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Research Article

Rough Multilinear Fractional Integrals on Weighted Morrey Spaces

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It is showed that a class of multilinear fractional operators with rough kernels, which are similar to the higher-order commutators for the rough fractional integrals, are bounded on the weighted Morrey spaces.

1. Introduction and Results

Suppose that $\mathbb{S}^{n-1}$ denotes the unit sphere of $\mathbb{R}^n$ ($n \geq 2$) equipped with the usual Lebesgue measure, $\Omega \in L^s(\mathbb{S}^{n-1})$ ($s > 1$) is homogeneous of degree zero, and A_j, $j = 1, \ldots, k$, are functions defined on $\mathbb{R}^n$. Consider the following multilinear fractional integral with rough kernel defined by

$$T_{\Omega,\alpha}^{A_1,\ldots,A_k}(f)(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha+N}} \prod_{j=1}^{k} R_{m_j}(A_j; x, y) f(y)\, dy, \tag{1}$$

where $0 < \alpha < n$, $N = \sum_{j=1}^{k}(m_j - 1)$, and

$$R_{m_j}(A_j; x, y) = A_j(x) - \sum_{|\gamma|<m_j} \frac{1}{\gamma!} D^\gamma A_j(y)(x-y)^\gamma. \tag{2}$$

When $k = 1$, $A_1 = A$, and $m_1 = 1$, then $T_{\Omega,\alpha}^{A}$ is just the commutator of the rough fractional integral $T_{\Omega,\alpha}$ with the function A:

$$[A, T_{\Omega,\alpha}] f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} (A(x) - A(y)) f(y)\, dy. \tag{3}$$

If $m_1 \geq 2$, then $T_{\Omega,\alpha}^{A}$ is nontrivial generalization of the above commutator. The weighted (L^p, L^q)-boundedness of the operator $T_{\Omega,\alpha}^{A}$ was given by Wu and Yang in [1].

When $k > 1$, $A_j = A$ for $j = 1, \ldots, k$, and $m_j = 1$, then $T_{\Omega,\alpha}^{A_1,\ldots,A_k}$ is the higher-order commutator of the fractional integral $T_{\Omega,\alpha}$ with the function A:

$$T_{\Omega,\alpha}^{A,k} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} (A(x) - A(y))^k f(y)\, dy. \tag{4}$$

The weighted (L^p, L^q)-boundedness of $T_{\Omega,\alpha}^{A,k}$ was given by Ding and Lu in [2].

Ding and Lu in [3] proved the following result.

Theorem 1. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, let $1/q = 1/p - \alpha/n$, and let Ω be homogeneous of degree zero with $\Omega \in L^s(\mathbb{S}^{n-1})$, $s > 1$. Moreover, for $1 \leq j \leq k$, $|\gamma| = m_j - 1$, $m_j \geq 2$, and $D^\gamma A_j \in \mathrm{BMO}(\mathbb{R}^n)$. If $\omega^{s'} \in A(p/s', q/s')$, then there exists a constant C, independent of A_j ($1 \leq j \leq k$) and f, such that*

$$\left\| T_{\Omega,\alpha}^{A_1,\ldots,A_k} f \right\|_{L^q(\omega^q,\mathbb{R}^n)} \leq C \left(\prod_{j=1}^{k} \sum_{|\gamma|=m_j-1} \left\| D^\gamma A_j \right\|_* \right) \|f\|_{L^p(\omega^p,\mathbb{R}^n)}. \tag{5}$$

Here and in the sequel, p' always denotes the conjugate index of any $p > 1$; that is, $1/p + 1/p' = 1$, and C stands for a constant which is independent of the main parameters, but it may vary from line to line.

The purpose of this paper is to discuss the boundedness properties of the rough fractional multilinear integral operators $T_{\Omega,\alpha}^{A_1,\dots,A_k}$ on appropriate weighted Morrey spaces.

The classical Morrey spaces were introduced by Morrey [4] in 1938 and have been studied intensively by various authors and together with weighted Lebesgue spaces play an important role in the theory of partial differential equations [5–7].

Let $1 \le p < \infty$ and $0 \le \lambda \le n$; Morrey spaces are defined by

$$L^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} < \infty \right\}, \tag{6}$$

where

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\lambda/p} \|f\|_{L^p(B(x,r))}. \tag{7}$$

Note that $L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $L^{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Let $\Phi(r)$, $r > 0$, be a growth function, that is, a positive increasing function in $(0,\infty)$, which satisfies doubling condition

$$\Phi(2r) \le D\Phi(r), \quad \forall r > 0, \tag{8}$$

where $D = D(\Phi) \ge 1$ is a doubling constant independent of r. Mizuhara in [8] gave generalization Morrey spaces $L^{p,\Phi}(\mathbb{R}^n)$ considering $\Phi(r)$ instead of r^λ in (7).

Komori and Shirai [9] introduced a version of the weighted Morrey space $L^{p,\kappa}(\omega,\mathbb{R}^n)$, which is a natural generalization of the weighted Lebesgue spaces $L^p(\omega,\mathbb{R}^n)$.

Let $1 \le p < \infty$, $0 < \kappa < 1$, and ω a weight function. The spaces $L^{p,\kappa}(\omega,\mathbb{R}^n)$ are defined by

$$L^{p,\kappa}(\omega,\mathbb{R}^n) = \left\{ f \in L^p_{\mathrm{loc}}(\omega) : \|f\|_{L^{p,\kappa}(\omega,\mathbb{R}^n)} < \infty \right\}, \tag{9}$$

where

$$\|f\|_{L^{p,\kappa}(\omega,\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} \left(\frac{1}{\omega(B(x,r))^\kappa} \cdot \int_{B(x,r)} |f(y)|^p \omega(y)\, dy \right)^{1/p}. \tag{10}$$

In order to deal with the fractional order case, we need to consider the weighted Morrey spaces with two weight functions; they were also introduced by Komori and Shirai in [9].

Let $1 \le p < \infty$, $0 < \kappa < 1$. For two weight functions u and v, the spaces $L^{p,\kappa}(u,v)(\mathbb{R}^n)$ are defined by

$$L^{p,\kappa}(u,v)(\mathbb{R}^n) = \left\{ f \in L^p_{\mathrm{loc}}(u) : \|f\|_{L^{p,\kappa}(u,v)} < \infty \right\}, \tag{11}$$

where

$$\|f\|_{L^{p,\kappa}(u,v)(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} \left(\frac{1}{v(B(x,r))^\kappa} \int_{B(x,r)} |f(y)|^p u(y)\, dy \right)^{1/p}. \tag{12}$$

Let $1 \le p < \infty$, φ a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, and ω a nonnegative measurable function on $\mathbb{R}^n$. We denote by $M^p_\varphi(\omega,\mathbb{R}^n)$ other weighted Morrey spaces, the spaces of all functions $f \in L^p_{\mathrm{loc}}(\omega)$ with finite norm

$$\begin{aligned} \|f\|_{M^p_\varphi(\omega,\mathbb{R}^n)} &= \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{\varphi(x,r)} \left(\frac{1}{w(B(x,r))} \|f\|^p_{L^p(\omega,B(x,r))} \right)^{1/p} \\ &< \infty, \end{aligned} \tag{13}$$

where

$$\|f\|_{L^p(\omega,B(x,r))} = \int_{B(x,r)} |f(y)|^p \omega(y)\, dy. \tag{14}$$

Remark 2. (1) If $\omega = 1$ and $\varphi(x,r) = r^{(\lambda-n)/p}$ with $0 \le \lambda \le n$, then $M^p_\varphi(\omega,\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n)$.

(2) If $\omega = 1, \varphi(x,r)^p = \Phi(r)r^{-n}$, then $M^p_\varphi(\omega,\mathbb{R}^n) = L^{p,\Phi}(\mathbb{R}^n)$.

(3) If $\varphi(x,r) = \omega(B(x,r))^{(\kappa-1)/p}$, then $M^p_\varphi(\omega,\mathbb{R}^n) = L^{p,\kappa}(\omega,\mathbb{R}^n)$.

(4) If $\varphi(x,r) = v(B(x,r))^{\kappa/p}\omega(B(x,r))^{-1/p}$, then $M^p_\varphi(\omega,\mathbb{R}^n) = L^{p,\kappa}(v,\omega)(\mathbb{R}^n)$.

Our main results can be formulated as follows.

Theorem 3. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, let $1/q = 1/p - \alpha/n$, and let Ω be homogeneous of degree zero with $\Omega \in L^s(\mathbb{S}^{n-1})$, $s > 1$. Suppose $\omega^{s'} \in A(p/s', q/s')$ and (φ_1,φ_2) satisfies the condition*

$$\begin{aligned} &\int_s^\infty \left(1 + \ln\frac{r}{s}\right)^k \\ &\quad \cdot \frac{\operatorname{ess\,inf}_{r<t<\infty} \varphi_1(x,t) \left(\omega^p(B(x,t))\right)^{1/p}}{\left(\omega^q(B(x,r))\right)^{1/q}} \frac{1}{r} dr \\ &\quad \le C_0 \varphi_2(x,s), \end{aligned} \tag{15}$$

where C_0 does not depend on x and r. If A_j have derivatives of order $m_j - 1$ in $BMO(\mathbb{R}^n)$, $j = 1,\dots,k$, then there is a constant $C > 0$, independent of A_j $(1 \le j \le k)$ and f, such that

$$\begin{aligned} &\left\| T_{\Omega,\alpha}^{A_1,\dots,A_k} f \right\|_{M^q_{\varphi_2}(\omega^q,\mathbb{R}^n)} \\ &\quad \le C \left(\prod_{j=1}^k \sum_{|\gamma|=m_j-1} \left\| D^\gamma A_j \right\|_* \right) \|f\|_{M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)}. \end{aligned} \tag{16}$$

Remark 4. Let $\varphi_1(x,t) = (\Phi(t)t^{-n})^{1/p}$, let $\varphi_2(x,t) = (\Phi(t))^{1/p}t^{n/q}$, and let $\omega = 1$. If $1 \le D(\Phi) \le 2^n$, it is easy to prove that (φ_1, φ_2) satisfies condition (15).

Remark 5. Let $\varphi_1(x,t) = (\omega^q(B(x,t)))^{\kappa/p}(\omega^p(B(x,t)))^{-1/p}$, $\varphi_2(x,t) = (\omega^q(B(x,t)))^{\kappa/p-1/q}$, $0 < \kappa < p/q$, and $w^q \in A_\infty(\mathbb{R}^n)$. By [10], we know that (φ_1, φ_2) satisfies condition (15).

Corollary 6. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, let $1/q = 1/p - \alpha/n$, and let Ω be homogeneous of degree zero with $\Omega \in L^s(\mathbb{S}^{n-1})$, $s > 1$. Suppose $1 \le D(\Phi) \le 2^n$, and A_j have derivatives of order $m_j - 1$ in $BMO(\mathbb{R}^n)$, $j = 1, \ldots, k$; then $T_{\Omega,\alpha}^{A_1,\ldots,A_k}$ is bounded from $L^{p,\Phi}(\mathbb{R}^n)$ to $L^{q,\Phi^{q/p}}(\mathbb{R}^n)$ and*

$$\left\|T_{\Omega,\alpha}^{A_1,\ldots,A_k} f\right\|_{L^{q,\Phi^{q/p}}(\mathbb{R}^n)} \le C\left(\prod_{j=1}^{k}\sum_{|\gamma|=m_j-1}\left\|D^\gamma A_j\right\|_*\right)\|f\|_{L^{p,\Phi}(\mathbb{R}^n)}. \tag{17}$$

Remark 7. In Corollary 6, let $k = 1$ and $m_1 = 1$; then we obtain the main result in [11].

Corollary 8. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, let $1/q = 1/p - \alpha/n$, and let Ω be homogeneous of degree zero with $\Omega \in L^s(\mathbb{S}^{n-1})$, $s > 1$. Suppose $0 < \kappa < p/q$, $\omega^{s'} \in A(p/s', q/s')$, and A_j have derivatives of order $m_j - 1$ in $BMO(\mathbb{R}^n)$, $j = 1, \ldots, k$; then $T_{\Omega,\alpha}^{A_1,\ldots,A_k}$ is bounded from $L^{p,\kappa}(\omega^p, \omega^q)(\mathbb{R}^n)$ to $L^{q,\kappa q/p}(\omega^q, \mathbb{R}^n)$ and*

$$\left\|T_{\Omega,\alpha}^{A_1,\ldots,A_k} f\right\|_{L^{q,\kappa q/p}(\omega^q,\mathbb{R}^n)} \le C\left(\prod_{j=1}^{k}\sum_{|\gamma|=m_j-1}\left\|D^\gamma A_j\right\|_*\right)\|f\|_{L^{p,\kappa}(\omega^p,\omega^q)(\mathbb{R}^n)}. \tag{18}$$

Remark 9. In Corollary 8, if $k = 1$ and $m_1 = 1$, then we obtain the main result in [12]; if $\Omega = 1$ and $\omega = 1$, then we obtain the main result in [13].

2. Some Preliminaries

We begin with some properties of $A_p(\mathbb{R}^n)$ weights which play a great role in the proofs of our main results.

Weight ω is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $B = B(x_0, r_B)$ denote the ball with the center x_0 and radius r_B and let $\lambda B = B(x_0, \lambda r_B)$ for any $\lambda > 0$. For a given weight function ω and a measurable set E, we also denote the Lebesgue measure of E by $|E|$ and set weighted measure $\omega(E) = \int_E \omega(x)dx$. For any given weight function ω on $\mathbb{R}^n$, $X \subseteq \mathbb{R}^n$, and $0 < p < \infty$, denote by $L^p(\omega, X)$ the space of all functions f satisfying

$$\|f\|_{L^p(\omega,X)} = \left(\int_X |f(x)|^p \omega(x)\,dx\right)^{1/p} < \infty. \tag{19}$$

Weight ω is said to belong to $A_p(\mathbb{R}^n)$ for $1 < p < \infty$, if there exists a constant C such that

$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{1-p'}\,dx\right)^{p-1} \le C. \tag{20}$$

$A_1(\mathbb{R}^n)$ denotes the set consisting of all weight functions ω:

$$\frac{1}{|B|}\int_B \omega(y)\,dy \le C \cdot \operatorname*{ess\,inf}_{x\in B} \omega(x) \quad \text{for every ball } B \subset \mathbb{R}^n. \tag{21}$$

Weight ω is said to belong to $A_\infty(\mathbb{R}^n)$ if there are positive numbers C and δ such that

$$\frac{\omega(E)}{\omega(B)} \le C\left(\frac{|E|}{|B|}\right)^\delta \tag{22}$$

for all balls B and all measurable $E \subset B$. It is well known that

$$A_\infty(\mathbb{R}^n) = \bigcup_{1\le p<\infty} A_p(\mathbb{R}^n). \tag{23}$$

The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^p-boundedness of Hardy-Littlewood maximal function in [14].

Lemma 10 (see [14, 15]). *Suppose $\omega \in A_p(\mathbb{R}^n)$. The following statements hold:*

(i) *For any $1 \le p < \infty$, there is a positive number C such that*

$$\frac{\omega(B_k)}{\omega(B_j)} \le C2^{np(k-j)} \quad \text{for } k > j. \tag{24}$$

(ii) *For any $1 \le p < \infty$, there is a positive number C and δ such that*

$$\frac{\omega(B_k)}{\omega(B_j)} \ge C2^{\delta(k-j)} \quad \text{for } k > j. \tag{25}$$

(iii) *For any $1 < p < \infty$, one has $\omega^{1-p'} \in A_{p'}$.*

We also need another weight class $A(p,q)$ introduced by Muckenhoupt and Wheeden in [16] to study weighted boundedness of fractional integral operators.

Given $1 \le p \le q < \infty$, we say that $\omega \in A(p,q)$ if there exists a constant C such that, for every ball $B \subset \mathbb{R}^n$, the inequality

$$\left(\frac{1}{|B|}\int_B \omega(y)^{-p'}\,dy\right)^{1/p'}\left(\frac{1}{|B|}\int_B \omega(y)^q\,dy\right)^{1/q} \le C \tag{26}$$

holds when $1 < p < \infty$, and for every ball $B \subset \mathbb{R}^n$ the inequality

$$\left(\frac{1}{|B|}\int_B \omega(y)^q\,dy\right)^{1/q} \le C \cdot \operatorname*{ess\,inf}_{x\in B} w(x) \tag{27}$$

holds when $p = 1$.

By (26), we have

$$\left(\int_B \omega(y)^{-p'}\,dy\right)^{1/p'}\left(\int_B \omega(y)^q\,dy\right)^{1/q} \le C|B|^{1/p'+1/q}. \tag{28}$$

We summarize some properties about weights $A(p,q)$; see [15, 16].

Lemma 11. *Given $1 \le p \le q < \infty$*

(i) *$\omega \in A(p,q)$ if and only if $\omega^q \in A_{1+q/p'}$;*

(ii) *$\omega \in A(p,q)$ if and only if $\omega^{-p'} \in A_{1+p'/q}$;*

(iii) *if $p_1 < p_2$ and $q_2 > q_1$, then $A(p_1,q_1) \subset A(p_2,q_2)$.*

John and Nirenberg introduced the function space of BMO in [17]. A locally integrable function b is said to be in $\mathrm{BMO}(\mathbb{R}^n)$ if

$$\sup_{B\subset\mathbb{R}^n}\frac{1}{|B|}\int_B |b(x)-b_B|\,dx = \|b\|_* < \infty, \tag{29}$$

where

$$b_B = \frac{1}{|B|}\int_B b(y)\,dy. \tag{30}$$

Lemma 12 (see [10]). *Suppose $\omega \in A_\infty$ and $b \in BMO(\mathbb{R}^n)$. Then for any $1 \le p < \infty$ and $r_1, r_2 > 0$, we have*

$$\left(\frac{1}{\omega(B(x_0,r_1))}\cdot\int_{B(x_0,r_1)}\left|b(x)-b_{B(x_0,r_2)}\right|^p\omega(x)\,dx\right)^{1/p} \le C\|b\|_*\cdot\left(1+\left|\ln\frac{r_1}{r_2}\right|\right). \tag{31}$$

Below we recall some conclusions about $R_m(A;x,y)$.

Lemma 13 (see [9]). *Let b be a function on $\mathbb{R}^n$ with the mth derivatives in $L^q(\mathbb{R}^n)$, $q > n$. Then*

$$|R_m(b;x,y)| \le C|x-y|^m\sum_{|\gamma|=m}\left(\frac{1}{B(x,5\sqrt{n}|x-y|)}\cdot\int_{B(x,5\sqrt{n}|x-y|)}|D^\gamma b(z)|^q\,dz\right)^{1/q}. \tag{32}$$

Lemma 14 (see [18]). *For fixed $x \in \mathbb{R}^n$, let*

$$\overline{A}(x) = A(x) - \sum_{|\gamma|=m-1}\frac{1}{\gamma!}(D^\gamma A)_{B(x,5\sqrt{n}|x-y|)}\,x^\gamma. \tag{33}$$

Then $R_m(A;x,y) = R_m(\overline{A};x,y)$.

Lemma 15. *Let $x \in B(x_0,l)$, and let $y \in B(x_0,2^{j+1}l)\setminus B(x_0,2^jl)$. Then*

$$|R_m(A;x,y)| \le C|x-y|^{m-1}\left(j\sum_{|\gamma|=m-1}\|D^\gamma A\|_* + \sum_{|\gamma|=m-1}\left|D^\gamma A(y)-(D^\gamma A)_{B(x_0,l)}\right|\right). \tag{34}$$

Proof. From Lemma 14, we have

$$\begin{aligned}|R_m(A;x,y)| &= \left|R_m(\overline{A};x,y)\right| \\ &\le \left|R_{m-1}(\overline{A};x,y)\right| + \sum_{|\gamma|=m-1}\frac{1}{\gamma!}\left|D^\gamma\overline{A}(y)\right||x-y|^{m-1}.\end{aligned} \tag{35}$$

By Lemma 13,

$$\left|R_{m-1}(\overline{A};x,y)\right| \le C|x-y|^{m-1}\sum_{|\gamma|=m-1}\|D^\gamma A\|_*. \tag{36}$$

If $x \in B(x_0,l)$, $y \in B(x_0,2^{j+1}l)\setminus B(x_0,2^jl)$, we can easily see that $2^{j-1}l \le |x-y| < 2^{j+2}l$. We get

$$\begin{aligned}B(x_0,2^{j-1}l) &\subset B(x,5\sqrt{n}|x-y|) \\ &\subset 100\sqrt{n}B(x_0,2^jl).\end{aligned} \tag{37}$$

Then

$$\frac{|100\sqrt{n}B(x_0,2^jl)|}{|B(x,5\sqrt{n}|x-y|)|} \le \frac{|100\sqrt{n}B(x_0,2^jl)|}{|B(x_0,2^{j-1}l)|} \le C. \tag{38}$$

Thus

$$\begin{aligned}&\left|(D^\gamma A)_{B(x,5\sqrt{n}|x-y|)}-(D^\gamma A)_{B(x_0,2^jl)}\right| \\ &\le \frac{1}{|B(x,5\sqrt{n}|x-y|)|}\cdot\int_{B(x,5\sqrt{n}|x-y|)}\left|D^\gamma A(y)-(D^\gamma A)_{B(x_0,2^jl)}\right|dy \\ &\le \frac{C}{|100\sqrt{n}B(x_0,2^jl)|}\cdot\int_{100\sqrt{n}B(x_0,2^jl)}\left|D^\gamma A(y)-(D^\gamma A)_{B(x_0,2^jl)}\right|dy \\ &\le C\|D^\gamma A\|_*.\end{aligned} \tag{39}$$

Note that

$$\begin{aligned}&\left|(D^\gamma A)_{B(x_0,2^jl)}-(D^\gamma A)_{B(x_0,l)}\right| \\ &\le \sum_{k=1}^{j}\left|(D^\gamma A)_{B(x_0,2^kl)}-(D^\gamma A)_{B(x_0,2^{k-1}l)}\right| \\ &\le 2^n j\|D^\gamma A\|_*.\end{aligned} \tag{40}$$

Then

$$\begin{aligned}&\left|\left(D^{\gamma}A\right)_{B(x,5\sqrt{n}|x-y|)}-\left(D^{\gamma}A\right)_{B(x_0,l)}\right|\\&\quad\le\left|\left(D^{\gamma}A\right)_{B(x,5\sqrt{n}|x-y|)}-\left(D^{\gamma}A\right)_{B(x_0,2^{j}l)}\right|\\&\qquad+\left|\left(D^{\gamma}A\right)_{B(x_0,2^{j}l)}-\left(D^{\gamma}A\right)_{B(x_0,l)}\right|\le Cj\left\|D^{\gamma}A\right\|_{*}.\end{aligned}\tag{41}$$

Thus

$$\begin{aligned}\left|D^{\gamma}\overline{A}(y)\right|&=\left|D^{\gamma}A(y)-\left(D^{\gamma}A\right)_{B(x,5\sqrt{n}|x-y|)}\right|\\&\le\left|D^{\gamma}A(y)-\left(D^{\gamma}A\right)_{B(x_0,l)}\right|\\&\quad+\left|\left(D^{\gamma}A\right)_{B(x,5\sqrt{n}|x-y|)}-\left(D^{\gamma}A\right)_{B(x_0,l)}\right|\\&\le\left|D^{\gamma}A(y)-\left(D^{\gamma}A\right)_{B(x_0,l)}\right|+Cj\left\|D^{\gamma}A\right\|_{*}.\end{aligned}\tag{42}$$

Combining with (35), (36), and (42), then (34) is proved. □

Finally, we recall a relationship between essential supremum and essential infimum.

Lemma 16 (see [19]). *Let f be a real-valued nonnegative function and measurable on E. Then*

$$\left(\operatorname*{ess\,inf}_{x\in E}f(x)\right)^{-1}=\operatorname*{ess\,sup}_{x\in E}\frac{1}{f(x)}.\tag{43}$$

3. A Local Estimate

To prove Theorem 3, we first investigate the following local estimate.

Theorem 17. *Let $0<\alpha<n$, $1<p<n/\alpha$, let $1/q=1/p-\alpha/n$, and let Ω be homogeneous of degree zero with $\Omega\in L^{s}(\mathbb{S}^{n-1})$, $s>1$. Suppose $\omega^{s'}\in A(p/s',q/s')$ and A_j have derivatives of order m_j-1 in $BMO(\mathbb{R}^n)$, $j=1,\dots,k$; then, for any $l>0$, there is a constant C independent of f such that*

$$\begin{aligned}&\left\|T_{\Omega,\alpha}^{A_1,\dots,A_k}(f)\right\|_{L^q(\omega^q,B(x_0,l))}\\&\quad\le C\left(\prod_{j=1}^{k}\sum_{|\gamma|=m_j-1}\left\|D^{\gamma}A_j\right\|_{*}\right)\left(\omega^q\left(B(x_0,l)\right)\right)^{1/q}\\&\quad\cdot\int_{2l}^{\infty}\|f\|_{L^p(\omega^p,B(x_0,r))}\left(\omega^q\left(B(x_0,r)\right)\right)^{-1/q}\frac{1}{r}dr.\end{aligned}\tag{44}$$

Proof. To simplify the proof process of Theorem 17 in the following discussion we consider only the case $k=2$. The method can be used to deal with the case $k>2$ without any essential difficulty.

We write f as $f=f_1+f_2$, where $f_1(y)=f(y)\chi_{B(x_0,2l)}(y)$, and $\chi_{B(x_0,2l)}$ denotes the characteristic function of $B(x_0,2l)$. Then

$$\begin{aligned}&\left\|T_{\Omega,\alpha}^{A_1,A_2}(f)\right\|_{L^p(\omega^p,B(x_0,l))}\\&\quad\le\left\|T_{\Omega,\alpha}^{A_1,A_2}(f_1)\right\|_{L^p(\omega^p,B(x_0,l))}\\&\qquad+\left\|T_{\Omega,\alpha}^{A_1,A_2}(f_2)\right\|_{L^p(\omega^p,B(x_0,l))}.\end{aligned}\tag{45}$$

Since $f_1\in L^p(\omega^p,\mathbb{R}^n)$, by the boundedness of $T_{\Omega,\alpha}^{A_1,A_2}$ from $L^p(\omega^p,\mathbb{R}^n)$ to $L^q(\omega^q,\mathbb{R}^n)$ (Theorem 1) we get

$$\begin{aligned}\left\|T_{\Omega,\alpha}^{A_1,A_2}(f_1)\right\|_{L^q(\omega^q,B(x_0,l))}&\le\left\|T_{\Omega,\alpha}^{A_1,A_2}(f_1)\right\|_{L^q(\omega^q,\mathbb{R}^n)}\\&\le C\left(\prod_{j=1}^{2}\sum_{|\gamma|=m_j-1}\left\|D^{\gamma}A_j\right\|_{*}\right)\|f_1\|_{L^p(\omega^p,\mathbb{R}^n)}\\&=C\left(\prod_{j=1}^{2}\sum_{|\gamma|=m_j-1}\left\|D^{\gamma}A_j\right\|_{*}\right)\|f\|_{L^p(\omega^p,B(x_0,2l))}.\end{aligned}\tag{46}$$

Noting that $q>p>1$ and $s'p/p'(p-s')\ge1$, then by Hölder's inequality

$$\begin{aligned}1&\le\left(\frac{1}{|B|}\int_B\omega(y)^p\,dy\right)^{1/p}\left(\frac{1}{|B|}\int_B\omega(y)^{-p'}\,dy\right)^{1/p'}\\&\le\left(\frac{1}{|B|}\int_B\omega(y)^q\,dy\right)^{1/q}\\&\quad\cdot\left(\frac{1}{|B|}\int_B\omega(y)^{-s'p/(p-s')}\,dy\right)^{(p-s')/s'p}.\end{aligned}\tag{47}$$

This means

$$l^{n/s'-\alpha}\le\left(\omega^q\left(B(x_0,l)\right)\right)^{1/q}\left\|\omega^{-1}\right\|_{L^{s'p/(p-s')}(B(x_0,l))}.\tag{48}$$

Thus

$$\begin{aligned}\|f\|_{L^p(\omega^p,B(x_0,2l))}&\le Cl^{n/s'-\alpha}\|f\|_{L^p(\omega,B(x_0,2l))}\\&\quad\cdot\int_{2l}^{\infty}\frac{1}{r^{n/s'-\alpha+1}}dr\le C\left(\omega^qB(x_0,l)\right)^{1/q}\\&\quad\cdot\left\|\omega^{-1}\right\|_{L^{s'p/(p-s')}(B(x_0,l))}\int_{2l}^{\infty}\|f\|_{L^p(\omega,B(x_0,r))}\\&\quad\cdot\frac{1}{r^{n/s'-\alpha+1}}dr\le C\left(\omega^qB(x_0,l)\right)^{1/q}\\&\quad\cdot\int_{2l}^{\infty}\|f\|_{L^p(\omega,B(x_0,r))}\left\|\omega^{-1}\right\|_{L^{s'p/(p-s')}(B(x_0,r))}\frac{1}{r^{n/s'-\alpha+1}}dr.\end{aligned}\tag{49}$$

Since $\omega^{s'}\in A(p/s',q/s')$, by (28), we get

$$\left(\omega^q\left(B(x_0,r)\right)\right)^{1/q}\left\|\omega^{-1}\right\|_{L^{s'p/(p-s')}(B(x_0,r))}\le Cr^{n/s'-\alpha}\tag{50}$$

holds for all $r > 0$. Then

$$\left\|T_{\Omega,\alpha}^{A_1,A_2}(f_1)\right\|_{L^q(\omega^q,B(x_0,l))} \le C\left(\prod_{j=1}^{2}\sum_{|\gamma|=m_j-1}\left\|D^\gamma A_j\right\|_*\right) \cdot \left(\omega^q\left(B\left(x_0,l\right)\right)\right)^{1/q} \cdot \int_{2l}^{\infty}\|f\|_{L^p(\omega^p,B(x_0,r))}\left(\omega^q\left(B\left(x_0,r\right)\right)\right)^{-1/q}\frac{1}{r}dr. \tag{51}$$

Let $N = m_1 + m_2 - 2$, and let $\Delta_i = (B(x_0, 2^{j+1}l)) \setminus (B(x_0, 2^j l))$. Suppose $x \in B(x_0, l)$. By Lemma 15,

$$\begin{aligned}
\left|T_{\Omega,\alpha}^{A_1,A_2}(f_2)(x)\right| &\le \left|\int_{(B(x_0,2l))^c}\frac{\Omega(x-y)}{|x-y|^{n-\alpha+N}}R_{m_1}(A_1;x,y)R_{m_2}(A_2;x,y)f(y)\,dy\right| \\
&\le C\sum_{j=1}^{\infty}\int_{\Delta_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}\prod_{k=1}^{2}\left(j\sum_{|\gamma|=m_k-1}\|D^\gamma A_k\|_* + \sum_{|\gamma|=m_k-1}\left|D^\gamma A_k(y)-(D^\gamma A_k)_{B(x_0,l)}\right|\right)dy \\
&\le C\left(\prod_{j=1}^{2}\sum_{|\gamma|=m_j-1}\left\|D^\gamma A_j\right\|_*\right)\sum_{j=1}^{\infty}j^2\int_{\Delta_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}dy \\
&\quad + C\sum_{|\gamma|=m_1-1}\|D^\gamma A_1\|_*\sum_{|\gamma|=m_2-1}\sum_{j=1}^{\infty}j\int_{\Delta_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}\left|D^\gamma A_2(y)-(D^\gamma A_2)_{B(x_0,l)}\right|dy \\
&\quad + C\sum_{|\gamma|=m_2-1}\|D^\gamma A_2\|_*\sum_{|\gamma|=m_1-1}\sum_{j=1}^{\infty}j\int_{\Delta_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}\left|D^\gamma A_1(y)-(D^{\gamma_1}A_1)_{B(x_0,l)}\right|dy \\
&\quad + C\sum_{|\gamma|=m_1-1}\sum_{|\gamma|=m_2-1}\sum_{j=1}^{\infty}\int_{\Delta_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}\prod_{k=1}^{2}\left|D^\gamma A_k(y)-(D^\gamma A_k)_{B(x_0,l)}\right|dy = I_1+I_2+I_3+I_4.
\end{aligned} \tag{52}$$

By Hölder's inequalities,

$$\int_{\Delta_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}dy \le \left(\int_{\Delta_i}|\Omega(x-y)|^s dy\right)^{1/s} \cdot \left(\int_{\Delta_i}\frac{|f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}}dy\right)^{1/s'}. \tag{53}$$

When $x \in B(x_0, s)$ and $y \in \Delta_i$, then by a direct calculation, we can see that $2^{j-1}l \le |y-x| < 2^{j+1}l$. Then

$$\left(\int_{\Delta_i}|\Omega(x-y)|^s dy\right)^{1/s} \le C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\left|B\left(x_0,2^{j+1}l\right)\right|^{1/s}. \tag{54}$$

We also note that if $x \in B(x_0, l)$, $y \in B(x_0, 2l)^c$, then $|y-x| \sim |y-x_0|$. Consequently

$$\left(\int_{\Delta_i}\frac{|f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}}dy\right)^{1/s'} \le \frac{1}{\left|B\left(x_0,2^{j+1}l\right)\right|^{1-\alpha/n}}\left(\int_{B(x_0,2^{j+1}l)}|f(y)|^{s'}dy\right)^{1/s'}. \tag{55}$$

Then

$$I_1 \le C\left(\prod_{j=1}^{2}\sum_{|\gamma|=m_j-1}\left\|D^\gamma A_j\right\|_*\right) \cdot \sum_{j=1}^{\infty}j^2\left(2^{j+1}l\right)^{\alpha-n/s'}\left(\int_{B(x_0,2^{j+1}l)}|f(y)|^{s'}dy\right)^{1/s'}. \tag{56}$$

Since $s' < p$, it follows from Hölder's inequality that

$$\left(\int_{B(x_0,2^{j+1}l)}|f(y)|^{s'}dy\right)^{1/s'} \le C\|f\|_{L^p(\omega^p,B(x_0,2^{j+1}l))}\left\|\omega^{-1}\right\|_{L^{s'p/(p-s')}(B(x_0,2^{j+1}l))}. \tag{57}$$

Then

$$\begin{aligned}
&\sum_{j=1}^{\infty}j^2\left(2^{j+1}l\right)^{\alpha-n/s'}\left(\int_{B(x_0,2^{j+1}s)}|f(y)|^{s'}dy\right)^{1/s'} \\
&\quad \le C\sum_{j=1}^{\infty}\left(1+\ln\frac{2^{j+1}l}{l}\right)^2\left(2^{j+1}l\right)^{\alpha-n/s'} \cdot \|f\|_{L^p(\omega^p,B(x_0,2^{j+1}l))}\left\|\omega^{-1}\right\|_{L^{s'p/(p-s')}(B(x_0,2^{j+1}l))}
\end{aligned}$$

$$\leq C\sum_{j=1}^{\infty}\int_{2^{j+1}l}^{2^{j+2}l}\left(1+\ln\frac{r}{l}\right)^{2}\|f\|_{L^{p}(\omega^{p},B(x_0,r))}\cdot\left\|\omega^{-1}\right\|_{L^{s'p/(p-s')}(B(x_0,r))}\frac{1}{r^{1-\alpha+n/s'}}dr\leq C\int_{2l}^{\infty}\left(1+\ln\frac{r}{l}\right)^{2}\|f\|_{L^{p}(\omega^{p},B(x_0,r))}\left\|\omega^{-1}\right\|_{L^{s'p/(p-s')}(B(x_0,r))}\cdot\frac{1}{r^{1-\alpha+n/s'}}dr. \tag{58}$$

From (50) we know

$$\left\|\omega^{-1}\right\|_{L^{s'p/(p-s')}(B(x_0,r))}\leq Cr^{n/s'-\alpha}\left(\omega^{q}\left(B\left(x_0,r\right)\right)\right)^{-1/q}. \tag{59}$$

Then

$$I_1\leq C\left(\prod_{j=1}^{2}\sum_{|\gamma|=m_j-1}\left\|D^{\gamma}A_j\right\|_{*}\right)\int_{2l}^{\infty}\left(1+\ln\frac{r}{l}\right)^{2}\cdot\|f\|_{L^{p}(\omega^{p},B(x_0,r))}\left(\omega^{q}\left(B\left(x_0,r\right)\right)\right)^{-1/q}\frac{1}{r}dr. \tag{60}$$

By Hölder's inequality and (54), (55) we have

$$\begin{aligned}\int_{\Delta_i}\frac{\left|\Omega\left(x-y\right)f\left(y\right)\right|}{\left|x-y\right|^{n-\alpha}}\left|D^{\gamma}A_2\left(y\right)-\left(D^{\gamma}A_2\right)_{B(x_0,l)}\right|dy&\leq\left(\int_{\Delta_i}\left|\Omega\left(x-y\right)\right|^{s}dy\right)^{1/s}\\&\cdot\left(\int_{\Delta_i}\frac{\left|\left(D^{\gamma}A_2\left(y\right)-\left(D^{\gamma}A_2\right)_{B(x_0,l)}\right)f\left(y\right)\right|^{s'}}{\left|x-y\right|^{(n-\alpha)s'}}dy\right)^{1/s'}\\&\leq C\sum_{j=1}^{\infty}\left(2^{j+1}l\right)^{\alpha-n/s'}\cdot\left(\int_{B(x_0,2^{j+1}l)}\left|D^{\gamma}A_2\left(y\right)-\left(D^{\gamma}A_2\right)_{B(x_0,l)}\right|^{s'}\cdot\left|f\left(y\right)\right|^{s'}dy\right)^{1/s'}.\end{aligned} \tag{61}$$

Applying Hölder's inequality again, we get

$$\begin{aligned}&\left(\int_{B(x_0,2^{j+1}l)}\left|D^{\gamma}A_2\left(y\right)-\left(D^{\gamma}A_2\right)_{B(x_0,l)}\right|^{s'}\cdot\left|f\left(y\right)\right|^{s'}dy\right)^{1/s'}\leq C\|f\|_{L^{p}(\omega^{p},B(x_0,2^{j+1}l))}\\&\cdot\left\|\left(D^{\gamma}A_2\left(y\right)-\left(D^{\gamma}A_2\right)_{B(x_0,l)}\right)\cdot\omega\left(\cdot\right)^{-1}\right\|_{L^{ps'/(p-s')}(B(x_0,2^{j+1}l))}.\end{aligned} \tag{62}$$

Consequently,

$$\begin{aligned}I_2&\leq C\sum_{|\gamma|=m_1-1}\left\|D^{\gamma}A_1\right\|_{*}\sum_{|\gamma|=m_2-1}\sum_{j=1}^{\infty}\int_{2^{j+1}l}^{2^{j+2}l}\left(1+\ln\frac{2^{j+1}l}{l}\right)\cdot\left(2^{j+1}l\right)^{\alpha-n/s'}\|f\|_{L^{p}(\omega^{p},B(x_0,r))}\\&\cdot\left\|\left(D^{\gamma}A_2\left(y\right)-\left(D^{\gamma}A_2\right)_{B(x_0,l)}\right)\omega\left(\cdot\right)^{-1}\right\|_{L^{ps'/(p-s')}(B(x_0,r))}dr\\&\leq C\sum_{|\gamma|=m_1-1}\left\|D^{\gamma}A_1\right\|_{*}\sum_{|\gamma|=m_2-1}\int_{2l}^{\infty}\|f\|_{L^{p}(\omega^{p},B(x_0,r))}\cdot\left\|\left(D^{\gamma}A_2\left(y\right)-\left(D^{\gamma}A_2\right)_{B(x_0,l)}\right)\omega\left(\cdot\right)^{-1}\right\|_{L^{ps'/(p-s')}(B(x_0,r))}\cdot\frac{1}{r^{1-\alpha+n/s'}}dr.\end{aligned} \tag{63}$$

By $\omega^{s'}\in A(p/s',q/s')$ and (ii) of Lemma 11 we know $\omega^{-s'p/(p-s')}\in A_{1+ps'/(p-s')q}$. Then it follows from (31) and (59) that

$$\begin{aligned}&\left\|\left(D^{\gamma}A_2\left(y\right)-\left(D^{\gamma}A_2\right)_{B(x_0,l)}\right)\omega\left(\cdot\right)^{-1}\right\|_{L^{ps'/(p-s')}(B(x_0,r))}\\&\leq\left(\int_{B(x_0,r)}\left|\left(D^{\gamma}A_2\left(y\right)-\left(D^{\gamma}A_2\right)_{B(x_0,l)}\right)\right|^{ps'/(p-s')}\cdot\omega^{-ps'/(p-s')}\left(y\right)dy\right)^{(p-s')/ps'}\leq C\left\|D^{\gamma}A_2\right\|_{*}\left(1+\ln\frac{r}{l}\right)\\&\cdot\left(\omega^{-ps'/(p-s')}\left(B\left(x_0,r\right)\right)\right)^{(p-s')/ps'}=C\left\|D^{\gamma}A_2\right\|_{*}\left(1+\ln\frac{r}{l}\right)\left\|\omega^{-1}\right\|_{L^{ps'/(p-s')}(B(x_0,r))}\leq C\left\|D^{\gamma}A_2\right\|_{*}\left(1+\ln\frac{r}{l}\right)\cdot r^{n/s'-\alpha}\left(\omega^{q}\left(B\left(x_0,r\right)\right)\right)^{-1/q}.\end{aligned} \tag{64}$$

Hence

$$I_2\leq C\left(\prod_{j=1}^{2}\sum_{|\gamma|=m_j-1}\left\|D^{\gamma}A_j\right\|_{*}\right)\int_{2l}^{\infty}\left(1+\ln\frac{r}{l}\right)\cdot\|f\|_{L^{p}(\omega^{p},B(x_0,r))}\left(\omega^{q}\left(B\left(x_0,r\right)\right)\right)^{-1/q}\frac{1}{r}dr. \tag{65}$$

Similar to the estimates for I_2, we have

$$I_3\leq C\left(\prod_{j=1}^{2}\sum_{|\gamma|=m_j-1}\left\|D^{\gamma}A_j\right\|_{*}\right)\int_{2l}^{\infty}\left(1+\ln\frac{r}{l}\right)\cdot\|f\|_{L^{p}(\omega^{p},B(x_0,r))}\left(\omega^{q}\left(B\left(x_0,r\right)\right)\right)^{-1/q}\frac{1}{r}dr. \tag{66}$$

Finally, we come to estimate I_4.

Similar to (59), we get

$$\begin{aligned}&\int_{\Delta_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}\prod_{k=1}^{2}\left|D^\gamma A_k(y)-(D^\gamma A_k)_{B(x_0,l)}\right|dy\\&\le C\left(2^{j+1}l\right)^{\alpha-n/s'}\\&\cdot\left(\int_{B(x_0,2^{j+1}l)}\left|\prod_{k=1}^{2}\left(D^\gamma A_k(y)-(D^\gamma A_k)_{B(x_0,l)}\right)\right|^{s'}\right.\\&\left.\cdot|f(y)|^{s'}dy\right)^{1/s'}.\end{aligned}\tag{67}$$

By Hölder's inequality we get

$$\begin{aligned}&\left(\int_{B(x_0,2^{j+1}l)}\left|\prod_{k=1}^{2}\left(D^\gamma A_k(y)-(D^\gamma A_k)_{B(x_0,l)}\right)\right|^{s'}\right.\\&\left.\cdot|f(y)|^{s'}dy\right)^{1/s'}\le C\|f\|_{L^p(\omega^p,B(x_0,2^{j+1}l))}\\&\cdot\left\|\left(\prod_{k=1}^{2}\left(D^\gamma A_k(y)-(D^\gamma A_k)_{B(x_0,l)}\right)\right)\right.\\&\left.\cdot\omega(\cdot)^{-1}\right\|_{L^{ps'/(p-s')}(B(x_0,2^{j+1}l))}\le C\|f\|_{L^p(\omega^p,B(x_0,2^{j+1}l))}\\&\cdot\prod_{k=1}^{2}\left\|\left|D^\gamma A_k(y)-(D^\gamma A_k)_{B(x_0,l)}\right|\right.\\&\left.\cdot\omega(\cdot)^{-1/2}\right\|_{L^{2ps'/(p-s')}(B(x_0,2^{j+1}l))}.\end{aligned}\tag{68}$$

Then

$$\begin{aligned}&\sum_{j=1}^{\infty}\int_{\Delta_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}\prod_{k=1}^{2}\left|D^\gamma A_k(y)-(D^\gamma A_k)_{B(x_0,l)}\right|dy\\&\le C\int_{2l}^{\infty}\|f\|_{L^p(\omega^p,B(x_0,r))}\prod_{k=1}^{2}\left\|\left|D^\gamma A_k(y)-(D^\gamma A_k)_{B(x_0,l)}\right|\right.\\&\left.\cdot\omega(\cdot)^{-1/2}\right\|_{L^{2ps'/(p-s')}(B(x_0,r))}\frac{1}{r^{1-\alpha+n/s'}}dr.\end{aligned}\tag{69}$$

Since $\omega^{-s'p/(p-s')}\in A_{1+ps'/(p-s')q}$, similar to (64), we can deduce

$$\begin{aligned}&\left\|\left|D^\gamma A_k(y)-(D^\gamma A_k)_{B(x_0,l)}\right|\right.\\&\left.\cdot\omega(\cdot)^{-1/2}\right\|_{L^{2ps'/(p-s')}(B(x_0,2^{j+1}l))}\le C\|D^\gamma A_k\|_*\left(1\right.\\&\left.+\ln\frac{r}{l}\right)\left(\omega^{-ps'/(p-s')}(B(x_0,r))\right)^{(p-s')/2ps'}.\end{aligned}\tag{70}$$

Then by (70) we get

$$\begin{aligned}&\sum_{j=1}^{\infty}\int_{\Delta_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}\prod_{k=1}^{2}\left|D^\gamma A_k(y)-(D^\gamma A_k)_{B(x_0,l)}\right|dy\\&\le C\prod_{k=1}^{2}\|D^\gamma A_k\|_*\int_{2l}^{\infty}\left(1+\ln\frac{r}{l}\right)^2\|f\|_{L^p(\omega^p,B(x_0,r))}\\&\cdot\left(\omega^q(B(x_0,r))\right)^{-1/q}\frac{1}{r}dr.\end{aligned}\tag{71}$$

Thus

$$\begin{aligned}I_4&\le C\left(\prod_{j=1}^{2}\sum_{|\gamma|=m_j-1}\|D^\gamma A_j\|_*\right)\int_{2l}^{\infty}\left(1+\ln\frac{r}{l}\right)\\&\cdot\|f\|_{L^p(\omega^p,B(x_0,r))}\left(\omega^q(B(x_0,r))\right)^{-1/q}\frac{1}{r}dr.\end{aligned}\tag{72}$$

Combining the estimates for I_1, I_2, I_3, and I_4, we get

$$\begin{aligned}&\left|T_{\Omega,\alpha}^{A_1,A_2}(f_2)(x)\right|\le C\left(\prod_{j=1}^{2}\sum_{|\gamma|=m_j-1}\|D^\gamma A_j\|_*\right)\\&\cdot\int_{2l}^{\infty}\left(1+\ln\frac{r}{l}\right)\|f\|_{L^p(\omega^p,B(x_0,r))}\left(\omega^q(B(x_0,r))\right)^{-1/q}\\&\cdot\frac{1}{r}dr.\end{aligned}\tag{73}$$

Then

$$\begin{aligned}&\left\|T_{\Omega,\alpha}^{A_1,A_2}(f_2)\right\|_{L^p(\omega^p,B(x_0,l))}\le C\left(\prod_{j=1}^{2}\sum_{|\gamma|=m_j-1}\|D^\gamma A_j\|_*\right)\\&\cdot\left(\omega^q(B(x_0,l))\right)^{1/q}\\&\cdot\int_{2l}^{\infty}\|f\|_{L^p(\omega^p,B(x_0,r))}\left(\omega^q(B(x_0,r))\right)^{-1/q}\frac{1}{r}dr.\end{aligned}\tag{74}$$

This completes the proof of Theorem 17. □

4. Proof of Theorem 3

Since $f\in M_{\varphi_1}^p(\omega^p,\mathbb{R}^n)$, then by Lemma 16 and the fact that $\|f\|_{L^p(\omega^p,B(x_0,r))}$ is a nondecreasing function of r, we get

$$\begin{aligned}&\frac{\|f\|_{L^p(\omega^p,B(x_0,r))}}{\operatorname{ess\,inf}_{0<r<t<\infty}\varphi_1(x_0,t)\left(\omega^p(B(x_0,t))\right)^{1/p}}\\&\le\operatorname{ess\,sup}_{0<r<t<\infty}\frac{\|f\|_{L^p(\omega^p,B(x_0,r))}}{\varphi_1(x_0,t)\left(\omega^p(B(x_0,t))\right)^{1/p}}\\&\le\sup_{t>0,x_0\in\mathbb{R}^n}\frac{\|f\|_{L^p(\omega^p,B(x_0,t))}}{\varphi_1(x_0,t)\left(\omega^p(B(x_0,t))\right)^{1/p}}\\&\le\|f\|_{M_{\varphi_1}^p(\omega^p,\mathbb{R}^n)}.\end{aligned}\tag{75}$$

Since (φ_1, φ_2) satisfies (15), we have

$$\begin{aligned}
&\int_l^\infty \left(1+\ln\frac{r}{l}\right)^k \|f\|_{L^p(\omega^p,B(x_0,r))} \left(\omega^q\left(B\left(x_0,r\right)\right)\right)^{-1/q} \\
&\quad\cdot \frac{1}{r}dr \\
&\quad\le \int_l^\infty \frac{\|f\|_{L^p(\omega^p,B(x_0,r))}}{\operatorname{ess\,inf}_{r<t<\infty}\varphi_1\left(x_0,t\right)\left(\omega^p\left(B\left(x_0,t\right)\right)\right)^{1/p}}\left(1\right. \\
&\quad\left.+\ln\frac{r}{l}\right)^k \\
&\quad\cdot \frac{\operatorname{ess\,inf}_{r<t<\infty}\varphi_1\left(x_0,t\right)\left(\omega^p\left(B\left(x_0,t\right)\right)\right)^{1/p}}{\left(\omega^q\left(B\left(x_0,r\right)\right)\right)^{1/q}}\frac{1}{r}dr \\
&\quad\le C\|f\|_{M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)}\int_l^\infty\left(1+\ln\frac{r}{l}\right)^k \\
&\quad\cdot \frac{\operatorname{ess\,inf}_{r<t<\infty}\varphi_1\left(x_0,t\right)\left(\omega^p\left(B\left(x_0,t\right)\right)\right)^{1/p}}{\left(\omega^q\left(B\left(x_0,r\right)\right)\right)^{1/q}}\frac{1}{r}dr \\
&\quad\le C\|f\|_{M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)}\varphi_2\left(x_0,l\right).
\end{aligned} \tag{76}$$

Then by Theorem 17 we get

$$\begin{aligned}
&\left\|T^{A_1,\dots,A_k}_{\Omega,\alpha}\left(f\right)\right\|_{M^q_{\varphi_2}(\omega^q,\mathbb{R}^n)} \le C\sup_{x_0\in\mathbb{R}^n,l>0}\frac{1}{\varphi_2\left(x_0,l\right)} \\
&\quad\cdot\left(\frac{1}{\omega^q B\left(x_0,l\right)}\right. \\
&\quad\left.\cdot\int_{B(x_0,l)}\left|T^{A_1,\dots,A_k}_{\Omega,\alpha}\left(f\right)\left(y\right)\right|^q\omega^q\left(y\right)dy\right)^{1/q} \\
&\quad\le C\left(\prod_{j=1}^k\sum_{|\gamma|=m_j-1}\|D^\gamma A\|_*\right)\sup_{x_0\in\mathbb{R}^n,l>0}\frac{1}{\varphi_2\left(x_0,l\right)} \\
&\quad\cdot\int_l^\infty\left(1+\ln\frac{r}{l}\right)^k\|f\|_{L^p(\omega^p,B(x_0,r))}\left(\omega^q\left(B\left(x_0,r\right)\right)\right)^{-1/q} \\
&\quad\cdot\frac{1}{r}dr \le C\left(\prod_{j=1}^k\sum_{|\gamma|=m_j-1}\|D^\gamma A\|_*\right)\|f\|_{M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)}.
\end{aligned} \tag{77}$$

Conflict of Interests

The authors declare that they have no conflict of interests.

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