Research Article

Topological Structures of Derivative Weighted Composition Operators on the Bergman Space

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Received 24 July 2015; Accepted 20 October 2015

Academic Editor: Nikolai L. Vasilevski

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We characterize the difference of derivative weighted composition operators on the Bergman space in the unit disk and determine when linear-fractional derivative weighted composition operators belong to the same component of the space of derivative weighted composition operators on the Bergman space under the operator norm topology.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane. The algebra of all holomorphic functions on domain $\mathbb{D}$ will be denoted by $H(\mathbb{D})$. Let $S(\mathbb{D})$ be the set of analytic self-maps of $\mathbb{D}$. Every $\varphi \in S(\mathbb{D})$ induces the composition operator C_φ defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$.

Let $u : \mathbb{D} \rightarrow \mathbb{C}$ be analytic, and the weighted composition operator, uC_φ , is defined by

$$uC_\varphi(f)(z) = u(z) \cdot f(\varphi(z)) \quad (1)$$

for any $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. When $u \equiv 1$, it becomes a composition operator.

Derivative weighted composition (d -composition) operator is defined by

$$D_\varphi(f) = \varphi' f \circ \varphi. \quad (2)$$

It is obvious that $D_\varphi = \varphi' C_\varphi$.

For $\alpha > -1$ and $0 < p < \infty$, we recall that a weighted Bergman space $A_\alpha^p(\mathbb{D})$ is the set of holomorphic functions on the unit disk for which

$$\int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha \frac{dA(z)}{\pi} < \infty, \quad (3)$$

where $dA(z)$ is the Lebesgue area measure on the unit disk and $\|f\|_p$ is the p th root of this integral. Moreover, the Bergman space $A^2(\mathbb{D}) = A_0^2(\mathbb{D})$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z) \overline{g(z)} \frac{dA(z)}{\pi}. \quad (4)$$

A weighted Dirichlet space $\mathcal{D}_\alpha^p$ is the set of holomorphic functions on the unit disk for which

$$\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^\alpha \frac{dA(z)}{\pi} < \infty \quad (5)$$

with norm given by

$$\|f\|_{\mathcal{D}_\alpha^p}^p = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^\alpha \frac{dA(z)}{\pi} \quad (6)$$

and the Dirichlet space $\mathcal{D} = \mathcal{D}_0^2$ is also a Hilbert space with the inner product

$$\langle f, g \rangle = f(0) \overline{g(0)} + \int_{\mathbb{D}} f'(z) \overline{g'(z)} \frac{dA(z)}{\pi}. \quad (7)$$

It is easy to show that (Exercise 2.1.6 in [1]) $A_\alpha^2(\mathbb{D}) = \mathcal{D}_{\alpha+2}$.

Much effort has been expended on characterizing those analytic maps which induce bounded or compact composition operators. Readers interested in this topic can refer to the books [2] by Shapiro, [1] by Cowen and MacCluer, and [3, 4] by Zhu, which are excellent sources for the development of the theory of composition operators and function spaces, and the recent papers [5–8].

Another area of particular interest is the topological structure of the space of composition operators acting on a given function space. When X is a Banach space of analytic functions, we write $C(X)$ for the space of composition operators and $C_d(X)$ for the space of derivative weighted composition operators on X under the operator norm topology. The investigation of the topological structure of $C(H^2)$ was initiated by Berkson [9] in 1981. Central problem focuses on the relations between the structure of $C(H^2)$ and the compactness properties of its members.

Continuing the work, in 1989, MacCluer [10] showed that, on the weighted Bergman space A_α^2 for $\alpha \geq -1$, the compact composition operators form an arcwise connected set in $C(A_\alpha^2)$ and gave necessary conditions for two composition operators to have compact difference. At about the same time, Shapiro and Sundberg [11] gave further results on compact difference and isolation and, among other things, posed the following fundamental question and conjectured that it had a positive answer:

(*) Do the compact composition operators form a connected component of the set $C(H^2)$?

In 2008, Gallardo-Gutiérrez et al. [12] gave a negative answer to the question for a variety of spaces in addition to H^p . In 2003, Bourdon [13] determined two linear-fractional self-maps of the disk having the same first-order data at a point ξ on the boundary of the disk and different second derivatives at ξ lie in the same component of $C(H^2)$, while the induced composition operators do not have compact difference. In 2005, Moorhouse [14] answers the question of compact difference for composition operators acting on A_α^2 , $\alpha > -1$, and gave a partial answer to the component structure of $C(A_\alpha^2)$. Later, Kriete and Moorhouse [15] extended their study to general linear combinations. Saukko [16, 17] obtained a complete characterization of bounded and compact differences between standard weighted Bergman spaces. Recently, Choe et al. [18, 19] extend Moorhouses characterization to the unit polydisk and unit ball in $\mathbb{C}^N$.

Topological structure of composition operators, which are from some analytic function spaces into the Bloch type spaces or the space of bounded analytic functions, has been studied intensively during the past decades. Interested readers can refer to [20, 21] and the references therein.

In 2004 Čučković and Zhao [5] characterized the bounded, compact, and Schatten class weighted composition operators on the Bergman space by generalized Berezin transforms. Derivative weighted composition operators are a special class of the weighted composition operators. The boundedness is characterized as follows.

Theorem 1 (Theorem 1 in [5]). *Let φ be an analytic self-map of $\mathbb{D}$. Then the d -weighted composition operator D_φ is bounded on A^2 if and only if*

$$\int_{\mathbb{D}} \frac{(1 - |z|^2)^2 |\varphi'(w)|^2}{|1 - \bar{z}\varphi(w)|^4} dA(w) < \infty. \quad (8)$$

Building on those foundations, the present paper continues this line of research. The remainder is assembled as follows. In Section 2, we characterize the compact difference of two derivative weighted composition operators on the Bergman space in the unit disk and discuss the isolation and component structure of the derivative weighted composition operator in $\mathcal{C}_d(A^2)$, and some similar results about composition operators on the Dirichlet space are also presented there. In Section 3, we show that the d -composition operators form an arcwise connected set in $\mathcal{C}_d(A^2)$. Finally, we determine when linear-fractional d -composition operators belong to the same component on the Bergman space under the operator norm topology in Section 4.

2. Difference of Derivative Weighted Composition Operators

Our discussion begins with the characterization of the compact difference of d -composition operators. When we accomplished the following theorem, we read a new paper [22], in which Allen et al. obtained the same estimation (Lemma 10). Nonetheless, we present our version in the following, since our method is totally different.

Theorem 2. *Suppose $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ has finite angular derivative $\varphi'(\zeta)$ at some point ζ in the unit circle. Let $\psi : \mathbb{D} \rightarrow \mathbb{D}$ and consider the difference of D_φ and D_ψ acting on the Bergman space $A^2(\mathbb{D})$. Then we have $\|D_\varphi - D_\psi\|_e^2 \geq 1$ if the symbol ψ satisfies one of the following two conditions:*

- (i) $\psi(\zeta) \neq \varphi(\zeta)$ (as radial limits).
- (ii) $\psi(\zeta) = \varphi(\zeta)$ but $\psi'(\zeta) \neq \varphi'(\zeta)$.

Proof. Since the Bergman space $A^2(\mathbb{D})$ is rotationally invariant, we may assume $\zeta = 1$, $\varphi(1) = 1$, and $\varphi'(1) = s < \infty$. We first estimate $\|D_\varphi - D_\psi\|$ by considering the adjoint of $D_\varphi - D_\psi$, $(D_\varphi - D_\psi)^*$ acting on reproducing kernel functions $K_z(w) = (1 - \bar{z}w)^{-2}$. Since

$$\begin{aligned} & \|(D_\varphi - D_\psi)^* K_z\|^2 \\ &= \langle (D_\varphi - D_\psi)^* K_z, (D_\varphi - D_\psi)^* K_z \rangle \\ &= \langle \overline{\varphi'(z)} K_{\varphi(z)} - \overline{\psi'(z)} K_{\psi(z)}, \overline{\varphi'(z)} K_{\varphi(z)} - \overline{\psi'(z)} K_{\psi(z)} \rangle \\ &= |\varphi'(z)|^2 \|K_{\varphi(z)}\|^2 + |\psi'(z)|^2 \|K_{\psi(z)}\|^2 - 2\operatorname{Re} \{ \overline{\varphi'(z)} \psi'(z) K_{\varphi(z)}(\psi(z)) \}, \end{aligned} \quad (9)$$

the formulas for the kernel functions K_z and their norms show that

$$\begin{aligned} \|D_\varphi - D_\psi\|^2 &\geq \frac{\|(D_\varphi - D_\psi)^* K_z\|^2}{\|K_z\|^2} \\ &= |\varphi'(z)|^2 \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^2 \\ &\quad + |\psi'(z)|^2 \left(\frac{1 - |z|^2}{1 - |\psi(z)|^2} \right)^2 \\ &\quad - 2\operatorname{Re} \left\{ \overline{\varphi'(z)\psi'(z)} \left(\frac{1 - |z|^2}{1 - \varphi(z)\psi(z)} \right)^2 \right\} \end{aligned} \quad (10)$$

for all z in $\mathbb{D}$.

By the Julia-Carathéodory Theorem, $(1 - |z|^2)/(1 - |\varphi(z)|^2)$ has nontangential limit $1/s$ as z goes to 1, so the first term of the right-hand side in the above expression has nontangential limit 1 as z goes to 1.

To deal with the third term of the right-hand side in (10) we distinguish two cases.

Case 1. Consider $\lim_{r \rightarrow 1} \psi(r) \neq 1$. We can find z_n approaching 1 nontangentially so that $\lim_{n \rightarrow \infty} \psi(z_n) = u \neq 1$.

From Schwarz-Pick Theorem we have that $|\psi'(z_n)| \leq (1 - |\psi(z_n)|^2)/(1 - |z_n|^2)$ for every $n = 1, 2, \dots$

It follows that

$$\begin{aligned} &2\operatorname{Re} \left\{ \overline{\varphi'(z_n)\psi'(z_n)} \left(\frac{1 - |z_n|^2}{1 - \varphi(z_n)\psi(z_n)} \right)^2 \right\} \\ &\leq 2|\varphi'(z_n)| |\psi'(z_n)| \left| \frac{1 - |z_n|^2}{1 - \varphi(z_n)\psi(z_n)} \right|^2 \\ &\leq \frac{2|\varphi'(z_n)|}{|1 - \overline{\varphi(z_n)\psi(z_n)}|^2} (1 - |\psi(z_n)|^2) (1 - |z_n|^2) \end{aligned} \quad (11)$$

has limit 0 as n tends to infinity since

$$\begin{aligned} \lim_{n \rightarrow \infty} |\varphi'(z_n)| &= s, \\ \lim_{n \rightarrow \infty} |1 - \overline{\varphi(z_n)\psi(z_n)}| &= |1 - u|, \\ \lim_{n \rightarrow \infty} (1 - |z_n|^2) &= 0. \end{aligned} \quad (12)$$

Combining with the above discussions and by (10), we can conclude that $\|D_\varphi - D_\psi\|^2 \geq 1$.

Case 2. Consider $\psi(1) = 1$ but $\psi'(1) \neq \varphi'(1) = s$. We first consider

$$\begin{aligned} \psi'(z)^{-1} \left[\frac{1 - \overline{\varphi(z)}\psi(z)}{1 - |z|^2} \right]^2 &= \psi'(z)^{-1} \left[\frac{1 - |\varphi(z)|^2}{1 - |z|^2} \right. \\ &\quad \left. + \frac{\overline{\varphi(z)}\varphi(z) - \psi(z)}{1 - |z|^2} \right]^2 = \left[\psi'(z)^{-1/2} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \right. \\ &\quad \left. + \frac{\overline{\varphi(z)}\frac{1 - z}{1 - |z|^2}}{\psi'(z)^{-1/2} \frac{1 - \psi(z)}{1 - z}} \right. \\ &\quad \left. - \psi'(z)^{-1/2} \frac{1 - \varphi(z)}{1 - z} \right]^2. \end{aligned} \quad (13)$$

Let $\gamma_M = \{z \in \mathbb{D} : |1 - z|/(1 - |z|^2) = M\}$, the boundary of a nontangential approach region at 1. As z approaches 1 along γ_M , the Julia-Carathéodory Theorem shows that

$$\begin{aligned} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} &\rightarrow s, \\ \frac{1 - \varphi(z)}{1 - z} &\rightarrow s \end{aligned} \quad (14)$$

and either

$$\frac{1 - \psi(z)}{1 - z} \rightarrow \psi'(1) \quad \text{when } \psi'(1) < \infty \quad (15)$$

or

$$\left| \frac{1 - \psi(z)}{1 - z} \right| \rightarrow \infty \quad \text{when } |\psi'(1)| = \infty. \quad (16)$$

It is easy to show that $\psi'(z)^{-1/2}((1 - \psi(z))/(1 - z)) - \psi'(z)^{-1/2}((1 - \varphi(z))/(1 - z)) \rightarrow 0$. For any given positive number N , we may, by choosing M large enough, find a sequence z_n approaching 1 along γ_M so that for n large enough

$$\begin{aligned} &\left| \overline{\varphi(z_n)} \frac{1 - z_n}{1 - |z_n|^2} \psi'(z_n)^{-1/2} \right. \\ &\quad \left. \cdot \left(\frac{1 - \psi(z_n)}{1 - z_n} - \frac{1 - \varphi(z_n)}{1 - z_n} \right) \right| > N + 1. \end{aligned} \quad (17)$$

It follows from (13) that

$$\psi'(z_n)^{-1} \left[\frac{1 - \overline{\varphi(z_n)}\psi(z_n)}{1 - |z_n|^2} \right]^2 > N + 1 \quad (18)$$

for n large enough. So, for arbitrary positive δ , choose N such that $2s/N < \delta$; we have

$$\begin{aligned} &2\operatorname{Re} \left\{ \overline{\varphi'(z_n)\psi'(z_n)} \left(\frac{1 - |z_n|^2}{1 - \varphi(z_n)\psi(z_n)} \right)^2 \right\} \leq \frac{2s}{N} \\ &< \delta \end{aligned} \quad (19)$$

from which and (10), we deduce that $\|D_\varphi - D_\psi\|^2 \geq 1 - \delta$ for arbitrary positive δ , so $\|D_\varphi - D_\psi\|^2 \geq 1$ by letting $\delta \rightarrow 0$.

It is standard to check the lower bound of the essential norm $\|D_\varphi - D_\psi\|_e^2$:

$$\begin{aligned} \|D_\varphi - D_\psi\|_e^2 &= \inf \left\{ \|D_\varphi - D_\psi - Q\|^2 : Q \text{ compact} \right\} \\ &\geq \lim_{|z| \rightarrow 1} \left\| (D_\varphi - D_\psi)^* \left(\frac{K_z}{\|K_z\|} \right) \right\|^2 \geq 1. \end{aligned} \quad (20)$$

This completes the proof of the theorem. $\square$

As a direct consequence, we can obtain the following corollary.

Corollary 3. *If φ has finite angular derivative on a set of positive measures, then D_φ is isolated in $\mathcal{E}_d(A^2)$. For any $\psi \neq \varphi$, $\|D_\varphi - D_\psi\|_e \geq 1$.*

Definition 4. Let φ and ψ be two self-mappings of the unit disk, and we say that φ and ψ have the *same first-order boundary data* or satisfy SBD, provided that whenever either $|\varphi(\zeta)| = 1$ or $|\psi(\zeta)| = 1$, then $\varphi(\zeta) = \psi(\zeta)$ and $\varphi'(\zeta) = \psi'(\zeta)$.

Theorem 5. *Suppose D_ψ is in the component of $\mathcal{E}_d(A^2)$ containing D_φ . Then φ and ψ must have the same first-order boundary data, where φ has finite angular derivative.*

Proof. Suppose either $|\varphi(\zeta)| = 1$ or $|\psi(\zeta)| = 1$ at ζ in the unit circle. If $\psi(\zeta) \neq \varphi(\zeta)$ or $\psi(\zeta) = \varphi(\zeta)$ but $\psi'(\zeta) \neq \varphi'(\zeta)$, by Theorem 2, $\|D_\varphi - D_\psi\|_e^2 \geq 1$, D_ψ is not in the component of $\mathcal{E}_d(A^2)$ containing D_φ . Hence $\psi(\zeta) = \varphi(\zeta)$ and $\psi'(\zeta) = \varphi'(\zeta)$; that is, φ and ψ must have the same first-order boundary data. $\square$

d -composition operators on the weighted Bergman space are highly connected with composition operators on weighted Dirichlet space. In the following, we will use the symbol C to denote a finite positive number which is not necessarily the same at each occurrence.

Lemma 6. *For $\alpha > -1$ and $1 < p \leq q < \infty$, let $\varphi, \psi \in \mathcal{D}_\alpha^p$ be such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\psi(\mathbb{D}) \subset \mathbb{D}$. Then $C_\varphi - C_\psi$ is bounded (compact) from $\mathcal{D}_\alpha^p$ into $\mathcal{D}_\alpha^q$ if and only if $D_\varphi - D_\psi$ is bounded (compact) from A_α^p into A_α^q .*

Proof. We just verify the condition of boundedness. Suppose $C_\varphi - C_\psi$ is bounded from $\mathcal{D}_\alpha^p$ into $\mathcal{D}_\alpha^q$. Then there exists a constant $C > 0$ such that

$$\|(C_\varphi - C_\psi)f\|_{\mathcal{D}_\alpha^q} \leq C \|f\|_{\mathcal{D}_\alpha^p} \quad (21)$$

for all $f \in \mathcal{D}_\alpha^p$. Take $f \in A_\alpha^p$ and let the function $g \in \mathcal{D}_\alpha^p$ be such that $g' = f$ and $g(0) = 0$. Then we have

$$\begin{aligned} \|(D_\varphi - D_\psi)f\|_{A_\alpha^q}^q &= \|\varphi' f \circ \varphi - \psi' f \circ \psi\|_{A_\alpha^q}^q \\ &= \|(g \circ \varphi - g \circ \psi)'\|_{A_\alpha^q}^q \end{aligned}$$

$$\begin{aligned} &= \|g \circ \varphi - g \circ \psi\|_{\mathcal{D}_\alpha^q}^q \\ &= \|(C_\varphi - C_\psi)g\|_{\mathcal{D}_\alpha^q}^q \leq C \|g\|_{\mathcal{D}_\alpha^p}^q \\ &= C \|f\|_{A_\alpha^p}^q. \end{aligned} \quad (22)$$

Thus $D_\varphi - D_\psi : A_\alpha^p \rightarrow A_\alpha^q$ is bounded.

Conversely, assume $D_\varphi - D_\psi : A_\alpha^p \rightarrow A_\alpha^q$ is bounded. Take $f \in \mathcal{D}_\alpha^p$ such that $f(0) = 0$. Then we have

$$\begin{aligned} \|(C_\varphi - C_\psi)f\|_{\mathcal{D}_\alpha^q}^q &= \|(f \circ \varphi - f \circ \psi)'\|_{A_\alpha^q}^q \\ &= \|(D_\varphi - D_\psi)(f')\|_{A_\alpha^q}^q \leq C \|f'\|_{A_\alpha^p}^p \\ &= C \|f\|_{\mathcal{D}_\alpha^p}^p. \end{aligned} \quad (23)$$

$\square$

Theorem 7. *Suppose φ is an analytic self-map of unit disk and has finite angular derivative $\varphi'(\zeta)$ at some point ζ in the unit circle. Let ψ be another analytic self-map of unit disk and consider C_φ and C_ψ acting on $\mathcal{D}$. Then unless both*

- (1) $\varphi(\zeta) = \psi(\zeta)$,
- (2) $\varphi'(\zeta) = \psi'(\zeta)$,

we have $\|C_\varphi - C_\psi\|_e^2 \geq C$, where the constant C depends only on the spaces $\mathcal{D}$.

Proof. Note that $\mathcal{D} = \mathcal{D}_0^2$ and $A^2 = A_0^2$. Setting $\alpha = 0$ and $p = q = 2$ in Lemma 6, from the proof of Lemma 6, we know there exist two constants C_1 and C_2 such that

$$C_1 \|D_\varphi - D_\psi\|_{A^2} \leq \|C_\varphi - C_\psi\|_{\mathcal{D}} \leq C_2 \|D_\varphi - D_\psi\|_{A^2}. \quad (24)$$

The desired result now follows from Theorem 2. $\square$

Corollary 8. *If φ has finite angular derivative on a set of positive measures, then C_φ is isolated in $\mathcal{E}(\mathcal{D})$. For any $\psi \neq \varphi$, $\|C_\varphi - C_\psi\|_e^2 \geq C$, where constant C depends only on the $\mathcal{D}$.*

Corollary 9. *Suppose C_ψ is in the component of $\mathcal{E}(\mathcal{D})$ containing C_φ . Then φ and ψ must have the same first-order boundary data, where φ has finite angular derivative.*

3. Compact Composition Operators on Dirichlet Space

The proof of the following lemma is routine, which is omitted here.

Lemma 10. *Let φ be an analytic self-map of $\mathbb{D}$ such that the induced composition operator C_φ is bounded on $\mathcal{D}$. Then C_φ acts on $\mathcal{D}$ compactly if and only if whenever $\{f_n\}$ is a bounded sequence in $\mathcal{D}$ converging to zero uniformly on compact subsets of $\mathbb{D}$, then $\|C_\varphi(f_n)\| \rightarrow 0$.*

It is well known (e.g., Proposition 9.9 in [1]) that the compact composition operators on $\mathcal{D}_\alpha$ form an arcwise connected set in $\mathcal{C}(\mathcal{D}_\alpha)$ for all $\alpha \geq 1$. By Lemma 10, we find the result is also true on the Dirichlet $\mathcal{D} = \mathcal{D}_0$ if we note that if $\|\varphi\|_\infty < 1$, then C_φ is compact on the Dirichlet space.

Theorem 11. *The compact composition operators on $\mathcal{D}$ form an arcwise connected set in $\mathcal{C}(\mathcal{D})$.*

The proof of this theorem is almost the same as the proof of Proposition 9.9 in [1], so we just give an outline here.

Proof. Given C_φ and C_ψ compact on $\mathcal{D}$ we first construct a continuous map of $[0, 1]$ into $\mathcal{C}(\mathcal{D})$ taking 1 to C_φ and 0 to C_{φ_0} , the composition operator whose symbol is the constant map $\varphi(0)$. The path $t \mapsto C_{\varphi_t}$, where $\varphi_t(z) = \varphi(tz)$, will satisfy our demand ($\|\varphi_t\|_\infty < 1$ implying the compactness of C_{φ_t}).

Secondly, we construct continuous arc joining.

C_{φ_0} to C_{ψ_0} . Let $L(t) = (1 - t)\varphi(0) + t\psi(0)$ be the line segment in $\mathbb{D}$ from $\varphi(0)$ to $\psi(0)$ and then map $[0, 1]$ into $\mathcal{C}(\mathcal{D})$ by $t \mapsto C_{L(t)}$, where $C_{L(t)}f(z) = f(L(t))$. It is easy to verify that this map is continuous. $\square$

It follows from Lemma 6 and Theorem 11 that we have following corollary.

Corollary 12. *The compact d -composition operators on $A^2(\mathbb{D})$ form an arcwise connected set in $\mathcal{C}_d(A^2)$.*

From Theorem 11, we find that the compact composition operators on $\mathcal{D}$ are in the same component, but we do not know whether there is any noncompact composition operator belonging to the component generated by compact ones on $\mathcal{D}$. The answers to those similar questions on different analytic function spaces appear somewhat different. One has positive answers when the spaces are classical Bergman space, Bloch space, and H^∞ , while the answer is negative on classical Hardy space.

Remark 13. Let $\varphi \in S(\mathbb{D})$ be a linear-fractional self-map of the unit disk with $\varphi(\zeta) = \eta$, where both ζ and η are unimodular. Then the induced composition operator C_φ cannot be in the component generated by compact composition operators on Dirichlet space. Suppose C_φ did. The linear-fractional self-map φ must have finite angular derivative at ζ . And the composition operator induced by $z/2$ is compact on $\mathcal{D}$. By Theorem 11 compact composition operators must be in the same component and C_φ and $C_{z/2}$ are in the same component. By Corollary 9 φ and $z/2$ must satisfy SBD. That is impossible.

Lemma 14 (Corollary 2 in [5]). *Let u be an analytic function on $\mathbb{D}$ and let φ be an analytic self-map of $\mathbb{D}$. Then the weighted composition operator uC_φ is compact on A^2 if and only if $\lim_{|a| \rightarrow 1} \|(uC_\varphi)k_a\|^2 = 0$, where $k_a(z)$ is the normalized reproducing kernel of A^2 at a .*

There is a practical corollary of Lemma 14.

Lemma 15 (Proposition 1 in [5]). *Let u be an analytic function on $\mathbb{D}$ and $\varphi \in S(\mathbb{D})$. If the weighted composition operator uC_φ is compact on A^2 , then*

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|^2) |u(z)|}{1 - |\varphi(z)|^2} = 0. \quad (25)$$

From this lemma, in next section we will see that some linear-fractional composition operators cannot be compact on Dirichlet space, and we will discuss that component in $\mathcal{C}(\mathcal{D})$ through the characterization of the component in $\mathcal{C}_d(A^2)$.

4. Linear-Fractional d -Composition Operators

Throughout this section, φ and ψ will denote linear-fractional self-mappings of $\mathbb{D}$. To determine when linear-fractional d -composition operators belong to the same component of $\mathcal{C}_d(A^2)$, we need to state some lemmas.

Lemma 16 (Theorem 4.2.9 in [3]). *Suppose $n \geq 1$ and f is analytic in $\mathbb{D}$; then $f \in A^2(\mathbb{D})$ if and only if $(1 - |z|^2)^n f^{(n)}(z)$ is in $A^2(\mathbb{D})$.*

If φ and ψ are linear-fractional self-maps of the unit disk with the same boundary data, $\varphi(\zeta) = \psi(\zeta) = \eta$, where ζ and η are in the unit circle. Then it is trivial that D_φ and D_ψ are in the same path component in $\mathcal{C}_d(A^2)$ if and only if $D_{\tilde{\eta}\varphi(\zeta z)}$ and $D_{\tilde{\eta}\psi(\zeta z)}$ are in the same path component in $\mathcal{C}_d(A^2)$.

So we can just discuss the situations when $\varphi(1) = 1$ and $\varphi'(1) = s \in \mathbb{R}$. The following lemma follows Lemma 3.2 in [13] and its proof.

Lemma 17. *Suppose that φ is a nonautomorphic, linear-fractional self-mapping of $\mathbb{D}$ such that $\varphi(1) = 1$ and $\varphi'(1) = s$. Denote $T(z) = (1+z)/(1-z)$ for $z \in \mathbb{D}$; then there is a complex constant a with positive real part such that*

$$\begin{aligned} \varphi(z) &= T^{-1} \left(\frac{1}{s} T(z) + a \right) \\ &= \frac{(-1 + as - s)z - 1 - as + s}{(s - 1 + as)z - s - 1 - as}. \end{aligned} \quad (26)$$

Conversely if we put $\varphi(z) = ((-1 + as - s)z - 1 - as + s)/((s - 1 + as)z - s - 1 - as)$, it is clear that φ is a linear-fractional self-mapping of $\mathbb{D}$ such that $\varphi(1) = 1$ and $\varphi'(1) = s$. By Lemma 15 and direct computation, it is easy to check all C_φ are noncompact on $\mathcal{D}$. So all D_φ are also noncompact on A^2 equivalently by Lemma 6.

Lemma 18. Let $r : [0, 1] \rightarrow \mathbb{C}$ be defined by $r(t) = a + t(b - a)$ so that r is the straight-line path in the open right half-plane joining a and b . Let φ_t be the self-map of $\mathbb{D}$ given by

$$\begin{aligned} \varphi_t(z) &= T^{-1} \left(\frac{1}{s} T(z) + r(t) \right) \\ &= \frac{(-1 + r(t)s - s)z - 1 - r(t)s + s}{(s - 1 + r(t)s)z - s - 1 - r(t)s}. \end{aligned} \quad (27)$$

Then $t \mapsto D_{\varphi_t}$ is continuous from the unit interval to $\mathcal{C}_d(A^2)$, placing both D_φ and D_ψ in the same path component.

Proof. Let t_1 and t_2 be distinct points in $[0, 1]$ with $t_1 \leq t_2$. This lemma will be proved if we proved that there is constant M such that

$$\|D_{\varphi_{t_2}} - D_{\varphi_{t_1}}\|^2 \leq M(t_2 - t_1)^2. \quad (28)$$

We require some preliminary estimates. Fix z in the closure of $\mathbb{D}$, let $\varphi_z(t) = \varphi_t(z)$, and calculate the derivative of φ_z with respect to t :

$$\varphi'_z(t) = \frac{2(b - a)s^2(1 - z)^2}{((s - 1 + r(t)s)z - s - 1 - r(t)s)^2}. \quad (29)$$

For any $z \in \overline{\mathbb{D}}$ and $t \in [0, 1]$, note that $|r(t)| \leq |a| + |b|$; we have

$$\begin{aligned} |(s - 1 + r(t)s)z| + |s + 1 + r(t)s| \\ \leq 2(s + 1 + (|a| + |b|)s), \\ |(s - 1 + r(t)s)z - s - 1 - r(t)s| \\ \geq |s + 1 + r(t)s| - |s - 1 + r(t)s|. \end{aligned} \quad (30)$$

Since $\operatorname{Re}(r(t)) > 0$,

$$\begin{aligned} |s + 1 + r(t)s|^2 - |s - 1 + r(t)s|^2 &= 4s + 4\operatorname{Re}(r(t))s \\ &> 4s, \end{aligned} \quad (31)$$

it follows from (30) that

$$\begin{aligned} |(s - 1 + r(t)s)z - s - 1 - r(t)s| \\ \geq \frac{|s + 1 + r(t)s|^2 - |s - 1 + r(t)s|^2}{|(s - 1 + r(t)s)z| + |s + 1 + r(t)s|} \\ \geq \frac{4s}{2(s + 1 + (|a| + |b|)s)} = \frac{2s}{s + 1 + (|a| + |b|)s}. \end{aligned} \quad (32)$$

Combining (29) and (32), there is a constant $K_1 = s|b - a|(s + 1 + s(|a| + |b|))$ such that

$$|\varphi'_z(t)| \leq K_1 |1 - z|^2 \quad (33)$$

for all $z \in \overline{\mathbb{D}}$ and $t \in [0, 1]$. Calculate the derivative of φ_t with respect to z :

$$\varphi'_t(z) = \frac{4s}{((s - 1 + r(t))z - s - 1 - r(t)s)^2}. \quad (34)$$

Similarly we have

$$|\varphi'_t(z)| \leq K_2 \quad (35)$$

for a constant K_2 independent of $z \in \overline{\mathbb{D}}$ and $t \in [0, 1]$.

An easy calculation shows that

$$\frac{1 - z}{1 - \varphi_t(z)} = \frac{s + 1 + r(t)s - (s - 1 + r(t)s)z}{2s}. \quad (36)$$

Thus, applying the triangle inequality, we see that, for every $z \in \mathbb{D}$,

$$\frac{|1 - z|}{|1 - \varphi_t(z)|} \leq \frac{s + 1 + (|a| + |b|)s}{s} = K_3. \quad (37)$$

Because $\varphi_t(\mathbb{D})$ is a proper subdisk of $\mathbb{D}$ internally tangent to the unit circle at 1, we may set

$$\varphi_t(\mathbb{D}) = \{w \in \mathbb{D} : |w - w_0| < R\}, \quad (38)$$

where $1 - w_0 = R$; that is, $w_0 = 1 - R$. It is easy to show that

$$\begin{aligned} w \in \varphi_t(\mathbb{D}) &\iff \\ |w - (1 - R)|^2 &< R^2 \iff \end{aligned} \quad (39)$$

$$|1 - w|^2 < \frac{R}{1 - R} (1 - |w|^2),$$

and this implies that

$$\frac{|1 - \varphi_t(z)|^2}{1 - |\varphi_t(z)|^2} < \frac{R}{1 - R} \quad (40)$$

for any $z \in \mathbb{D}$.

It follows from (27) that $T(\varphi_t(z)) = T(z)/s + r(t)$. And the image of the circle

$$\frac{|1 - w|^2}{1 - |w|^2} = \frac{R}{1 - R} \quad (41)$$

under $T(z) = (1 + z)/(1 - z)$ is the line $\operatorname{Re}(r(t)) + ki$, where $k \in \mathbb{R}$ in right half-plane, and we put $T^{-1}(\operatorname{Re}(r(t)) + ki)$ back into (41). We have that

$$\begin{aligned} \frac{1 - R}{R} &= \frac{|1 - T^{-1}(\operatorname{Re}(r(t)) + ki)|^2}{|1 - T^{-1}(\operatorname{Re}(r(t)) + ki)|^2} \\ &= \frac{4}{|\operatorname{Re}(r(t)) + 1 + ki|^2 - |\operatorname{Re}(r(t)) - 1 + ki|^2} \\ &= \frac{1}{\operatorname{Re}(r(t))}. \end{aligned} \quad (42)$$

Consequently,

$$\begin{aligned} \frac{|1 - \varphi_t(z)|^2}{1 - |\varphi_t(z)|^2} &\leq (\operatorname{Re}(r(t)))^{-1} \\ &\leq (\min\{\operatorname{Re}(a), \operatorname{Re}(b)\})^{-1} = K_4. \end{aligned} \quad (43)$$

For any $f \in A^2$, we have that

$$f(\varphi_{t_2}(z)) - f(\varphi_{t_1}(z)) = \int_{t_1}^{t_2} f'(\varphi_z(t)) \varphi'_z(t) dt. \quad (44)$$

This produces the crucial estimate:

$$\begin{aligned} \|(D_{\varphi_{t_2}} - D_{\varphi_{t_1}})f\|^2 &= \int_{\mathbb{D}} |\varphi'_{t_2}(z) f(\varphi_{t_2}(z)) - \varphi'_{t_1}(z) \\ &\quad \cdot f(\varphi_{t_1}(z))|^2 dA(z) \\ &= \int_{\mathbb{D}} |[\varphi'_{t_2}(z) - \varphi'_{t_1}(z)] f(\varphi_{t_2}(z)) + \varphi'_{t_1}(z) \\ &\quad \cdot [f(\varphi_{t_2}(z)) - f(\varphi_{t_1}(z))]|^2 dA(z) \\ &\leq 2 \int_{\mathbb{D}} \left| 1 - \frac{\varphi'_{t_1}(z)}{\varphi'_{t_2}(z)} \right|^2 |\varphi'_{t_2}(z)|^2 \\ &\quad \cdot |f(\varphi_{t_2}(z))|^2 dA(z) \\ &\quad + 2 \int_{\mathbb{D}} |\varphi'_{t_1}(z)|^2 |f(\varphi_{t_2}(z)) \\ &\quad - f(\varphi_{t_1}(z))|^2 dA(z) = 2(I_1 + I_2), \end{aligned} \quad (45)$$

$$\begin{aligned} I_1 &= \int_{\mathbb{D}} \left| 1 - \left(\frac{(s-1+r(t_2)s)z - s - 1 - r(t_2)s}{(s-1+r(t_1)s)z - s - 1 - r(t_1)s} \right) \right|^2 \\ &\quad \cdot |\varphi'_{t_2}(z)|^2 |f(\varphi_{t_2}(z))|^2 dA(z) \leq K_5^4 (t_2 \\ &\quad - t_1)^2 \int_{\mathbb{D}} |\varphi'_{t_2}(z)|^2 |f(\varphi_{t_2}(z))|^2 dA(z) \leq K_5^4 (t_2 \\ &\quad - t_1)^2 \|f\|^2, \end{aligned}$$

where the constant K_5 denotes upper bound of $|(s-1+r(t)s)z - s - 1 - r(t)s|^{-1}$ multiplied by an absolute constant:

$$\begin{aligned} I_2 &\leq K_2 \int_{\mathbb{D}} |f(\varphi_{t_2}(z)) - f(\varphi_{t_1}(z))|^2 dA(z) \\ &= K_2 \int_{\mathbb{D}} \left| \int_{t_1}^{t_2} f'(\varphi_z(t)) \varphi'_z(t) dt \right|^2 dA(z) \\ &\leq K_2 (t_2 - t_1) \int_{\mathbb{D}} \int_{t_1}^{t_2} |f'(\varphi_z(t)) \varphi'_z(t)|^2 dt dA(z) \\ &\leq 4|b-a|^2 s^4 K_2 (t_2 - t_1) \int_{t_1}^{t_2} \int_{\mathbb{D}} |f'(\varphi_t(z))|^2 \end{aligned}$$

$$\begin{aligned} &\cdot |\varphi'_t(z)|^2 |1-z|^4 dA(z) dt \leq K_3^4 K_4^2 K_6 (t_2 \\ &- t_1) \int_{t_1}^{t_2} \int_{\mathbb{D}} |f'(\varphi_t(z))|^2 |\varphi'_t(z)|^2 \\ &\cdot (1-|\varphi_t(z)|^2)^2 dA(z) dt \leq K_3^4 K_4^2 K_6 (t_2 \\ &- t_1) \int_{t_1}^{t_2} dt \int_{\mathbb{D}} |f'(w)|^2 (1-|w|^2)^2 dA(w) \\ &= K_3^4 K_4^2 K_6 (t_2 - t_1)^2 \|(1-|w|^2) f'(w)\|^2 \\ &\leq K_3^4 K_4^2 K_6 (t_2 - t_1)^2 C \|f\|^2, \end{aligned} \quad (46)$$

where $K_6 = 4|b-a|^2 s^4 K_2$ and last inequality follows by using Lemma 16. Thus we have

$$\begin{aligned} \|(D_{\varphi_{t_2}} - D_{\varphi_{t_1}})f\|^2 \\ \leq 2(K_5^4 + K_3^4 K_4^2 K_6 C) \|f\|^2 (t_2 - t_1)^2, \end{aligned} \quad (47)$$

where K_j ($j = 3, 4, 5, 6$) and C depend only on a, b, s . This completes the proof of the lemma. $\square$

Theorem 19. Suppose that φ and ψ are linear-fractional self-mappings of $\mathbb{D}$. The following are equivalent:

- (a) D_φ and D_ψ lie in the same component of $\mathcal{C}_d(A^2)$.
- (b) φ and ψ satisfy SBD.
- (c) D_φ and D_ψ are joined by a continuous path in $\mathcal{C}_d(A^2)$.

Proof. Clearly (c) implies (a). That (a) implies (b) is proved in Theorem 5. Thus to complete the proof of Theorem 19, we establish (b) implies (c).

If φ is an automorphism, then, for any ζ in the unit circle, $|\varphi(\zeta)| = 1$. That φ and ψ satisfy SBD implies $\psi(\zeta) = \varphi(\zeta)$ for any ζ in the unit circle. It turns out that $\psi = \varphi$ by Maximum Principle, and the conclusion is obvious.

Suppose that φ is not an automorphism; then $\varphi(\mathbb{D})$ is a proper subdisk of $\mathbb{D}$; there are exactly two possibilities:

- (i) $|\varphi(\zeta)| < 1$ for any $\zeta \in \partial\mathbb{D}$.
- (ii) There exists a point $\zeta \in \partial\mathbb{D}$ such that $|\varphi(\zeta)| = 1$.

Suppose that (i) applies and φ and ψ satisfy SBD. Then ψ also must map the closure of $\mathbb{D}$ into $\mathbb{D}$. In this situation, both C_φ and C_ψ are compact operators on $\mathcal{D}$. By Lemma 17, we have both D_φ and D_ψ are compact on $A^2(\mathbb{D})$. Compact d -composition operators are shown to belong to the same arcwise connected subset in Corollary 12. Thus, to complete the proof of the theorem, we need to handle case (ii).

We may suppose that $\varphi(1) = 1$. Our setting condition is that ψ share the same first-order boundary data with φ so $\psi(1) = 1$ and we denote $s = \varphi'(1) = \psi'(1)$. By Lemma 17,

there are two constants a and b satisfying $\operatorname{Re}(a) > 0$ and $\operatorname{Re}(b) > 0$, such that

$$\begin{aligned} T(\varphi(z)) &= \frac{1}{s}T(z) + a, \\ T(\psi(z)) &= \frac{1}{s}T(z) + b. \end{aligned} \quad (48)$$

Denote φ_t described by Lemma 18; then $t \mapsto D_{\varphi_t}$ is continuous from the unit interval to $\mathcal{C}_d(A^2)$, placing both D_φ and D_ψ in the same path component by Lemma 18. This ends the proof of the theorem. $\square$

Next corollary is an obvious consequence of Theorem 19 and gives a partial description of component in $\mathcal{C}(\mathcal{D})$.

Corollary 20. *Suppose that φ and ψ are linear-fractional self-mappings of $\mathbb{D}$. The following are equivalent:*

- (a) C_φ and C_ψ lie in the same component of $\mathcal{C}(\mathcal{D})$.
- (b) φ and ψ satisfy SBD.
- (c) C_φ and C_ψ are joined by a continuous path in $\mathcal{C}(\mathcal{D})$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The work was supported in part by National Science Foundation of China (Grant nos. 11301132; 11371276; 11501415; and 11171087) and Natural Science Foundation of Hebei Province (Grant no. A2013202265) and Tianjin Advanced Education Development Fund (Grant no. 20111005).

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