Research Article

On Mann's Type Method for Nonexpansive and Strongly Quasinonexpansive Mappings in Hilbert Spaces

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In the setting of Hilbert spaces, we study Mann's type method to approximate strong solutions of variational inequalities. We show that these solutions are fixed points of a nonexpansive mapping and/or a strongly quasinonexpansive mapping, depending on the coefficients involved in the algorithm.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induce a norm $\| \cdot \|$. Let $T : H \rightarrow H$ be a nonlinear mapping. $\text{Fix}(T) := \{x \in H : x = Tx\}$ denotes the fixed point set of T .

The problem of approximating fixed points of nonlinear mapping has been widely investigated by many authors.

The starting point of the present paper is the following scheme introduced by Iemoto and Takahashi in [1]:

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n [\beta_n T x_n + (1 - \beta_n) S x_n], \quad (1)$$

where T is a nonexpansive mapping and S is a nonspreading mapping.

Depending on the coefficients α_n and β_n , in [1] was obtained weak convergence of the method either to a fixed point of T or to a fixed point of S or to a common fixed point of T and S .

Motivated by this result, with the aim of obtaining strong convergence, in [2] we introduced a viscosity in a previous scheme and we have been able to prove the following result, in which the most important role was played by the average type mappings $T_\delta = (1 - \delta)I + \delta T$ and $S_\delta = (1 - \delta)I + \delta S$, $\delta \in (0, 1)$.

Theorem 1 (see [2]). *Let $T : H \rightarrow H$ be a nonexpansive mapping and let $S : H \rightarrow H$ be a nonspreading mapping. Let*

$$x_1 \in C,$$

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) [\beta_n T_\omega x_n + (1 - \beta_n) S_\omega x_n], \quad (2)$$

$$n \geq 1,$$

with $(\alpha_n)_n \subset (0, 1)$, $\alpha_n \rightarrow 0$, and $\sum_n \alpha_n = \infty$. Then

(i) *if $\sum_n (1 - \beta_n) < \infty$, $\sum_n |\alpha_n - \alpha_{n+1}| < \infty$, and $\text{Fix}(T) \neq \emptyset$, then $(x_n)_n$ strongly converges to $\bar{p} = P_{\text{Fix}(T)} f(\bar{p})$ which is the unique point in $\text{Fix}(T)$ that satisfies the variational inequality*

$$\langle (I - f) \bar{p}, x - \bar{p} \rangle \geq 0, \quad \forall x \in \text{Fix}(T); \quad (3)$$

(ii) *if $\sum_n \beta_n < \infty$ and $\text{Fix}(S) \neq \emptyset$, then $(x_n)_n$ converges strongly to $\tilde{p} = P_{\text{Fix}(S)} f(\tilde{p})$ which is the unique point in $\text{Fix}(S)$ that satisfies the variational inequality*

$$\langle (I - f) \tilde{p}, x - \tilde{p} \rangle \geq 0, \quad \forall x \in \text{Fix}(S); \quad (4)$$

(iii) *if $\liminf_n \beta_n (1 - \beta_n) > 0$ and $\text{Fix}(T) \cap \text{Fix}(S) \neq \emptyset$, then $(x_n)_n$ strongly converges to $p_0 = P_{\text{Fix}(T) \cap \text{Fix}(S)} f(p_0)$*

which is the unique point in $\text{Fix}(T) \cap \text{Fix}(S)$ that satisfies the variational inequality

$$\langle (I - f)p_0, x - p_0 \rangle \geq 0, \quad \forall x \in \text{Fix}(T) \cap \text{Fix}(S). \quad (5)$$

At this point, three questions can be posed:

- (1) Is it possible to replace the coefficient $(1 - \alpha_n)$ with the term $(I - \alpha_n A)$ where $A : H \rightarrow H$ is a linear, bounded, strongly positive operator? In doing so, we would be able to solve some minimization problem.
- (2) Is it possible to replace the type average mappings T_δ and S_δ with the original mappings T and S ?
- (3) Is it possible to replace the nonspreading mapping S with a more general type of mappings?

We will see here that the answers are all positive.

A natural class of mappings containing the nonspreading mappings is given by the class of hybrid mappings. Following Aoyama et al. [3] we say that $S : H \rightarrow H$ is an L -hybrid mapping, with $L \geq 0$, if

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + L \langle x - Sx, y - Sy \rangle \quad (6)$$

$\forall x, y \in H$

or equivalently

$$2\|Sx - Sy\|^2 \leq \|x - Sy\|^2 + \|y - Sx\|^2 - 2\left(1 - \frac{L}{2}\right) \langle x - Sx, y - Sy \rangle. \quad (7)$$

We denote by $\mathcal{H}_L$ the family of L -hybrid mappings defined on the space H .

Note that particular choices of L give important families of nonlinear mappings:

- (i) $\mathcal{H}_0$ is the family of nonexpansive mappings (i.e., $\|Sx - Sy\| \leq \|x, y\|$ for all $x, y \in H$).
- (ii) $\mathcal{H}_2$ is the family of nonspreading mappings (i.e., $\|Sx - Sy\|^2 \leq \|x - y\|^2 + 2\langle x - Sx, y - Sy \rangle$, $\forall x, y \in H$).

Relevant properties of L -hybrid mappings are the following:

- ($\mathcal{H}_L$ -1) If $S \in \mathcal{H}_L$ and $\text{Fix}(S) \neq \emptyset$, then S is quasinonexpansive (i.e., $\|Tx - p\| \leq \|x - p\|$ for all $p \in \text{Fix}(T)$, for all $x \in H$) (the proof is immediate by the definition).
- ($\mathcal{H}_L$ -2) If $S \in \mathcal{H}_L \Rightarrow I - S$ is demiclosed in $0 \in H$ (for the proof see [1]).
- ($\mathcal{H}_L$ -3) If $S \in \mathcal{H}_L \Rightarrow \text{Fix}(S)$ is a closed and convex set (for the proof, see [4]).
- ($\mathcal{H}_L$ -4) If $S \in \mathcal{H}_L \Rightarrow S_\delta \in \mathcal{H}_{L/\delta}$, where S_δ is the type average mapping

$$S_\delta = (1 - \delta)I + \delta S \quad (8)$$

(for the proof, see [5]).

A first improvement of Theorem 1 [2] that one can obtain is to replace S with an L -hybrid mapping. (This will be a corollary of our main result, thanks to ($\mathcal{H}_L$ -1), ($\mathcal{H}_L$ -3), and next Proposition 2.)

But we can do better. Inspired by the paper of Wongchan and Saejung [6], we can consider a mapping S that has only the properties of being strongly quasinonexpansive and demiclosed.

We recall that the concept of strongly nonexpansive mapping was introduced by Bruck and Reich in 1977 [7] as follows: a mapping T is said to be strongly nonexpansive if T is nonexpansive and whenever $(x_n - y_n)$ is bounded and $\|x_n - y_n\| - \|Tx_n - Ty\| \rightarrow 0$, it follows that $(x_n - y_n) - (Tx_n - Ty_n) \rightarrow 0$.

To our knowledge, Saejung [8] in 2010 introduced the concept of strong quasi-nonexpansivity: a mapping S is said strongly quasi-nonexpansive if $\text{Fix}(S) \neq \emptyset$, S is quasinonexpansive and $x_n - Sx_n \rightarrow 0$ whenever (x_n) is a bounded sequence such that $\|x_n - p\| - \|Sx_n - p\| \rightarrow 0$ for some $p \in \text{Fix}(S)$.

In [7] was proved that an average mapping defined on a uniformly convex Banach space is strongly nonexpansive.

Following the same line on the proof, one can show that a type average mapping $S_\delta = (1 - \delta)I + \delta S$ of a quasinonexpansive mapping is strongly quasinonexpansive. For the sake of completeness, we propose here the following proof, lightly different, valid in Hilbert spaces.

Proposition 2. Let $S : C \rightarrow H$ be a quasinonexpansive mapping, where C is a closed convex subset of H . Then the type average mapping $S_\delta = (1 - \delta)I + \delta S$ is a strongly quasinonexpansive mapping.

Proof. Suppose $(x_n) \subset C$ is a bounded sequence such that $\|x_n - p\| - \|S_\delta x_n - p\| \rightarrow 0$ for some $p \in \text{Fix}(S)$. Now

$$\begin{aligned} \|S_\delta x_n - p\| &= \|(1 - \delta)x_n + \delta Sx_n - p\| \\ &\leq (1 - \delta)\|x_n - p\| + \delta\|Sx_n - p\| \\ &\implies \|x_n - p\| - \|S_\delta x_n - p\| \\ &\geq \delta(\|x_n - p\| - \|Sx_n - p\|). \end{aligned} \quad (9)$$

So

$$\begin{aligned} &(\text{by quasinonexpansivity of } S) \\ 0 &\leq \|x_n - p\| - \|Sx_n - p\| \\ &\leq \frac{1}{\delta} [\|x_n - p\| - \|S_\delta x_n - p\|] \rightarrow 0 \\ &\implies \|x_n - p\| - \|Sx_n - p\| \rightarrow 0 \text{ too.} \end{aligned} \quad (10)$$

Now, let $(x_{n_k} - p)$ be a subsequence of $(x_n - p)$ that admits limit, $\|x_{n_k} - p\| \rightarrow d$, as $k \rightarrow \infty$. Then, from the assumption that $\|x_n - p\| - \|S_\delta x_n - p\| \rightarrow 0$, it follows that $\|S_\delta x_{n_k} - p\| \rightarrow d$

as $k \rightarrow \infty$; that is, $\|(1 - \delta)(x_n - p) + \delta(Sx_n - p)\| \rightarrow d$. But then

$$\begin{aligned} d^2 &= \lim_k \|(1 - \delta)(x_{n_k} - p) + \delta(Sx_{n_k} - p)\|^2 \\ &= \lim_k \left[(1 - \delta) \|x_{n_k} - p\|^2 + \delta \|Sx_{n_k} - p\|^2 \right. \\ &\quad \left. - \delta(1 - \delta) \|Sx_{n_k} - x_{n_k}\|^2 \right] \\ &\Rightarrow \lim_k \|x_{n_k} - Sx_{n_k}\| = 0 \\ &\Rightarrow \lim_k \|x_{n_k} - S_\delta x_{n_k}\| = \delta \lim_k \|x_{n_k} - Sx_{n_k}\| = 0. \end{aligned} \quad (11)$$

By the boundness of the sequence (x_n) , this is sufficient, by the standard argument of compactness, to ensure $x_n - S_\delta x_n \rightarrow 0$ as $n \rightarrow \infty$. $\square$

So, by the quasinonexpansivity of L -hybrid mappings, it follows that in Theorem 1 [2] the assumption that S is a strongly quasinonexpansive mapping instead of type average nonspreeding (or also L -hybrid) mapping is a better assumption.

Of course, one can ask if some important L -hybrid mapping, as a nonspreeding mapping, or a nonexpansive mapping, is already strongly quasinonexpansive. This is not always true, as shown in the following proposition.

Proposition 3. *There exist nonexpansive mappings that are not strongly quasinonexpansive. Moreover, there exist nonspreeding mappings that are not strongly quasinonexpansive.*

Proof. Let $T : H \rightarrow H$ be such that $Tx = -x$. Then T is nonexpansive but not strongly quasinonexpansive.

Moreover let $X = A \cup B \cup C \subset H$, where

$$\begin{aligned} A &= \{x \in H : \|x\| \leq 1\}, \\ B &= \{x \in H : 1 < \|x\| < 2\}, \\ C &= \{x \in H : 2 \leq \|x\| \leq 3\}. \end{aligned} \quad (12)$$

Define $S : X \rightarrow X$ by

$$Sx := \begin{cases} x, & \text{if } x \in A, \\ \frac{x}{\|x\|}, & \text{if } x \in B, \\ 0, & \text{if } x \in C. \end{cases} \quad (13)$$

One can see that S is a nonspreeding mapping, distinguishing three cases: $(x \in A, y \in B)$, $(x \in A, y \in C)$, and $(x \in B, y \in C)$.

To see that S is not strongly quasinonexpansive, take x_0 with $\|x_0\| = 1$.

Then $x_0 \in \text{Fix}(S)$. Moreover, define $z_n = (2 + (1/n))x_0$. Then $Sz_n = 0$ and $\|z_n - x_0\| - \|Sz_n - x_0\| = 1 + (1/n) - 1 \rightarrow 0$ but $Sz_n - z_n = (2 + (1/n))x_0 \rightarrow 2x_0$. $\square$

Conversely, we have the following.

Proposition 4. *There exist strongly quasinonexpansive mappings that are not L -hybrid mappings for any L (and so, thanks to $(\mathcal{H}_L-4)$, that are not ever type average S_δ with S L -hybrid).*

Proof. Let $H = \mathbb{R}$. Define

$$\begin{aligned} T(0) &= 0, \\ T(n) &= -n + \frac{1}{n+1}, \quad \forall n \in \mathbb{N}, n > 0, \\ T(-n) &= n - \frac{1}{n+1}, \quad \forall n \in \mathbb{N}, n > 0. \end{aligned} \quad (14)$$

Then define in linear way T on each interval $[n, n+1]$, $n \in \mathbb{Z}$.

One can see easily that $\text{Fix}(T) = 0$ and T is strongly quasinonexpansive.

Moreover, from the fact that, for large $n \in \mathbb{Z}$, T is defined almost as $-I$, one can prove that can not be L -hybrid for any $L \geq 0$. $\square$

Finally, we show that the hypothesis on demiclosedness of $I - T$ is necessary in Theorem 2.3 (the main Theorem) of [6].

Proposition 5. *There exist strongly quasinonexpansive mappings such that $(I - S)$ are not demiclosed in 0.*

Proof. Let $H = \mathbb{R}$. Define $S : [0, 1] \rightarrow [0, 1]$ by

$$Sx := \begin{cases} x^2, & \text{if } 0 \leq x < 1, \\ \frac{1}{2}, & \text{if } x = 1. \end{cases} \quad (15)$$

Then $\text{Fix}(S) = \{0\}$, S is strongly quasinonexpansive, but $I - S$ is not demiclosed on 0 ($x_n = 1 - (1/n)$ is such that $x_n \rightarrow 1$ but $x_n - Sx_n \rightarrow 0$). $\square$

Reassuming, the main result of [6] has allowed us to improve Theorem 1 [2] with an emphasis on the concept of strong quasinonexpansivity instead of nonspreeding.

Our reasoning is inspired by the ideas contained in [2, 6, 9].

The solid bases on which our proof rests are given by the following lemmas.

Lemma 6 (Xu's Lemma [10]). *Assume $(a_n)_{n \in \mathbb{N}}$ is a sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1 - s_n)a_n + s_n\sigma_n + \gamma_n, \quad n \geq 0, \quad (16)$$

where $(s_n)_n$ is a sequence in $[0, 1]$ and $(\delta_n)_n$ is a sequence in $\mathbb{R}$ such that

- (1) $\sum_{n=1}^{\infty} s_n = \infty$;
- (2) $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$;
- (3) $\gamma_n \geq 0$, $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 7 (Maingé's Lemma [11]). *Let $(\gamma_n)_n$ be real sequence that has a subsequence (γ_{n_j}) which satisfies $\gamma_{n_j} < \gamma_{n_j+1}$ for all j . Then there exists an increasing sequence of integers $(\tau(n))_{n \geq n_0}$ satisfying*

- (1) $\lim_n \tau_n = +\infty$;
- (2) $\gamma_{\tau(n)} < \gamma_{\tau(n)+1}$, for all $n \geq n_0$;
- (3) $\gamma_n < \gamma_{\tau(n)+1}$, for all $n \geq n_0$.

2. Main Results

In all this section we denote by $O(1)$ any bounded real sequence (so, e.g., $O(1) + O(1) = O(1)$).

Theorem 8. *Let H be a Hilbert space. Suppose*

- (i) $T : H \rightarrow H$ is a nonexpansive mapping and $S : H \rightarrow H$ is a strongly quasinonexpansive mapping such that $I - S$ is demiclosed in 0, for which $\text{Fix}(T) \cap \text{Fix}(S) \neq \emptyset$;
- (ii) $A : H \rightarrow H$ is a linear bounded self-adjoint $\bar{\gamma}$ -strongly positive operator, $\bar{\gamma} < 1$;
- (iii) $f : H \rightarrow H$ is a contraction, with $\rho < \bar{\gamma}$ (of course this is not a real restriction, but this is useful for convenience of notations);
- (iv) $(\alpha_n)_n, (\beta_n)_n$ are two real sequences in $[0, 1]$.

Consider the iteration scheme

$$\begin{aligned} x_1 &\in C, \\ x_{n+1} &= \alpha_n f(x_n) + (I - \alpha_n A) [\beta_n T x_n + (1 - \beta_n) S x_n], \quad (17) \\ n &\geq 1. \end{aligned}$$

Then

- (1) if $\sum_n (1 - \beta_n) < \infty$, $\alpha_n \rightarrow 0$, $\sum_n \alpha_n = \infty$, and $\sum_n |\alpha_n - \alpha_{n+1}| < \infty$, then (x_n) strongly converges to $\bar{p} \in \text{Fix}(T)$ which is the unique point in $\text{Fix}(T)$ that solves the variational inequality

$$\langle (A - f) \bar{p}, x - \bar{p} \rangle \geq 0, \quad \forall x \in \text{Fix}(T), \quad (18)$$

which is the optimality condition for the minimization problem

$$\min_{x \in \text{Fix}(T)} \frac{1}{2} \langle Ax, x \rangle - h(x), \quad (19)$$

where h is a potential function for f (i.e., $h'(x) = f(x)$, for all $x \in H$);

- (2) if $\sum_n \beta_n < \infty$, $\alpha_n \rightarrow 0$, $\sum_n \alpha_n = +\infty$, and $\beta_n/\alpha_n \rightarrow 0$, then $(x_n)_n$ converges strongly to $\bar{p} \in \text{Fix}(S)$ which is the unique solution in $\text{Fix}(S)$ of the variational inequality

$$\langle (A - f) \bar{p}, x - \bar{p} \rangle \geq 0, \quad \forall x \in \text{Fix}(S); \quad (20)$$

- (3) if $\liminf_n \beta_n (1 - \beta_n) > 0$, $\alpha_n \rightarrow 0$, and $\sum_n \alpha_n = +\infty$, then (x_n) strongly converges to $p_0 \in \text{Fix}(T) \cap \text{Fix}(S)$ which is the unique solution in $\text{Fix}(T) \cap \text{Fix}(S)$ of the variational inequality

$$\langle (A - f) p_0, x - p_0 \rangle \geq 0, \quad \forall x \in \text{Fix}(T) \cap \text{Fix}(S). \quad (21)$$

Proof. First of all, we see that (x_n) is a bounded sequence.

Indeed, let $U_n = \beta_n T + (1 - \beta_n) S$ and $z \in \text{Fix}(T) \cap \text{Fix}(S)$. Then $\|x_{n+1} - z\| = \|\alpha_n f(x_n) + (I - \alpha_n A) U_n x_n - p\|$ (by linearity of A) $= \|(I - \alpha_n A)(U_n x_n - z) + \alpha_n(f(z) - Az) - \alpha_n(f(z) - f(x_n))\| \leq (\text{since } \|I - \alpha_n A\| \leq 1 - \alpha_n \bar{\gamma}, [9]) \leq (1 - \alpha_n \bar{\gamma} + \alpha_n \rho) \|x_n - z\| + \alpha_n \|f(z) - Az\| = [1 - \alpha_n(\bar{\gamma} - \rho)] \|x_n - z\| + \alpha_n(\bar{\gamma} - \rho) (\|f(z) - Az\|/\bar{\gamma} \rho)$ (from convexity) $\leq \max\{\|x_n - z\|, \|f(z) - Az\|/\bar{\gamma} \rho\}$ (from induction) $\leq \max\{\|x_1 - z\|, \|f(z) - Az\|/\bar{\gamma} \rho\}$.

Then (x_n) is bounded.

Moreover

$$x_{n+1} - U_n x_n = \alpha_n (f(x_n) - A U_n x_n) \rightarrow 0, \quad (22)$$

$n \rightarrow \infty,$

since $\alpha_n \rightarrow 0$.

Proof of (1). The key is to prove that

$$x_{n+1} - x_n \rightarrow 0. \quad (23)$$

To see this, we calculate

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(I - \alpha_n A)(U_n x_n - U_{n-1} x_{n-1}) - (\alpha_n \\ &\quad - \alpha_{n-1}) A U_{n-1} x_{n-1} + \alpha_n (f(x_n) - f(x_{n-1})) + (\alpha_n \\ &\quad - \alpha_{n-1}) f(x_{n-1})\| = \|(I - \alpha_n A)(U_n x_n - U_{n-1} x_{n-1}) \\ &\quad + (\alpha_{n-1} - \alpha_n) [A U_{n-1} x_{n-1} - f(x_{n-1})] \\ &\quad + \alpha_n (f(x_n) - f(x_{n-1}))\| = \|(\alpha_{n-1} - \alpha_n) \\ &\quad \cdot (A U_{n-1} x_{n-1} - f(x_{n-1})) + \alpha_n (f(x_n) \\ &\quad - f(x_{n-1})) + (I - \alpha_n A) [\beta_n T x_n + (1 - \beta_n) S x_n \\ &\quad - \beta_{n-1} T x_{n-1} - (1 - \beta_{n-1}) S x_{n-1}]\| = \|(\alpha_{n-1} - \alpha_n) \\ &\quad \cdot (A U_{n-1} x_{n-1} - f(x_{n-1})) + \alpha_n (f(x_n) \\ &\quad - f(x_{n-1})) + (I - \alpha_n A) [\beta_n (T x_n - T x_{n-1}) \\ &\quad + (1 - \beta_n) (S x_n - S x_{n-1}) + (\beta_n - \beta_{n-1}) T x_{n-1} \\ &\quad + (\beta_{n-1} - \beta_n) S x_{n-1}]\| \\ &\Rightarrow (\text{by nonexpansivity of } T), \end{aligned} \quad (24)$$

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq |\alpha_{n-1} - \alpha_n| O(1) + \alpha_n \rho \|x_n - x_{n-1}\| \\ &\quad + (1 - \alpha_n \bar{\gamma}) [\beta_n \|x_n - x_{n-1}\| + (1 - \beta_n) O(1) + |\beta_n \\ &\quad - \beta_{n-1}| O(1)] = (1 - s_n) \|x_n - x_{n-1}\| + \gamma_n, \end{aligned}$$

where $s_n = 1 - \beta_n + \alpha_n(\bar{\gamma} \beta_n - \rho) \geq \alpha_n O(1)$; definitively, $\gamma_n = |\alpha_{n-1} - \alpha_n| O(1) + [(1 - \alpha_n \bar{\gamma})(1 - \beta_n) + |\beta_n - \beta_{n-1}|] O(1)$.

Thanks to hypotheses on α_n, β_n , we see that $s_n \rightarrow 0$, $\sum_n s_n = +\infty$, and $\sum_n \gamma_n < \infty$. This is sufficient, from Xu's Lemma, to conclude $x_{n+1} - x_n \rightarrow 0$.

From this and (22), it follows immediately that

$$x_n - U_n x_n \rightarrow 0. \quad (25)$$

Indeed, $x_n - U_n x_n = (x_n - x_{n+1}) - (x_{n+1} - U_n x_n)$.

Moreover

$$\begin{aligned} \|x_n - U_n x_n\| &= \|x_n - \beta_n T x_n - (1 - \beta_n) S x_n\| \\ &\geq \|x_n - \beta_n T x_n\| - (1 - \beta_n) \|S x_n\| \\ &\implies \|x_n - \beta_n T x_n\| \\ &\leq \|x_n - U_n x_n\| + (1 - \beta_n) O(1), \end{aligned} \quad (26)$$

and so from (25) and hypotheses $\sum_n (1 - \beta_n) < \infty$, we have also

$$x_n - \beta_n T x_n \rightarrow 0. \quad (27)$$

From this we deduce also that

$$x_n - T x_n \rightarrow 0, \quad (28)$$

and this gives that any weak limit of (x_n) is in $\text{Fix}(T)$, since T is nonexpansive, and thus the Principle of Demiclosedness is satisfied.

Now we can show that $x_n \rightarrow \bar{p}$, where $\bar{p}$ is the unique solution in $\text{Fix}(T)$ of variational inequality (18). We show first that

$$\limsup_n \langle U_n x_n - \bar{p}, f(\bar{p}) - A\bar{p} \rangle \leq 0. \quad (29)$$

Indeed, let (x_{n_k}) be a subsequence of (x_n) such that

$$\begin{aligned} \limsup_n \langle x_n - \bar{p}, f(\bar{p}) - A\bar{p} \rangle \\ = \lim_k \langle x_{n_k} - \bar{p}, f(\bar{p}) - A\bar{p} \rangle, \end{aligned} \quad (30)$$

and $x_{n_k} \rightharpoonup z$. Then $z \in \text{Fix}(T)$ and so, from (30),

$$\limsup_n \langle x_n - \bar{p}, f(\bar{p}) - A\bar{p} \rangle = \langle z - \bar{p}, f(\bar{p}) - A\bar{p} \rangle, \quad (31)$$

and this is nonpositive by definition of $\bar{p}$.

So,

$$\begin{aligned} \limsup_n \langle U_n x_n - \bar{p}, f(\bar{p}) - A\bar{p} \rangle \\ = \limsup_n [\langle x_n - \bar{p}, f(\bar{p}) - A\bar{p} \rangle \\ + \langle U_n x_n - x_n, f(\bar{p}) - A\bar{p} \rangle] \quad (\text{from (25)}) \\ = \limsup_n \langle x_n - \bar{p}, f(\bar{p}) - A\bar{p} \rangle \quad (\text{by (30)}) \\ \leq 0. \end{aligned} \quad (32)$$

Finally

$$\begin{aligned} \|x_{n+1} - \bar{p}\| &= \|(I - \alpha_n A)(U_n x_n - \bar{p}) + \alpha_n (f(x_n) - A\bar{p})\|^2 \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|U_n x_n - \bar{p}\|^2 + \alpha_n^2 \|f(x_n) - A\bar{p}\|^2 \\ &\quad + 2\alpha_n \langle (I - \alpha_n A)(U_n x_n - \bar{p}), f(x_n) - A\bar{p} \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|\beta_n (T x_n - \bar{p}) + (1 - \beta_n) (S x_n - \bar{p})\|^2 \\ &\quad + \alpha_n^2 \|f(x_n) - A\bar{p}\|^2 \\ &\quad + 2\alpha_n \langle U_n x_n - \bar{p}, f(x_n) - A\bar{p} \rangle \\ &\quad - 2\alpha_n^2 \langle A(U_n x_n - \bar{p}), f(x_n) - A\bar{p} \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 [\beta_n \|x_n - \bar{p}\| + (1 - \beta_n) O(1)]^2 \\ &\quad + \alpha_n^2 O(1) + 2\alpha_n \langle U_n x_n - \bar{p}, f(x_n) - A\bar{p} \rangle \\ &\quad + 2\alpha_n^2 O(1) \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - \bar{p}\|^2 + (1 - \beta_n) O(1) + \alpha_n^2 O(1) \\ &\quad + 2\alpha_n \langle U_n x_n - \bar{p}, f(x_n) - f(\bar{p}) \rangle \\ &\quad + 2\alpha_n \langle U_n x_n - \bar{p}, f(\bar{p}) - A\bar{p} \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - \bar{p}\|^2 + (1 - \beta_n) O(1) + \alpha_n^2 O(1) \\ &\quad + 2\alpha_n \langle U_n x_n - \bar{p} + f(\bar{p}) - A\bar{p} \rangle \\ &\quad + 2\alpha_n [\beta_n \|x_n - \bar{p}\| + (1 - \beta_n) O(1)] \rho \|x_n - \bar{p}\| \\ &\leq [(1 - \alpha_n \bar{\gamma})^2 + 2\alpha_n \rho] \|x_n - \bar{p}\|^2 \\ &\quad + \alpha_n [\alpha_n O(1) + 2 \langle U_n x_n - \bar{p}, f(\bar{p}) - A\bar{p} \rangle] \\ &\quad + (1 - \beta_n) O(1) \\ &= (1 - s_n) \|x_n - \bar{p}\|^2 + \alpha_n \sigma_n + \gamma_n, \end{aligned} \quad (33)$$

where

$$\begin{aligned} s_n &= 1 - (1 - \alpha_n \bar{\gamma})^2 - 2\alpha_n \rho \\ &= 1 - 1 - \alpha_n^2 \bar{\gamma}^2 + 2\alpha_n \bar{\gamma} - 2\alpha_n \rho \\ &= \alpha_n [\alpha_n \bar{\gamma}^2 + 2(\bar{\gamma} - \rho)], \\ \sigma_n &= \alpha_n O(1) + 2 \langle U_n x_n - \bar{p}, f(\bar{p}) - A\bar{p} \rangle, \\ \gamma_n &= (1 - \beta_n) O(1). \end{aligned} \quad (34)$$

Thanks to the hypotheses on α_n, β_n and (29), from Xu's Lemma again, we obtain $x_n \rightarrow \bar{p}$.

Proof of (2). Let $\tilde{p}$ be the unique solution of variational inequality (20).

We want to show that $x_n \rightarrow \tilde{p}$. Now,

$$\begin{aligned}
& \|x_{n+1} - \tilde{p}\|^2 \\
&= \|\alpha_n f(x_n) + (I - \alpha_n A) U_n x_n - \tilde{p} + \alpha_n \tilde{p} - \alpha_n \tilde{p} + \alpha_n A z - \alpha_n A z\|^2 \\
&= \|\alpha_n (f(x_n) - A z) + (I - \alpha_n A) (U_n x_n - \tilde{p})\|^2 \\
& \text{(by the subdifferential inequality } \|x + y\|^2 \leq \|x\|^2 + 2 \langle y, x + y \rangle) \\
&\leq (1 - \alpha_n \bar{\gamma})^2 \|U_n x_n - \tilde{p}\|^2 + 2\alpha_n \langle f(x_n) - A \tilde{p}, x_{n+1} - \tilde{p} \rangle \\
&= (1 - \alpha_n \bar{\gamma})^2 \|\beta_n (Tx_n - \tilde{p}) + (1 - \beta_n) (Sx_n - \tilde{p})\|^2 \\
& \quad + 2\alpha_n \langle f(x_n) - f(\tilde{p}), x_{n+1} - \tilde{p} \rangle \\
& \quad + 2\alpha_n \langle f(\tilde{p}) - A \tilde{p}, x_{n+1} - \tilde{p} \rangle \\
& \quad \text{(by the quasinonexpansivity of } S) \\
&\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - \tilde{p}\|^2 + \beta_n O(1) + 2\alpha_n \rho \|x_n - \tilde{p}\| \|x_{n+1} - \tilde{p}\| \\
& \quad + 2\alpha_n \langle x_{n+1} - \tilde{p}, f(\tilde{p}) - A \tilde{p} \rangle.
\end{aligned} \tag{35}$$

At this point we distinguish two cases: or the sequence $\|x_n - \tilde{p}\|$ is definitively not increasing or not.

Alternative 1. ($\|x_n - \tilde{p}\|$) is definitively not increasing, so $\|x_{n+1} - \tilde{p}\| \leq \|x_n - \tilde{p}\|$, $\forall n \geq N$.

Then (35) furnish

$$\begin{aligned}
\|x_{n+1} - \tilde{p}\|^2 &\leq [(1 - \alpha_n \bar{\gamma})^2 + 2\alpha_n \rho] \|x_n - \tilde{p}\| \\
& \quad + \beta_n O(1) \\
& \quad + 2\alpha_n \langle x_{n+1} - \tilde{p} - f(\tilde{p}) - A \tilde{p} \rangle.
\end{aligned} \tag{36}$$

Putting $s_n = 1 - (1 - \alpha_n \bar{\gamma})^2 - 2\alpha_n \rho = \alpha_n \bar{\gamma}^2 - 2\alpha_n \bar{\gamma} + 2\alpha_n \rho = \alpha_n [\alpha_n \bar{\gamma}^2 + 2(\bar{\gamma} - \rho)] \geq 0$, $\sigma_n = \langle x_{n+1} - \tilde{p}, f(\tilde{p}) - A \tilde{p} \rangle$, and $\gamma_n = \beta_n O(1)$, we can rewrite (36) as

$$\|x_{n+1} - \tilde{p}\|^2 \leq (1 - s_n) \|x_n - \tilde{p}\|^2 + \alpha_n \sigma_n + \gamma_n, \tag{37}$$

so the thesis $x_n \rightarrow \tilde{p}$ will follow again by Xu's Lemma if we are able to show that

$$\limsup_n \langle x_{n+1} - \tilde{p}, f(\tilde{p}) - A \tilde{p} \rangle \leq 0. \tag{38}$$

(Note that until now we have not used the hypothesis of strong quasinonexpansivity of S .) Now, since ($\|x_n - \tilde{p}\|$) is definitively not increasing, there exists the $\lim_n \|x_n - \tilde{p}\|$. Then

$$\begin{aligned}
0 &= \lim_n \|x_{n+1} - \tilde{p}\| - \|x_n - \tilde{p}\| \\
&\leq \liminf_n [\alpha_n \|f(x_n) - A \tilde{p}\| \\
& \quad + (1 - \alpha_n \bar{\gamma}) \|U_n x_n - \tilde{p}\| - \|x_n - \tilde{p}\|] \\
& \quad \text{(since } \alpha_n \rightarrow 0)
\end{aligned}$$

$$\begin{aligned}
&= \liminf_n [\|U_n x_n - \tilde{p}\| - \|x_n - \tilde{p}\|] \\
&= \liminf_n \|\beta_n (Tx_n - \tilde{p}) + (1 - \beta_n) (Sx_n - \tilde{p})\| \\
& \quad - \|x_n - \tilde{p}\| \text{ (by hypothesis } \beta_n \rightarrow 0) \\
&= \liminf_n \|Sx_n - \tilde{p}\| - \|x_n - \tilde{p}\| \leq \liminf_n \|Sx_n - \tilde{p}\| \\
& \quad - \|x_n - \tilde{p}\| \text{ (by quasinonexpansivity of } S) \\
&\leq \limsup_n \|x_n - \tilde{p}\| - \|x_n - \tilde{p}\| = 0.
\end{aligned} \tag{39}$$

Thus

$$\lim_n \|Sx_n - \tilde{p}\| - \|x_n - \tilde{p}\| = 0. \tag{40}$$

From the strong quasinonexpansivity of S , we deduce

$$Sx_n - x_n \rightarrow 0. \tag{41}$$

At this point, by using the demiclosedness of $(I - S)$ in 0 , we can proceed as in the Proof of (1) to show (38).

The statement is proved when Alternative 1 holds.

Alternative 2. ($\|x_n - \tilde{p}\|$) is not definitively not increasing; that is, there exists a subsequence ($\|x_{n_j} - \tilde{p}\|$) such that

$$\|x_{n_j} - \tilde{p}\| < \|x_{n_j+1} - \tilde{p}\|, \quad \forall j \in \mathbb{N}. \tag{42}$$

From Maingé's Lemma, it follows that there exists an increasing sequence of integers $(\tau(n))_{n \in \mathbb{N}}$ satisfying

$$\lim_n \tau(n) = +\infty, \tag{43}$$

$$\|x_{\tau(n)} - \tilde{p}\| < \|x_{\tau(n)+1} - \tilde{p}\|,$$

$$\|x_n - \tilde{p}\| < \|x_{\tau(n)} - \tilde{p}\|, \quad \forall n \geq n_0. \tag{44}$$

Then

$$0 \leq \liminf_n \|x_{\tau(n)+1} - \tilde{p}\| - \|x_{\tau(n)} - \tilde{p}\|. \tag{45}$$

Retracing the same inequalities used to have (41) with $\tau(n)$ instead of n , we obtain

$$\lim_n \|Sx_{\tau(n)} - \tilde{p}\| - \|x_{\tau(n)} - \tilde{p}\| = 0. \tag{46}$$

Again the strong quasinonexpansivity yields

$$Sx_{\tau(n)} - x_{\tau(n)} \rightarrow 0, \tag{47}$$

And, from the demiclosedness of $I - S$ in 0 , we deduce as above that

$$\limsup_n \langle x_{\tau(n)+1} - \tilde{p}, f(\tilde{p}) - A \tilde{p} \rangle \leq 0. \tag{48}$$

Incidentally, we observe that

$$\begin{aligned}
\|x_{\tau(n)+1} - x_{\tau(n)}\| &= \alpha_{\tau(n)} \|(f(x_{\tau(n)}) - Ax_{\tau(n)})\| \\
& \quad + (1 - \alpha_{\tau(n)} \bar{\gamma}) \beta_{\tau(n)} O(1) \\
& \quad + (1 - \alpha_{\tau(n)} \bar{\gamma}) \|Sx_{\tau(n)} - x_{\tau(n)}\|,
\end{aligned} \tag{49}$$

and so, from (47), it follows also that $x_{\tau(n)+1} - x_{\tau(n)} \rightarrow 0$.

Now rewrite (35) with $\tau(n)$ instead of n :

$$\begin{aligned}
& \|x_{\tau(n)+1} - \tilde{p}\| \\
& \leq (1 - \alpha_{\tau(n)}\bar{\gamma})^2 \|x_{\tau(n)} - \tilde{p}\|^2 \\
& \quad + 2\alpha_{\tau(n)}\rho \|x_{\tau(n)} - \tilde{p}\| \|x_{\tau(n)+1} - \tilde{p}\| \\
& \quad + 2\alpha_{\tau(n)} \langle x_{\tau(n)+1} - \tilde{p}, f(\tilde{p}) - A\tilde{p} \rangle + \beta_{\tau(n)}O(1) \\
& \quad \quad \quad \text{(by property (43))} \\
& \leq [(1 - \alpha_{\tau(n)}\bar{\gamma})^2 + 2\alpha_{\tau(n)}\rho] \|x_{\tau(n)+1} - \tilde{p}\|^2 \\
& \quad + 2\alpha_{\tau(n)} \langle x_{\tau(n)+1} - \tilde{p}, f(\tilde{p}) - A\tilde{p} \rangle + \beta_{\tau(n)}O(1) \\
& \Rightarrow 2\alpha_{\tau(n)} (\bar{\gamma} - \rho) \|x_{\tau(n)+1} - \tilde{p}\|^2 \\
& \leq (\alpha_{\tau(n)}\bar{\gamma})^2 \|x_{\tau(n)+1} - \tilde{p}\|^2 \\
& \quad + 2\alpha_{\tau(n)} \langle x_{\tau(n)+1} - \tilde{p}, f(\tilde{p}) - A\tilde{p} \rangle + \beta_{\tau(n)}O(1) \Rightarrow
\end{aligned} \tag{50}$$

dividing by $\alpha_{\tau(n)}$

$$\begin{aligned}
0 & \leq 2(\bar{\gamma} - \rho) \|x_{\tau(n)+1} - \tilde{p}\|^2 \\
& \leq \alpha_{\tau(n)}\bar{\gamma}^2 \|x_{\tau(n)+1} - \tilde{p}\|^2 \\
& \quad + 2 \langle x_{\tau(n)+1} - \tilde{p}, f(\tilde{p}) - A\tilde{p} \rangle + \frac{\beta_{\tau(n)}}{\alpha_{\tau(n)}}O(1).
\end{aligned} \tag{51}$$

Passing to limsup and recalling the hypothesis $\beta_n/\alpha_n \rightarrow 0$ and (48), we obtain $\lim_n \|x_{\tau(n)} - \tilde{p}\| \leq \lim_n \|x_{\tau(n)+1} - \tilde{p}\| = 0$. Equation (44) ensures that also $x_n \rightarrow \tilde{p}$.

Proof of (3). Let p_0 be the unique point in $\text{Fix}(S) \cap F$ that satisfies variational inequality (21). Then

$$\begin{aligned}
& \|U_n x_n - p_0\|^2 \\
& = \|\beta_n (Tx_n - p_0) + (1 - \beta_n) (Sx_n - p_0)\|^2 \\
& = \beta_n \|Tx_n - p_0\|^2 + (1 - \beta_n) \|Sx_n - p_0\|^2 \\
& \quad - \beta_n (1 - \beta_n) \|Tx_n - Sx_n\|^2 \\
& \leq \|x_n - p_0\|^2 - \beta_n (1 - \beta_n) \|Tx_n - Sx_n\|^2 \Rightarrow \\
& \|x_{n+1} - p_0\|^2 \\
& = \|\alpha_n (f(x_n) - Ap_0) + (I - \alpha_n A) (U_n x_n - p_0)\|^2 \\
& \leq (1 - \alpha_n \bar{\gamma})^2 \|U_n x_n - p_0\|^2 + \alpha_n^2 O(1) + \alpha_n O(1) \\
& \leq \|x_n - p_0\|^2 - \beta_n (1 - \beta_n) \|Tx_n - Sx_n\|^2 \\
& \quad + \alpha_n^2 O(1) + \alpha_n O(1).
\end{aligned} \tag{52}$$

Also now we distinguish two cases.

Alternative 1. ($\|x_n - p_0\|$) is definitively not increasing, $\|x_{n+1} - p_0\| \leq \|x_n - p_0\|$, $\forall n \geq N$.

Then there exists $\lim_n \|x_n - \tilde{p}\|$, so (52) furnish

$$\begin{aligned}
& \beta_n (1 - \beta_n) \|Tx_n - Sx_n\| \\
& \leq \|x_n - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_n^2 O(1) + \alpha_n O(1),
\end{aligned} \tag{53}$$

and so, by hypothesis $\liminf_n \beta_n (1 - \beta_n) > 0$, we deduce

$$Tx_n - Sx_n \rightarrow 0. \tag{54}$$

Moreover,

$$\begin{aligned}
0 & = \lim_n \|x_{n+1} - p_0\| - \|x_n - p_0\| \\
& \leq \liminf [\alpha_n \|f(x_n) - Ap_0\| \\
& \quad + (1 - \alpha_n \bar{\gamma}) \|U_n x_n - p_0\| - \|x_n - p_0\|] \\
& \quad \quad \quad \text{(since } \alpha_n \rightarrow 0) \\
& \leq \liminf \|U_n x_n - p_0\| - \|x_n - p_0\| \\
& = \liminf \|\beta_n (Tx_n - p_0) + (1 - \beta_n) (Sx_n - p_0)\| \\
& \quad - \|x_n - p_0\| \leq \liminf \beta_n [\|Tx_n - p_0\| - \|x_n - p_0\|] \\
& \quad + (1 - \beta_n) [\|Sx_n - p_0\| - \|x_n - p_0\|] \\
& \leq \limsup_n \beta_n [\|Tx_n - p_0\| - \|x_n - p_0\|] + (1 - \beta_n) \\
& \quad \cdot [\|Sx_n - p_0\| - \|x_n - p_0\|] \\
& \quad \quad \quad \text{(by the quasinonexpansivity of } T \text{ and } S) \\
& \leq 0.
\end{aligned} \tag{55}$$

But then

$$\begin{aligned}
& \lim_n \{\beta_n [\|Tx_n - p_0\| - \|x_n - p_0\|] \\
& \quad + (1 - \beta_n) [\|Sx_n - p_0\| - \|x_n - p_0\|]\} = 0.
\end{aligned} \tag{56}$$

Since both of the addends are nonpositive and the limit of the sum is zero, it follows that

$$\begin{aligned}
& \lim_n \beta_n [\|Tx_n - p_0\| - \|x_n - p_0\|] \\
& = \lim_n (1 - \beta_n) [\|Sx_n - p_0\| - \|x_n - p_0\|] = 0.
\end{aligned} \tag{57}$$

But then the hypothesis $\liminf_n \beta_n (1 - \beta_n) > 0$ implies that

$$\begin{aligned}
& \lim_n \|Tx_n - p_0\| - \|x_n - p_0\| \\
& = \lim_n \|Sx_n - p_0\| - \|x_n - p_0\| = 0.
\end{aligned} \tag{58}$$

From strong quasinonexpansivity of S , it follows that

$$Sx_n - x_n \rightarrow 0. \tag{59}$$

Again $x_n - Tx_n = x_n - Sx_n + Sx_n - Tx_n$, so, by (54) and (57),

$$x_n - Tx_n \rightarrow 0. \tag{60}$$

We show now that

$$\limsup_n \langle x_n - p_0, f(p_0) - Ap_0 \rangle \leq 0. \quad (61)$$

Indeed, select a subsequence $x_{n_k} \rightharpoonup z$ such that

$$\begin{aligned} \limsup_n \langle x_n - p_0, f(p_0) - Ap_0 \rangle \\ = \lim_k \langle x_{n_k} - p_0, f(p_0) - Ap_0 \rangle \\ = \langle z - p_0, f(p_0) - Ap_0 \rangle. \end{aligned} \quad (62)$$

But by the demiclosedness of both T and S and by (59) and (60), one deduces that

$$z \in \text{Fix}(S) \cap \text{Fix}(T), \text{ and so, by definition of } p_0, \quad (63)$$

(61) is obtained.

Moreover, from (59) and (60) at once, it follows that

$$U_n x_n - x_n \longrightarrow 0. \quad (64)$$

Finally we are able to show $x_n \rightarrow p_0$. Indeed,

$$\begin{aligned} \|x_{n+1} - p_0\|^2 &= \|(I - \alpha_n A)(U_n x_n - p_0) \\ &\quad + \alpha_n (f(x_n) - Ap_0)\|^2 = (1 - \alpha_n \bar{\gamma})^2 \|U_n x_n - p_0\|^2 \\ &\quad + \alpha_n^2 \|f(x_n) - Ap_0\|^2 \\ &\quad + 2\alpha_n \langle (I - \alpha_n A)(U_n x_n - p_0), f(x_n) - Ap_0 \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - p_0\|^2 + \alpha_n^2 [\|f(x_n) - Ap_0\|^2 \\ &\quad - 2 \langle A(U_n x_n - p_0), f(x_n) - Ap_0 \rangle] + 2\alpha_n \langle U_n x_n \\ &\quad - x_n, f(x_n) - Ap_0 \rangle + 2\alpha_n \langle x_n - p_0, f(x_n) \\ &\quad - f(p_0) \rangle + 2\alpha_n \langle x_n - p_0, f(p_0) - Ap_0 \rangle \\ &\leq [(1 - \alpha_n \bar{\gamma})^2 + 2\alpha_n \rho] \|x_n - p_0\|^2 + \alpha_n^2 O(1) \\ &\quad + 2\alpha_n \langle U_n x_n - x_n, f(x_n) - Ap_0 \rangle + 2\alpha_n \langle x_n \\ &\quad - p_0, f(p_0) - p_0 \rangle. \end{aligned} \quad (65)$$

Put

$$\begin{aligned} s_n &= 1 - (1 - \alpha_n \bar{\gamma})^2 - 2\alpha_n \rho, \\ \sigma_n &= 2 \langle U_n x_n - x_n, f(x_n) - Ap_0 \rangle \\ &\quad + 2 \langle x_n - p_0, f(p_0) - p_0 \rangle + \alpha_n O(1), \end{aligned} \quad (66)$$

and the thesis follows again by Xu's Lemma, taking account of (61) and (64).

Alternative 2. ($\|x_n - p_0\|$) is not definitively not increasing; that is, there exists a subsequence $(\|x_{n_j} - p_0\|)$ such that $\|x_{n_j} - p_0\| < \|x_{n_{j+1}} - p_0\|, \forall j \in \mathbb{N}$.

From Maingé's Lemma, it follows that there exists an increasing sequence of integers $(\tau(n))_{n \in \mathbb{N}}$ satisfying (43) and (44).

Then

$$\begin{aligned} 0 &\leq \liminf_n \|x_{\tau(n)+1} - p_0\| - \|x_{\tau(n)} - p_0\| \\ &\leq \limsup_n \|x_{\tau(n)+1} - p_0\| - \|x_{\tau(n)} - p_0\| \\ &\leq \limsup_n \|x_{n+1} - p_0\| - \|x_n - p_0\| \\ &\leq \limsup_n \|(I - \alpha_n A)(U_n x_n - p_0) \\ &\quad + \alpha_n (f(x_n) - Ap_0)\| - \|x_n - p_0\| \leq (1 - \alpha_n \bar{\gamma}) \|x_n \\ &\quad - p_0\| + \alpha_n O(1) - \|x_n - p_0\| \longrightarrow 0, \text{ as } n \longrightarrow \infty. \end{aligned} \quad (67)$$

Hence

$$\lim_n [\|x_{\tau(n)+1} - p_0\| - \|x_{\tau(n)} - p_0\|] = 0. \quad (68)$$

Now, retracing the same inequalities used to obtain (59) with $\tau(n)$ instead of n , we have

$$Sx_{\tau(n)} - x_{\tau(n)} \longrightarrow 0. \quad (69)$$

Moreover, we can rewrite (52) as

$$\begin{aligned} 0 &\leq \beta_{\tau(n)} (1 - \beta_{\tau(n)}) \|Tx_{\tau(n)} - Sx_{\tau(n)}\|^2 \\ &\leq \|x_{\tau(n)} - p_0\|^2 - \|x_{\tau(n)+1} - p_0\|^2 + \alpha_n O(1), \end{aligned} \quad (70)$$

and so, from (68) and the hypothesis $\liminf_n \beta_n (1 - \beta_n) > 0$,

$$Tx_{\tau(n)} - Sx_{\tau(n)} \longrightarrow 0. \quad (71)$$

Again,

$$x_{\tau(n)} - Tx_{\tau(n)} = x_{\tau(n)} - Sx_{\tau(n)} + Sx_{\tau(n)} - Tx_{\tau(n)}, \quad (72)$$

so, by (69) and (71),

$$x_{\tau(n)} - Tx_{\tau(n)} \longrightarrow 0, \quad (73)$$

so also

$$x_{\tau(n)} - U_{\tau(n)} x_{\tau(n)} \longrightarrow 0. \quad (74)$$

The same reasoning used to have (61) can be now repeated with $\tau(n)$ instead of n obtaining

$$\limsup_n \langle x_{\tau(n)} - p_0, f(p_0) - Ap_0 \rangle \leq 0. \quad (75)$$

Analogously, we can repeat all the reasoning used to deduce (68) with $\tau(n)$ instead of n and obtain

$$\|x_{\tau(n)+1} - p_0\|^2 \leq (1 - s_{\tau(n)}) \|x_{\tau(n)} - p_0\|^2 + s_{\tau(n)} \sigma_{\tau(n)}, \quad (76)$$

where

$$\begin{aligned} s_{\tau(n)} &= 1 - (1 - \alpha_{\tau(n)} \bar{\gamma})^2 - 2\alpha_{\tau(n)} \rho, \\ \sigma_{\tau(n)} &= \frac{\alpha_{\tau(n)}}{s_{\tau(n)}} [2 \langle U_{\tau(n)} x_{\tau(n)} - x_{\tau(n)}, f(x_{\tau(n)}) - Ap_0 \rangle \\ &\quad + 2 \langle x_{\tau(n)} - p_0, f(p_0) - Ap_0 \rangle] + \alpha_n O(1), \end{aligned} \quad (77)$$

So, from Xu's Lemma, taking in account (74) and (75), we have

$$\|x_{\tau(n)} - p_0\|^2 \longrightarrow 0. \quad (78)$$

Once again by (44) we deduce $x_n \rightarrow p_0$. $\square$

Remark 9. Theorem 1.1 in [2] is a corollary of our main result, but the technique of proof is completely different. Here we used the strong quasiconvexity of the mapping S . In Theorem 1 we used the following properties of a type average $S_\delta = (1 - \delta)I + \delta S$ of a nonspreading mapping S :

- (i) $\|S_\delta x - S_\delta y\|^2 \leq \|x - y\|^2 + (2/\delta)\langle x - S_\delta x, y - S_\delta y \rangle - (1 - \delta)\|(x - S_\delta x) - (S_\delta y)\|^2$.
- (ii) $(\delta/2)\|x - Sx\|^2 \leq \langle x - S_\delta x, x - q \rangle, \forall q \in \text{Fix}(S)$.

From Proposition 3 we know that there exist nonspreading mappings that are not strongly quasiconvex mappings.

So we do not know if the main result remains true if we take a nonspreading mapping S instead of a strongly quasiconvex mapping, or of type average S_δ of a nonspreading mapping.

The question remains open. However we would like to note that, in all the examples that we know, the algorithm works also with nonspreading mappings.

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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