, z)| \leq C t_B^{-n/2} \leq C |2B|^{-1}. \quad (42)$$

Thus,

$$\begin{aligned} I & \leq \frac{C}{|2B|} \int_{2B} |(b(z) - b_B)_\sigma| |f(z)| dz \\ & = \frac{C}{|2B|} \int_{2B} \prod_{\sigma_j \in \sigma} |b_{\sigma_j}(z) - (b_{\sigma_j})_B| |f(z)| dz. \end{aligned} \quad (43)$$

For simplicity, we only consider the case of $j = 2$. We also want to point out that although we state our results on the case of $j = 2$, all results are valid on the multilinear case ($j > 2$) without any essential difference and difficulty in the proof. So it follows that

$$\begin{aligned} I & \leq \frac{C}{|2B|} \int_{2B} (|b_{\sigma_1}(z) - (b_{\sigma_1})_{2B}| + |(b_{\sigma_1})_{2B} - (b_{\sigma_1})_B|) \\ & \quad \times (|b_{\sigma_2}(z) - (b_{\sigma_2})_{2B}| + |(b_{\sigma_2})_{2B} - (b_{\sigma_2})_B|) |f(z)| dz \\ & \leq \frac{C}{|2B|} \int_{2B} |b_{\sigma_1}(z) - (b_{\sigma_1})_{2B}| |b_{\sigma_2}(z) - (b_{\sigma_2})_{2B}| |f(z)| dz \\ & \quad + \frac{C}{|2B|} \int_{2B} |b_{\sigma_1}(z) - (b_{\sigma_1})_{2B}| |(b_{\sigma_2})_{2B} - (b_{\sigma_2})_B| |f(z)| dz \\ & \quad + \frac{C}{|2B|} \int_{2B} |(b_{\sigma_1})_{2B} - (b_{\sigma_1})_B| |b_{\sigma_2}(z) - (b_{\sigma_2})_{2B}| |f(z)| dz \\ & \quad + \frac{C}{|2B|} \int_{2B} |(b_{\sigma_1})_{2B} - (b_{\sigma_1})_B| |(b_{\sigma_2})_{2B} - (b_{\sigma_2})_B| |f(z)| dz \\ & := I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (44)$$

We split I_1 as follows:

$$\begin{aligned} I_1 & \leq \frac{C}{|2B|} \int_{2B} \left\{ |b_{\sigma_1}(z) - (b_{\sigma_1})_{2B,w}| + |(b_{\sigma_1})_{2B,w} - (b_{\sigma_1})_{2B}| \right\} \\ & \quad \times \left\{ |b_{\sigma_2}(z) - (b_{\sigma_2})_{2B,w}| \right. \\ & \quad \left. + |(b_{\sigma_2})_{2B,w} - (b_{\sigma_2})_{2B}| \right\} |f(z)| dz \end{aligned}$$

$$\begin{aligned} & = \frac{C}{|2B|} \int_{2B} |b_{\sigma_1}(z) - (b_{\sigma_1})_{2B,w}| \\ & \quad \times |b_{\sigma_2}(z) - (b_{\sigma_2})_{2B,w}| |f(z)| dz \\ & \quad + \frac{C}{|2B|} \int_{2B} |b_{\sigma_1}(z) - (b_{\sigma_1})_{2B,w}| \\ & \quad \times |(b_{\sigma_2})_{2B,w} - (b_{\sigma_2})_{2B}| |f(z)| dz \\ & \quad + \frac{C}{|2B|} \int_{2B} |(b_{\sigma_1})_{2B,w} - (b_{\sigma_1})_{2B}| \\ & \quad \times |b_{\sigma_2}(z) - (b_{\sigma_2})_{2B,w}| |f(z)| dz \\ & \quad + \frac{C}{|2B|} \int_{2B} |(b_{\sigma_1})_{2B,w} - (b_{\sigma_1})_{2B}| \\ & \quad \times |(b_{\sigma_2})_{2B,w} - (b_{\sigma_2})_{2B}| |f(z)| dz \\ & := I_{11} + I_{12} + I_{13} + I_{14}. \end{aligned} \quad (45)$$

We now consider the four terms, respectively. Choose $\tau_1, \tau_2, \tau, s > 1$ satisfying $1/\tau_1 + 1/\tau_2 + 1/\tau + 1/s = 1$. According to Hölder's inequality and $w \in A_1$, we have

$$\begin{aligned} I_{11} & = \frac{C}{|2B|} \int_{2B} |b_{\sigma_1}(z) - (b_{\sigma_1})_{2B,w}| w(z)^{1/\tau_1} \\ & \quad \times |b_{\sigma_2}(z) - (b_{\sigma_2})_{2B,w}| w(z)^{1/\tau_2} |f(z)| \\ & \quad \times w(z)^{1/\tau} w(z)^{-1+1/s} dz \\ & \leq \frac{C}{|2B|} \left(\int_{2B} |b_{\sigma_1}(z) - (b_{\sigma_1})_{2B,w}|^{\tau_1} w(z) dz \right)^{1/\tau_1} \\ & \quad \times \left(\int_{2B} |b_{\sigma_2}(z) - (b_{\sigma_2})_{2B,w}|^{\tau_2} w(z) dz \right)^{1/\tau_2} \\ & \quad \times \left(\int_{2B} |f(z)|^\tau w(z) dz \right)^{1/\tau} \left(\int_{2B} w(z)^{-s+1} dz \right)^{1/s} \\ & \leq \frac{C}{|2B|} \|\vec{b}_\sigma\|_{*,w} M_{\tau,w} f(x) w(2B)^{1-1/s} w(x)^{-1+1/s} |2B|^{1/s} \\ & = C \|\vec{b}_\sigma\|_* M_{\tau,w} f(x) \left(\frac{w(2B)}{|2B|} \right)^{1-1/s} w(x)^{-1+1/s} \\ & \leq C \|\vec{b}_\sigma\|_* M_{\tau,w} f(x). \end{aligned} \quad (46)$$

In the above inequalities, we use the fact that if $w \in A_1$, then $w \in A_\infty$. Thus, the norm of $\text{BMO}(w)$ is equivalent to the norm of $\text{BMO}(\mathbb{R}^n)$ (see Lemma 16).

For I_{12} , we first estimate the term that contains $(b_{\sigma_2})_{2B}$. In fact, it follows from the John-Nirenberg lemma that there exist $C_1 > 0$ and $C_2 > 0$ such that, for any ball B and $\alpha > 0$,

$$|\{z \in 2B : |b_{\sigma_2}(z) - (b_{\sigma_2})_{2B}| > \alpha\}| \leq C_1 |2B| e^{-C_2 \alpha / \|b_{\sigma_2}\|_*}, \quad (47)$$

since $b \in \text{BMO}(\mathbb{R}^n)$. Using the definition of A_∞ , we get

$$\begin{aligned} & w\left(\left\{z \in 2B : \left|b_{\sigma_2}(z) - (b_{\sigma_2})_{2B}\right| > \alpha\right\}\right) \\ & \leq Cw(2B)e^{-C_2\alpha\delta/\|b_{\sigma_2}\|_*}, \end{aligned} \quad (48)$$

for some $\delta > 0$. We can see that (48) yields

$$\begin{aligned} & \int_{2B} \left|b_{\sigma_2}(z) - (b_{\sigma_2})_{2B}\right| w(z) dz \\ & = \int_0^\infty w\left(\left\{z \in 2B : \left|b_{\sigma_2}(z) - (b_{\sigma_2})_{2B}\right| > \alpha\right\}\right) d\alpha \\ & \leq Cw(2B) \int_0^\infty e^{-C_2\alpha\delta/\|b_{\sigma_2}\|_*} d\alpha \\ & = Cw(2B) \|b_{\sigma_2}\|_*. \end{aligned} \quad (49)$$

Thus,

$$\begin{aligned} & \left|(b_{\sigma_2})_{2B,w} - (b_{\sigma_2})_{2B}\right| \\ & \leq \frac{1}{w(2B)} \int_{2B} \left|b_{\sigma_2}(z) - (b_{\sigma_2})_{2B}\right| w(z) dz \\ & \leq \frac{C}{w(2B)} w(2B) \|b_{\sigma_2}\|_* = C \|b_{\sigma_2}\|_*. \end{aligned} \quad (50)$$

On the basis of (50), we now estimate I_{12} . For the above τ , select u, v such that $1/u + 1/v + 1/\tau = 1$; by virtue of Hölder's inequality and $w \in A_1$, we have

$$\begin{aligned} I_{12} & \leq C \|b_{\sigma_2}\|_* \frac{1}{|2B|} \int_{2B} \left|b_{\sigma_1}(z) - (b_{\sigma_1})_{2B,w}\right| |f(z)| dz \\ & = C \|b_{\sigma_2}\|_* \frac{1}{|2B|} \int_{2B} \left|b_{\sigma_1}(z) - (b_{\sigma_1})_{2B,w}\right| \\ & \quad \times w(z)^{1/u} |f(z)| w(z)^{1/\tau} w(z)^{-1+1/v} dz \\ & \leq C \|b_{\sigma_2}\|_* \frac{1}{|2B|} \left(\int_{2B} \left|b_{\sigma_1}(z) - (b_{\sigma_1})_{2B,w}\right|^u w(z) dz\right)^{1/u} \\ & \quad \times \left(\int_{2B} |f(z)|^\tau w(z) dz\right)^{1/\tau} \left(\int_{2B} w(z)^{-v+1} dz\right)^{1/v} \\ & \leq C \|\vec{b}_\sigma\|_* M_{\tau,w}f(x) \frac{1}{|2B|} w(2B)^{1/u+1/\tau} w(x)^{-1+1/v} |2B|^{1/v} \\ & = C \|\vec{b}_\sigma\|_* M_{\tau,w}f(x) \left(\frac{w(2B)}{|2B|}\right)^{1-1/v} w(x)^{-1+1/v} \\ & = C \|\vec{b}_\sigma\|_* M_{\tau,w}f(x). \end{aligned} \quad (51)$$

Analogous to the estimate of I_{12} , we also have $I_{13} \leq C \|\vec{b}_\sigma\|_* M_{\tau,w}f(x)$.

As for I_{14} , taking advantage of Hölder's inequality and (50), we get

$$\begin{aligned} I_{14} & \leq C \|\vec{b}_\sigma\|_* \frac{1}{|2B|} \int_{2B} |f(z)| dz \\ & = C \|\vec{b}_\sigma\|_* \frac{1}{|2B|} \int_{2B} |f(z)| w(z)^{1/\tau} w(z)^{1/\tau'-1} dz \\ & \leq C \|\vec{b}_\sigma\|_* \frac{1}{|2B|} \left(\int_{2B} |f(z)|^\tau w(z) dz\right)^{1/\tau} \\ & \quad \times \left(\int_{2B} w(z)^{1-\tau'} dz\right)^{1/\tau'} \\ & = C \|\vec{b}_\sigma\|_* \frac{1}{|2B|} M_{\tau,w}f(x) w(2B)^{1/\tau} w(x)^{1/\tau'-1} |2B|^{1/\tau'} \\ & = C \|\vec{b}_\sigma\|_* M_{\tau,w}f(x) \left(\frac{w(2B)}{|2B|}\right)^{1/\tau} w(x)^{1/\tau'-1} \\ & \leq C \|\vec{b}_\sigma\|_* M_{\tau,w}f(x). \end{aligned} \quad (52)$$

Collecting the estimates of I_{11} , I_{12} , I_{13} , and I_{14} , it is easy to see that

$$I_1 \leq C \|\vec{b}_\sigma\|_* M_{\tau,w}f(x). \quad (53)$$

For term I_2 , using the fact that $|b_{2B} - b_B| \leq C\|b\|_*$ (see [12]), we now get

$$\begin{aligned} I_2 & \leq \|b_{\sigma_2}\|_* \frac{C}{|2B|} \int_{2B} \left(\left|b_{\sigma_1}(z) - (b_{\sigma_1})_{2B,w}\right| \right. \\ & \quad \left. + \left|(b_{\sigma_1})_{2B,w} - (b_{\sigma_1})_{2B}\right|\right) \\ & \quad \times |f(z)| dz \\ & = \frac{C\|b_{\sigma_2}\|_*}{|2B|} \int_{2B} \left|b_{\sigma_1}(z) - (b_{\sigma_1})_{2B,w}\right| |f(z)| dz \\ & \quad + \frac{C\|b_{\sigma_2}\|_*}{|2B|} \int_{2B} \left|(b_{\sigma_1})_{2B,w} - (b_{\sigma_1})_{2B}\right| |f(z)| dz \\ & := I_{21} + I_{22}. \end{aligned} \quad (54)$$

By some estimates similar to those used in the estimate for I_{12} , we conclude that

$$I_{21} \leq C \|b_{\sigma_2}\|_* M_{\tau,w}f(x). \quad (55)$$

For I_{22} , using the same method as in dealing with I_{14} , we get

$$I_{22} \leq C \|b_{\sigma_2}\|_* M_{\tau,w}f(x). \quad (56)$$

Therefore,

$$I_2 \leq C \|b_{\sigma_2}\|_* M_{\tau,w}f(x). \quad (57)$$

Analogously,

$$I_3 \leq C \|b_{\sigma_2}\|_* M_{\tau,w}f(x). \quad (58)$$

An argument similar to that used in the estimate for I_{14} leads to

$$I_4 \leq C \|b_\sigma\|_* M_{\tau,w} f(x). \quad (59)$$

Hence,

$$\begin{aligned} I &\leq I_1 + I_2 + I_3 + I_4 \\ &\leq C \|b_\sigma\|_* M_{\tau,w} f(x). \end{aligned} \quad (60)$$

Next, we will consider the second term II in (41). For any $y \in B$, $z \in 2^{k+1}B \setminus 2^k B$, it is easy to get that $|y - z| \geq 2^{k-1}r_B$ and

$$|p_{t_B}(y, z)| \leq C \frac{e^{-C2^{2(k-1)}2^{(k+1)n}}}{|2^{k+1}B|}. \quad (61)$$

Thus,

$$\begin{aligned} II &\leq C \sum_{k=1}^{\infty} \frac{e^{-C2^{2(k-1)}2^{(k+1)n}}}{|2^{k+1}B|} \int_{2^{k+1}B} |(b(z) - b_B)_\sigma| |f(z)| dz \\ &= C \sum_{k=1}^{\infty} \frac{e^{-C2^{2(k-1)}2^{(k+1)n}}}{|2^{k+1}B|} \\ &\quad \times \int_{2^{k+1}B} |b_{\sigma_1}(z) - (b_{\sigma_1})_B| \cdots |b_{\sigma_j}(z) - (b_{\sigma_j})_B| |f(z)| dz. \end{aligned} \quad (62)$$

For simplicity, we also consider the case of $j = 2$.

$$\begin{aligned} II &\leq C \sum_{k=1}^{\infty} \frac{e^{-C2^{2(k-1)}2^{(k+1)n}}}{|2^{k+1}B|} \\ &\quad \times \int_{2^{k+1}B} (|b_{\sigma_1}(z) - (b_{\sigma_1})_{2^{k+1}B}| \\ &\quad + |(b_{\sigma_1})_{2^{k+1}B} - (b_{\sigma_1})_B|) \\ &\quad \times (|b_{\sigma_2}(z) - (b_{\sigma_2})_{2^{k+1}B}| \\ &\quad + |(b_{\sigma_2})_{2^{k+1}B} - (b_{\sigma_2})_B|) |f(z)| dz \\ &= C \sum_{k=1}^{\infty} \frac{e^{-C2^{2(k-1)}2^{(k+1)n}}}{|2^{k+1}B|} \\ &\quad \times \int_{2^{k+1}B} |b_{\sigma_1}(z) - (b_{\sigma_1})_{2^{k+1}B}| \\ &\quad \times |b_{\sigma_2}(z) - (b_{\sigma_2})_{2^{k+1}B}| |f(z)| dz \\ &\quad + C \sum_{k=1}^{\infty} \frac{e^{-C2^{2(k-1)}2^{(k+1)n}}}{|2^{k+1}B|} \\ &\quad \times \int_{2^{k+1}B} |(b_{\sigma_1})_{2^{k+1}B} - (b_{\sigma_1})_B| \\ &\quad \times |b_{\sigma_2}(z) - (b_{\sigma_2})_{2^{k+1}B}| |f(z)| dz \end{aligned}$$

$$\begin{aligned} &+ C \sum_{k=1}^{\infty} \frac{e^{-C2^{2(k-1)}2^{(k+1)n}}}{|2^{k+1}B|} \\ &\quad \times \int_{2^{k+1}B} |b_{\sigma_1}(z) - (b_{\sigma_1})_{2^{k+1}B}| \\ &\quad \times |(b_{\sigma_2})_{2^{k+1}B} - (b_{\sigma_2})_B| |f(z)| dz \\ &+ C \sum_{k=1}^{\infty} \frac{e^{-C2^{2(k-1)}2^{(k+1)n}}}{|2^{k+1}B|} \\ &\quad \times \int_{2^{k+1}B} |(b_{\sigma_1})_{2^{k+1}B} - (b_{\sigma_1})_B| \\ &\quad \times |(b_{\sigma_2})_{2^{k+1}B} - (b_{\sigma_2})_B| |f(z)| dz \\ &:= II_1 + II_2 + II_3 + II_4. \end{aligned} \quad (63)$$

For II_1 , similar to the estimate of I_1 , we have

$$\begin{aligned} II_1 &\leq C \|\vec{b}_\sigma\|_* M_{\tau,w} f(x) \sum_{k=1}^{\infty} e^{-C2^{2(k-1)}2^{(k+1)n}} \\ &\leq C \|\vec{b}_\sigma\|_* M_{\tau,w} f(x). \end{aligned} \quad (64)$$

Noticing that we could use similar methods of the estimates of I_2 , I_3 , and I_4 in the estimates of II_2 , II_3 , and II_4 , respectively. From this together with the fact that $|b_{2^{j+1}B} - b_B| \leq 2^n(j+1)\|b\|_*$ (see [12]), it is easy to get

$$II_2 + II_3 + II_4 \leq C \|\vec{b}_\sigma\|_* M_{\tau,w} f(x). \quad (65)$$

Therefore,

$$II \leq C \|\vec{b}_\sigma\|_* M_{\tau,w} f(x). \quad (66)$$

According to the estimates of I and II , the lemma has been proved. $\square$

Now, we will establish a lemma which plays an important role in the proof of Theorem 10.

Lemma 25. Let $0 < \alpha < n$, $w \in A_1$, and $b \in BMO(w)$; then, for all $r > 1$, $\tau > 1$, and $x \in \mathbb{R}^n$, we have

$$\begin{aligned} &M_L^\#(L_b^{-\alpha/2} f)(x) \\ &\leq C \left\{ \|\vec{b}\|_* M_{r,w}(L^{-\alpha/2} f)(x) \right. \\ &\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} C_{j,m} \|\vec{b}_\sigma\|_* M_{\tau,w}(L_{\vec{b}_\sigma}^{-\alpha/2} f)(x) \\ &\quad + \|\vec{b}\|_* w(x)^{-\alpha/n} M_{\alpha,r,w} f(x) \\ &\quad \left. + \|\vec{b}\|_* M_{\alpha,1} f(x) \right\}. \end{aligned} \quad (67)$$

Proof. For any given $x \in \mathbb{R}^n$, take a ball $B = B(x_0, r_B)$ which contains x . For $f \in L^p(\mathbb{R}^n)$, let $f_1 = f \chi_{2B}$, $f_2 = f - f_1$.

Denote the kernel of $L^{-\alpha/2}$ by $K_\alpha(x, y)$, $\vec{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_m)$, where $\lambda_j \in \mathbb{R}^n$, $j = 1, 2, \dots, m$. Then $L_{\vec{b}}^{-\alpha/2} f$ can be written in the following form:

$$\begin{aligned} L_{\vec{b}}^{-\alpha/2} f(y) &= \int_{\mathbb{R}^n} \prod_{j=1}^m (b_j(y) - b_j(z)) K_\alpha(y, z) f(z) dz \\ &= \int_{\mathbb{R}^n} \prod_{j=1}^m ((b_j(y) - \lambda_j) - (b_j(z) - \lambda_j)) K_\alpha(y, z) f(z) dz \\ &= \sum_{i=0}^m \sum_{\sigma \in C_i^m} (-1)^{m-i} (b(y) - \lambda)_\sigma \\ &\quad \times \int_{\mathbb{R}^n} (b(z) - \lambda)_{\sigma'} K_\alpha(y, z) f(z) dz. \end{aligned} \quad (68)$$

Now expanding $(b(z) - \lambda)_{\sigma'}$ as

$$(b(z) - \lambda)_{\sigma'} = ((b(z) - b(y)) + (b(y) - \lambda))_{\sigma'}, \quad (69)$$

and combining (68) with (69), it is easy to see that

$$\begin{aligned} e^{-t_B L} (L_{\vec{b}}^{-\alpha/2} f)(y) &= e^{-t_B L} \left(\prod_{j=1}^m (b_j - \lambda_j) L^{-\alpha/2} f \right)(y) \\ &\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} C_{j,m} e^{-t_B L} \left((b - \lambda)_\sigma L_{\vec{b}_{\sigma'}}^{-\alpha/2} f \right)(y) \\ &\quad + (-1)^m e^{-t_B L} \left(L^{-\alpha/2} \left(\prod_{j=1}^m (b_j - \lambda_j) f_1 \right) \right)(y) \\ &\quad + (-1)^m e^{-t_B L} \left(L^{-\alpha/2} \left(\prod_{j=1}^m (b_j - \lambda_j) f_2 \right) \right)(y), \end{aligned} \quad (70)$$

where $t_B = r_B^2$ and r_B is the radius of ball B .

Take $\lambda_j = (b_j)_B$, $j = 1, 2, \dots, m$, and denote $\vec{b}_B = ((b_1)_B, (b_2)_B, \dots, (b_m)_B)$; then

$$\begin{aligned} &\frac{1}{|B|} \int_B \left| L_{\vec{b}}^{-\alpha/2} f(y) - e^{-t_B L} (L_{\vec{b}}^{-\alpha/2} f)(y) \right| dy \\ &\leq \frac{1}{|B|} \int_B \left| \prod_{j=1}^m (b_j(y) - (b_j)_B) L^{-\alpha/2} f(y) \right| dy \\ &\quad + \frac{1}{|B|} \int_B \left| \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} C_{j,m} (b(y) - b_B)_\sigma L_{\vec{b}_{\sigma'}}^{-\alpha/2} f(y) \right| dy \end{aligned}$$

$$\begin{aligned} &+ \frac{1}{|B|} \int_B \left| L^{-\alpha/2} \left(\prod_{j=1}^m (b_j - (b_j)_B) f_1 \right)(y) \right| dy \\ &+ \frac{1}{|B|} \int_B \left| e^{-t_B L} \left(\prod_{j=1}^m (b_j - (b_j)_B) L^{-\alpha/2} f \right)(y) \right| dy \\ &+ \frac{1}{|B|} \int_B \left| \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} C_{j,m} e^{-t_B L} \left((b - b_B)_\sigma L_{\vec{b}_{\sigma'}}^{-\alpha/2} f \right)(y) \right| dy \\ &+ \frac{1}{|B|} \int_B \left| e^{-t_B L} \left(L^{-\alpha/2} \left(\prod_{j=1}^m (b_j - (b_j)_B) f_1 \right) \right)(y) \right| dy \\ &+ \frac{1}{|B|} \int_B \left| L^{-\alpha/2} \left(\prod_{j=1}^m (b_j - (b_j)_B) f_2 \right)(y) \right. \\ &\quad \left. - e^{-t_B L} \left(L^{-\alpha/2} \left(\prod_{j=1}^m (b_j - (b_j)_B) f_2 \right) \right)(y) \right| dy \\ &:= \sum_{i=1}^7 G_i. \end{aligned} \quad (71)$$

We now estimate the above seven terms, respectively. We take $m = 2$ as an example; the estimate for the case $m > 2$ is the same. For the first term G_1 , we split it as follows:

$$\begin{aligned} G_1 &= \frac{1}{|B|} \int_B |b_1(y) - (b_1)_{B,w}| |b_2(y) - (b_2)_{B,w}| |L^{-\alpha/2} f(y)| dy \\ &\quad + \frac{1}{|B|} \int_B |b_1(y) - (b_1)_{B,w}| |(b_2)_{B,w} - (b_2)_B| |L^{-\alpha/2} f(y)| dy \\ &\quad + \frac{1}{|B|} \int_B |(b_1)_{B,w} - (b_1)_B| |b_2(y) - (b_2)_{B,w}| |L^{-\alpha/2} f(y)| dy \\ &\quad + \frac{1}{|B|} \int_B |(b_1)_{B,w} - (b_1)_B| |(b_2)_{B,w} - (b_2)_B| |L^{-\alpha/2} f(y)| dy \\ &:= G_{11} + G_{12} + G_{13} + G_{14}. \end{aligned} \quad (72)$$

Choose $r_1, r_2, r, q > 1$ such that $1/r_1 + 1/r_2 + 1/r + 1/q = 1$. Noticing that $w \in A_1$, by Hölder's inequality and the similar estimate of I_1 , we have

$$G_1 \leq C \|\vec{b}\|_* M_{r,w} (L^{-\alpha/2} f)(x). \quad (73)$$

For G_2 , take $\tau_1, \dots, \tau_j, \tau, \nu > 1$ that satisfy $1/\tau_1 + \dots + 1/\tau_j + 1/\tau + 1/\nu = 1$; then following Hölder's inequality and the same idea as that of G_1 yields

$$G_2 \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} C_{j,m} \|\vec{b}_\sigma\|_* M_{\tau,w} (L_{\vec{b}_{\sigma'}}^{-\alpha/2} f)(x). \quad (74)$$

To estimate G_3 , applying Lemma 15 (Kolmogorov's inequality), weak $(1, n/(n - \alpha))$ boundedness of $L^{-\alpha/2}$ (see Remark 22), and Hölder's inequality, we have

$$\begin{aligned} G_3 &= \frac{1}{|B|} \int_B \left| L^{-\alpha/2} \left(\prod_{j=1}^m (b_j - (b_j)_B) f_1 \right) (y) \right| dy \\ &\leq \frac{C}{|B|^{1-\alpha/n}} \left\| L^{-\alpha/2} \left(\prod_{j=1}^m (b_j - (b_j)_B) f_1 \right) \right\|_{L^{n/(n-\alpha), \infty}} \quad (75) \\ &\leq \frac{C}{|B|^{1-\alpha/n}} \int_{2B} \left| \prod_{j=1}^m (b_j(y) - (b_j)_B) f(y) \right| dy. \end{aligned}$$

We consider the case of $m = 2$, for example, and we split G_3 as follows:

$$\begin{aligned} G_3 &\leq \frac{C}{|B|^{1-\alpha/n}} \int_{2B} (|b_1(y) - (b_1)_{2B}| + |(b_1)_{2B} - (b_1)_B|) \\ &\quad \times (|b_2(y) - (b_2)_{2B}| + |(b_2)_{2B} - (b_2)_B|) |f(y)| dy \\ &\leq \frac{C}{|B|^{1-\alpha/n}} \int_{2B} |b_1(y) - (b_1)_{2B}| |b_2(y) - (b_2)_{2B}| |f(y)| dy \\ &\quad + \frac{C}{|B|^{1-\alpha/n}} \int_{2B} |b_1(y) - (b_1)_{2B}| |(b_2)_{2B} - (b_2)_B| |f(y)| dy \\ &\quad + \frac{C}{|B|^{1-\alpha/n}} \int_{2B} |(b_1)_{2B} - (b_1)_B| |b_2(y) - (b_2)_{2B}| |f(y)| dy \\ &\quad + \frac{C}{|B|^{1-\alpha/n}} \int_{2B} |(b_1)_{2B} - (b_1)_B| |(b_2)_{2B} - (b_2)_B| |f(y)| dy \\ &:= G_{31} + G_{32} + G_{33} + G_{34}. \quad (76) \end{aligned}$$

Take $r_1, r_2, r, q > 1$ such that $1/r_1 + 1/r_2 + 1/r + 1/q = 1$; by virtue of Hölder's inequality and the same manner as that used in dealing with I_1, I_2, I_3 in Lemma 24, we get

$$G_{31} + G_{32} + G_{33} \leq C \|\vec{b}\|_* M_{\alpha, r, w} f(x). \quad (77)$$

For G_{34} ,

$$\begin{aligned} G_{34} &\leq C \|\vec{b}\|_* \frac{1}{|B|^{1-\alpha/n}} \int_{2B} |f(y)| dy \\ &\leq C \|\vec{b}\|_* M_{\alpha, 1} f(x). \quad (78) \end{aligned}$$

Therefore,

$$G_3 \leq C \|\vec{b}\|_* M_{\alpha, r, w} f(x) + C \|\vec{b}\|_* M_{\alpha, 1} f(x). \quad (79)$$

By Lemma 24, we have

$$\begin{aligned} G_4 &\leq C \|\vec{b}\|_* M_{r, w} (L^{-\alpha/2} f)(x), \\ G_5 &\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} C_{j, m} \|\vec{b}_\sigma\|_* M_{\tau, w} (L_{b_\sigma}^{-\alpha/2} f)(x). \quad (80) \end{aligned}$$

Next, we consider the term G_6 :

$$\begin{aligned} G_6 &= \frac{1}{|B|} \int_B \left| \int_{\mathbb{R}^n} p_{t_B}(y, z) \right. \\ &\quad \times \left. \left(L^{-\alpha/2} \left(\prod_{j=1}^m (b_j - (b_j)_B) f_1 \right) (z) \right) dz \right| dy \\ &\leq \frac{1}{|B|} \int_B \int_{2B} |p_{t_B}(y, z)| \\ &\quad \times \left| L^{-\alpha/2} \left(\prod_{j=1}^m (b_j - (b_j)_B) f_1 \right) (z) \right| dz dy \\ &\leq \frac{1}{|B|} \int_B \int_{\mathbb{R}^n \setminus 2B} |p_{t_B}(y, z)| \\ &\quad \times \left| L^{-\alpha/2} \left(\prod_{j=1}^m (b_j - (b_j)_B) f_1 \right) (z) \right| dz dy \\ &:= G_{61} + G_{62}. \quad (81) \end{aligned}$$

For G_{61} , since $y \in B, z \in 2B$, and $|p_{t_B}(y, z)| \leq C|2B|^{-1}$, it follows that

$$G_{61} \leq \frac{C}{|2B|} \int_{2B} \left| L^{-\alpha/2} \left(\prod_{j=1}^m (b_j - (b_j)_B) f_1 \right) (z) \right| dz. \quad (82)$$

Analogous to the estimate of G_3 , we have

$$G_{61} \leq C \|\vec{b}\|_* M_{\alpha, r, w} f(x) + C \|\vec{b}\|_* M_{\alpha, 1} f(x). \quad (83)$$

For G_{62} , note that $y \in B, z \in 2^{k+1}B \setminus 2^k B$, and $|p_{t_B}(y, z)| \leq C(e^{-C2^{2(k-1)}} 2^{(k+1)n}/|2^{k+1}B|)$. Hence, the estimate of G_{62} runs as that of G_{61} yields that

$$G_{62} \leq C \|\vec{b}\|_* M_{\alpha, r, w} f(x) + C \|\vec{b}\|_* M_{\alpha, 1} f(x). \quad (84)$$

For the last term G_7 , applying Lemma 23, we have

$$\begin{aligned} G_7 &\leq \frac{1}{|B|} \int_B \left| \left(L^{-\alpha/2} - e^{-t_B L} L^{-\alpha/2} \right) \right. \\ &\quad \times \left. \left(\prod_{j=1}^m (b_j - (b_j)_B) f_2 \right) (y) \right| dy \\ &\leq \frac{1}{|B|} \int_B \int_{\mathbb{R}^n \setminus 2B} |\tilde{K}_{\alpha, t_B}(y, z)| \\ &\quad \times \left| \prod_{j=1}^m (b_j(z) - (b_j)_B) f(z) \right| dz dy \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{t_B}{|x_0 - z|^{n-\alpha+2}} \\
&\quad \times \left| \prod_{j=1}^m (b_j(z) - (b_j)_B) f(z) \right| dz \\
&\leq C \sum_{k=1}^{\infty} \frac{2^{-2k}}{|2^k B|^{1-\alpha/n}} \int_{2^{k+1}B} \left| \prod_{j=1}^m (b_j(z) - (b_j)_B) \right| |f(z)| dz.
\end{aligned} \tag{85}$$

The same manner as that of G_3 gives us that

$$\begin{aligned}
G_7 &\leq C \sum_{k=1}^{\infty} 2^{-2k} (\|\vec{b}\|_* M_{\alpha,r,w} f(x) + \|\vec{b}\|_* M_{\alpha,1} f(x)) \\
&\leq C \|\vec{b}\|_* (M_{\alpha,r,w} f(x) + M_{\alpha,1} f(x)).
\end{aligned} \tag{86}$$

According to the above estimates, we have completed the proof of Lemma 25. $\square$

3. Proof of Theorems

At first, we give the proof of Theorem 10.

Proof. It follows from Lemmas 13, 14, 25, and 17–21 that

$$\begin{aligned}
&\|L_{\vec{b}}^{-\alpha/2} f\|_{L^{q,\kappa q/p}(w^{q/p},w)} \\
&\leq \|ML_{\vec{b}}^{-\alpha/2} f\|_{L^{q,\kappa q/p}(w^{q/p},w)} \\
&\leq \|M_L^{\sharp}(L_{\vec{b}}^{-\alpha/2} f)\|_{L^{q,\kappa q/p}(w^{q/p},w)} \\
&\leq C \left\{ \|\vec{b}\|_* \|M_{r,w}(L^{-\alpha/2} f)\|_{L^{q,\kappa q/p}(w^{q/p},w)} \right. \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} C_{j,m} \|\vec{b}_{\sigma}\|_* \\
&\quad \times \|M_{\tau,w}(L_{\vec{b}_{\sigma'}}^{-\alpha/2} f)\|_{L^{q,\kappa q/p}(w^{q/p},w)} \\
&\quad + \|\vec{b}\|_* \|w^{-\alpha/n} M_{\alpha,r,w}(f)\|_{L^{q,\kappa q/p}(w^{q/p},w)} \\
&\quad \left. + \|\vec{b}\|_* \|M_{\alpha,1}(f)\|_{L^{q,\kappa q/p}(w^{q/p},w)} \right\} \\
&\leq C \|\vec{b}\|_* \|L^{-\alpha/2} f\|_{L^{q,\kappa q/p}(w^{q/p},w)} \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} C_{j,m} \|\vec{b}_{\sigma}\|_* \|L_{\vec{b}_{\sigma'}}^{-\alpha/2} f\|_{L^{q,\kappa q/p}(w^{q/p},w)} \\
&\quad + C \|\vec{b}\|_* \|M_{\alpha,r,w} f\|_{L^{q,\kappa q/p}(w)} + C \|\vec{b}\|_* \|f\|_{L^{p,\kappa}(w)} \\
&\leq C \|\vec{b}\|_* \|f\|_{L^{p,\kappa}(w)} \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} C_{j,m} \|\vec{b}_{\sigma}\|_* \|L_{\vec{b}_{\sigma'}}^{-\alpha/2} f\|_{L^{q,\kappa q/p}(w^{q/p},w)}.
\end{aligned} \tag{87}$$

Then, we can make use of an induction on $\sigma \subseteq \{1, 2, \dots, m\}$ to get that

$$\|L_{\vec{b}}^{-\alpha/2} f\|_{L^{q,\kappa q/p}(w^{q/p},w)} \leq C \|\vec{b}\|_* \|f\|_{L^{p,\kappa}(w)}. \tag{88}$$

This completes the proof of Theorem 10. $\square$

Now, we are in the position of proving Theorem 11. If we take $\kappa = 0$ in Theorem 10, we will immediately get our desired results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to express their gratitude to the anonymous referees for reading the paper carefully and giving some valuable comments. The research was partially supported by the National Nature Science Foundation of China under Grants nos. 11171306 and 11071065 and sponsored by the Scientific Project of Zhejiang Provincial Science Technology Department under Grant no. 2011C33012 and the Scientific Research Fund of Zhejiang Provincial Education Department under Grant no. Z201017584.

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