Research Article

Ulam-Hyers Stability of Trigonometric Functional Equation with Involution

Jaeyoung Chung,¹ Chang-Kwon Choi,² and Jongjin Kim³

¹Department of Mathematics, Kunsan National University, Kunsan 573-701, Republic of Korea

²Department of Mathematics, Chonbuk National University, Jeonju 561-756, Republic of Korea

³Department of Mathematics and Institute of Pure and Applied Mathematics, Chonbuk National University, Jeonju 561-756, Republic of Korea

Correspondence should be addressed to Chang-Kwon Choi; ck38@jbnu.ac.kr

Received 15 July 2015; Accepted 10 September 2015

Academic Editor: Ajda Fošner

Copyright © 2015 Jaeyoung Chung et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Let S and G be a commutative semigroup and a commutative group, respectively, $\mathbb{C}$ and $\mathbb{R}^+$ the sets of complex numbers and nonnegative real numbers, respectively, and $\sigma : S \rightarrow S$ or $\sigma : G \rightarrow G$ an involution. In this paper, we first investigate general solutions of the functional equation $f(x + \sigma y) = f(x)g(y) - g(x)f(y)$ for all $x, y \in S$, where $f, g : S \rightarrow \mathbb{C}$. We then prove the Hyers-Ulam stability of the functional equation; that is, we study the functional inequality $|f(x + \sigma y) - f(x)g(y) + g(x)f(y)| \leq \psi(y)$ for all $x, y \in G$, where $f, g : G \rightarrow \mathbb{C}$ and $\psi : G \rightarrow \mathbb{R}^+$.

1. Introduction

The following four functional equations

$$f(x + y) = f(x)g(y) + g(x)f(y), \quad (1)$$

$$g(x + y) = g(x)g(y) - f(x)f(y), \quad (2)$$

$$f(x - y) = f(x)g(y) - g(x)f(y), \quad (3)$$

$$g(x - y) = g(x)g(y) + f(x)f(y) \quad (4)$$

are called *trigonometric functional equations*. The above functional equations have been investigated together or individually. For early historical developments of the above functional equations we refer the reader to [1, p. 177] and [2, pp. 209–217]. In particular, general solutions of (3) are described in [2, pp. 216–217] and general solutions of (4) and regular solutions of (1) and (2) are described in [1, pp. 177–180]. Several authors [3–5] have studied some functional equations with involutions, particular cases $\sigma(y) = -y$ of which are the usual d'Alembert's functional equation [6], Wilson's functional equation [7], or some related classical functional equations with usual operations. In this paper,

generalizing the functional equation (3) we first consider the functional equation

$$f(x + \sigma y) = f(x)g(y) - g(x)f(y) \quad (5)$$

for all $x, y \in S$, where $f, g : S \rightarrow \mathbb{C}$. As a result we first find all general solutions of functional equation (5). Secondly, we investigate the Hyers-Ulam stability of functional equation (5). The Hyers-Ulam stability problems of functional equations go back to 1940 when Ulam proposed a question concerning the approximate homomorphisms from a group to a metric group (see [8]). Székelyhidi [9] investigated the Hyers-Ulam stability of trigonometric functional equations (1) and (2). As a particular case of the result in [9] he obtained the stability of functional inequality

$$\begin{aligned} |f(x + y) - f(x)g(y) - g(x)f(y)| &\leq \psi(y), \\ |f(x + y) - f(x)f(y) + g(x)g(y)| &\leq \psi(y) \end{aligned} \quad (6)$$

for all $x, y \in G$, where $f, g : G \rightarrow \mathbb{C}$ and $\psi : G \rightarrow \mathbb{R}^+$. On the other hand, we obtained a Székelyhidi type stability of functional equations (3) and (4) (see [10, 11] and also [12–14] for related results). In this paper we obtain a parallel

Hyers-Ulam stability to that of [9] for functional equation (3); that is, we obtain the Hyers-Ulam stability of the functional inequality

$$|f(x + \sigma y) - f(x)g(y) + g(x)f(y)| \leq \psi(y) \quad (7)$$

for all $x, y \in G$, where $f, g : G \rightarrow \mathbb{C}$ and $\psi : G \rightarrow \mathbb{R}^+$. Unfortunately, we have no idea yet how to obtain all general solutions of equation

$$g(x + \sigma y) = g(x)g(y) + f(x)f(y) \quad (8)$$

for all $x, y \in S$, where $f, g : S \rightarrow \mathbb{C}$ and its stability, which are the parallel generalizations of the functional equation and stability of (4).

2. General Solutions of (5)

Throughout this section we denote by S a 2-divisible commutative semigroup with an identity element. A function $\sigma : S \rightarrow S$ is said to be an *involution* if $\sigma(x + y) = \sigma(x) + \sigma(y)$ for all $x, y \in S$ and $\sigma(\sigma(x)) = x$ for all $x \in S$. For simplicity we write σx instead of $\sigma(x)$. A function $m : S \rightarrow \mathbb{C}$ is called an *exponential function* provided that $m(x + y) = m(x)m(y)$ for all $x, y \in S$ and $a : S \rightarrow \mathbb{C}$ is called an *additive function* provided that $a(x + y) = a(x) + a(y)$ for all $x, y \in S$. In this section, we present the general solutions (f, g) of functional equation (5).

Lemma 1 (see [15, Theorem 2]). *Let $f, g : S \rightarrow \mathbb{C}$ satisfy the functional equation*

$$f(x + y) + f(x + \sigma y) = 2f(x)g(y) \quad (9)$$

for all $x, y \in S$. Then either (g, f) has the form

$$g(x) = \frac{m(x) + m(\sigma x)}{2}, \quad (10)$$

$$f(x) = \lambda_1 m(x) + \lambda_2 m(\sigma x)$$

for all $x \in S$, where $m : S \rightarrow \mathbb{C}$ is an exponential function satisfying $m \neq m \circ \sigma$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, or

$$\begin{aligned} g(x) &= m(x), \\ f(x) &= m(x)(\beta + a(x)), \end{aligned} \quad (11)$$

where $m : S \rightarrow \mathbb{C}$ is an exponential function satisfying $m = m \circ \sigma$, and $a : S \rightarrow \mathbb{C}$ is an additive function on $S^* = \{x \in S : m(x) \neq 0\}$ satisfying $a(x) = -a(\sigma x)$ for all $x \in S^*$ and taking arbitrary values on $S \setminus S^*$, and $\beta \in \mathbb{C}$.

In the following, we exclude the trivial cases when $f(x) = 0$ for all $x \in S$ or $g(x) = 0$ for all $x \in S$.

Theorem 2. *Let $f : S \rightarrow \mathbb{C}$ satisfy the functional equation*

$$f(x + \sigma y) = f(x)g(y) - g(x)f(y) \quad (12)$$

for all $x, y \in S$. Then either (f, g) has the form

$$\begin{aligned} f(x) &= \lambda(m(x) - m(\sigma x)), \\ g(x) &= \gamma m(x) + (1 - \gamma)m(\sigma x) \end{aligned} \quad (13)$$

for all $x \in S$, where $m : S \rightarrow \mathbb{C}$ is an exponential function satisfying $m \neq m \circ \sigma$ and $\lambda, \gamma \in \mathbb{C}$, or

$$\begin{aligned} f(x) &= a(x)m(x), \\ g(x) &= (1 + \beta a(x))m(x) \end{aligned} \quad (14)$$

for all $x \in S$, where $m : S \rightarrow \mathbb{C}$ is an exponential function satisfying $m = m \circ \sigma$, and $a : S \rightarrow \mathbb{C}$ is an additive function on $S^* = \{x \in S : m(x) \neq 0\}$ satisfying $a(x) = -a(\sigma x)$ for all $x \in S^*$ and taking arbitrary values on $S \setminus S^*$, and $\beta \in \mathbb{C}$.

Proof. Let g_e and g_o be the even part and odd part of g with respect to σ ; that is,

$$\begin{aligned} g_e(x) &= \frac{g(x) + g(\sigma x)}{2}, \\ g_o(x) &= \frac{g(x) - g(\sigma x)}{2} \end{aligned} \quad (15)$$

for all $x \in S$. Then we have

$$\begin{aligned} g(x) &= g_e(x) + g_o(x), \\ g_e(\sigma x) &= g_e(x), \\ g_o(\sigma x) &= -g_o(x) \end{aligned} \quad (16)$$

for all $x \in S$. Replacing (x, y) by (y, x) in (12), we have

$$f(y + \sigma x) = f(y)g(x) - g(y)f(x) \quad (17)$$

for all $x, y \in S$. From (12) and (17) we have

$$f(x + \sigma y) = -f(y + \sigma x) \quad (18)$$

for all $x, y \in S$. Putting $y = 0$ in (18) we have

$$f(\sigma x) = -f(x) \quad (19)$$

for all $x \in S$. Substituting (16) into (12) we have

$$\begin{aligned} f(x + \sigma y) &= f(x)g_e(y) + f(x)g_o(y) \\ &\quad - g_e(x)f(y) - g_o(x)f(y) \end{aligned} \quad (20)$$

for all $x, y \in S$. Replacing (x, y) by $(\sigma x, \sigma y)$ in (20) and using (18) and (19) we have

$$\begin{aligned} -f(x + \sigma y) &= -f(x)g_e(y) + f(x)g_o(y) \\ &\quad + g_e(x)f(y) - g_o(x)f(y) \end{aligned} \quad (21)$$

for all $x, y \in S$. Adding (21) and (20) we have

$$f(x)g_o(y) = g_o(x)f(y) \quad (22)$$

for all $x, y \in S$. Choosing z_0 such that $f(z_0) \neq 0$, putting $y = z_0$ in (22), and dividing the result by $f(z_0)$ we have

$$g_o(x) = \beta f(x) \quad (23)$$

for all $x \in S$, where $\beta = g(z_0)/f(z_0)$. Thus, we have

$$g(x) = g_e(x) + g_o(x) = g_e(x) + \beta f(x) \quad (24)$$

for all $x, y \in S$. Thus, it remains to find g_e and f . Replacing y by σy in (20) and using (19) we have

$$f(x+y) = f(x)g_e(y) - f(x)g_o(y) + g_e(x)f(y) - g_o(x)f(y) \quad (25)$$

for all $x, y \in S$. Adding (25) and (20) we have

$$f(x+y) + f(x+\sigma y) = 2f(x)g_e(y) \quad (26)$$

for all $x, y \in S$. By Lemma 1, (g_e, f) has the form (10) or (11). First, we consider the case when (g_e, f) has the form (10); that is, $g_e(x) = (1/2)(m(x) + m(\sigma x))$, $f(x) = \lambda_1 m(x) + \lambda_2 m(\sigma x)$ for all $x \in S$ and $m \neq m \circ \sigma$. From (19) we have $\lambda_1 m(x) + \lambda_2 m(\sigma x) = -\lambda_1 m(\sigma x) - \lambda_2 m(x)$ for all $x \in S$, which implies $(\lambda_1 + \lambda_2)m(x) = -(\lambda_1 + \lambda_2)m(\sigma x)$ for all $x \in S$. If $\lambda_1 + \lambda_2 \neq 0$, we have $m(x) = -m(\sigma x)$ for all $x \in S$, and hence $m(x)^2 = m(2x) = -m(2\sigma x) = -m(\sigma x)^2 = -m(x)^2$ for all $x \in S$, which leads to the contradiction $m(x) = m(\sigma x) = 0$ for all $x \in S$. Thus, we have $\lambda_2 = -\lambda_1 := \lambda$ and

$$f(x) = \lambda(m(x) - m(\sigma x)) \quad (27)$$

for all $x \in S$. From (24) and (27) we have

$$\begin{aligned} g(x) &= \frac{m(x) + m(\sigma x)}{2} + \beta f(x) \\ &= \frac{m(x) + m(\sigma x)}{2} + \beta \lambda(m(x) - m(\sigma x)) \\ &= \gamma m(x) + (1 - \gamma)m(\sigma x) \end{aligned} \quad (28)$$

for all $x \in S$, where $\gamma = 1/2 + \beta\lambda$. Thus, we get (13). Secondly, we consider the case when (g_e, f) has the form (11); that is, $g_e(x) = m(x)$, $f(x) = m(x)(\beta + a(x))$, $m(x) = m(\sigma x)$, and $a(x) = -a(\sigma x)$ for all $x \in S$. From (19) we have $m(x)(\beta + a(x)) = -m(\sigma x)(\beta + a(\sigma x)) = -m(x)(\beta - a(x))$ for all $x \in S$, which implies $\beta = 0$. Thus, we have $f(x) = m(x)a(x)$ for all $x \in S$. From (23) we have $g(x) = g_e(x) + g_o(x) = m(x) + \beta m(x)a(x)$ for all $x \in S$. Thus, we get (14). This completes the proof. $\square$

In particular if $S = G$ a commutative group, then since $S^* = S$, we have the following.

Corollary 3. Let $f : G \rightarrow \mathbb{C}$ satisfy the functional equation

$$f(x-y) - f(x)g(y) + g(x)f(y) = 0 \quad (29)$$

for all $x, y \in G$. Then either (f, g) has the form

$$\begin{aligned} f(x) &= \lambda(m(x) - m(-x)), \\ g(x) &= \gamma m(x) + (1 - \gamma)m(-x) \end{aligned} \quad (30)$$

for all $x \in G$, where $m : G \rightarrow \mathbb{C}$ is an exponential function satisfying $m(x) \neq m(-x)$ for some $x \in G$ and $\lambda, \gamma \in \mathbb{C}$, or else

$$\begin{aligned} f(x) &= a(x)m(x), \\ g(x) &= (1 + \beta a(x))m(x) \end{aligned} \quad (31)$$

for all $x \in G$, where $m : G \rightarrow \mathbb{C}$ is an exponential function satisfying $m(x) = m(-x)$ for all $x \in G$ and $a : G \rightarrow \mathbb{C}$ is an additive function, and $\beta \in \mathbb{C}$.

In particular, if G is a 2-divisible commutative group, then $m(x) = m(-x)$ for all $x \in G$ if and only if $m(x) = 1$ for all $x \in G$. Thus, we have the following.

Corollary 4. Let $f : G \rightarrow \mathbb{C}$ satisfy the functional equation

$$f(x-y) - f(x)g(y) + g(x)f(y) = 0 \quad (32)$$

for all $x, y \in G$. Then either (f, g) has the form

$$\begin{aligned} f(x) &= \lambda(m(x) - m(-x)), \\ g(x) &= \gamma m(x) + (1 - \gamma)m(-x) \end{aligned} \quad (33)$$

for all $x \in G$, where $m : G \rightarrow \mathbb{C}$ is an exponential function with $m \neq 1$, or else

$$\begin{aligned} f(x) &= a(x), \\ g(x) &= 1 + \beta a(x) \end{aligned} \quad (34)$$

for all $x \in G$, where $a : G \rightarrow \mathbb{C}$ is an additive function, and $\beta \in \mathbb{C}$.

Let $\langle H, + \rangle$ be a commutative semigroup, and $f, g : H \times H \rightarrow \mathbb{C}$. As an application of Theorem 2, we determine all general solutions of the functional equation

$$\begin{aligned} f(x_1 + y_2, x_2 + y_1) &= f(x_1, x_2)g(y_1, y_2) \\ &\quad - g(x_1, x_2)f(y_1, y_2) \end{aligned} \quad (35)$$

for all $x_1, x_2, y_1, y_2 \in H$.

In the following we exclude the trivial cases when $f = 0$ or $g = 0$.

Theorem 5. Let $f, g : H \times H \rightarrow \mathbb{C}$ satisfy the functional equation (35). Then either (f, g) has the form

$$f(x_1, x_2) = \lambda(m_1(x_1)m_2(x_2) - m_1(x_2)m_2(x_1)), \quad (36)$$

$$\begin{aligned} g(x_1, x_2) &= \gamma m_1(x_1)m_2(x_2) \\ &\quad + (1 - \gamma)m_1(x_2)m_2(x_1) \end{aligned} \quad (37)$$

for all $x_1, x_2 \in H$, where $m_1, m_2 : H \rightarrow \mathbb{C}$ are exponential functions satisfying $m_1 \neq m_2$ and $\lambda, \gamma \in \mathbb{C}$, or else

$$f(x_1, x_2) = (a(x_1) - a(x_2))m(x_1 + x_2), \quad (38)$$

$$g(x_1, x_2) = (1 + \beta(a(x_1) - a(x_2)))m(x_1 + x_2) \quad (39)$$

for all $x_1, x_2 \in H$, where $m : H \rightarrow \mathbb{C}$ is an exponential function and $a : H \rightarrow \mathbb{C}$ is an additive function on $H^* := \{x \in H : m(x) \neq 0\}$ taking arbitrary values on $H \setminus H^*$, and $\beta \in \mathbb{C}$.

Proof. Let $\sigma(x_1, x_2) = (x_2, x_1)$ for all $x_1, x_2 \in H$. Then by Theorem 2, all general solutions (f, g) of functional equation (35) are given by

$$f(x_1, x_2) = \lambda(m(x_1, x_2) - m(x_2, x_1)), \quad (40)$$

$$g(x_1, x_2) = \gamma m(x_1, x_2) + (1 - \gamma)m(x_2, x_1) \quad (41)$$

for all $x_1, x_2 \in H$, where $m : H \times H \rightarrow \mathbb{C}$ is exponential functions satisfying $m(x_1, x_2) \neq m(x_2, x_1)$ for some $(x_1, x_2) \in H \times H$ and $\lambda, \gamma \in \mathbb{C}$, or else

$$f(x_1, x_2) = a(x_1, x_2) m(x_1, x_2), \quad (42)$$

$$g(x_1, x_2) = (1 + \beta a(x_1, x_2)) m(x_1, x_2) \quad (43)$$

for all $x_1, x_2 \in H$, where $m : H \times H \rightarrow \mathbb{C}$ is exponential functions satisfying $m(x_1, x_2) = m(x_2, x_1)$ for all $x_1, x_2 \in H$, $a : H \times H \rightarrow \mathbb{C}$ is an additive function on $(H \times H)^* := \{(x_1, x_2) : m(x_1, x_2) \neq 0\}$, and $\beta \in \mathbb{C}$. Since m can be written in the form $m(x_1, x_2) = m_1(x_1)m_2(x_2)$ for some exponential functions $m_1, m_2 : H \rightarrow \mathbb{C}$, from (40) and (41) we get (36) and (37), respectively. Now, we consider the case when $m = m \circ \sigma$. Since $m(x_1, x_2) = m(x_2, x_1)$ for all $x_1, x_2 \in H$ we have

$$\begin{aligned} m(x_1, x_2) &= m_1(x_1)m_2(x_2) = m_1(x_2)m_2(x_1) \\ &= m(x_2, x_1) \end{aligned} \quad (44)$$

for all $x_1, x_2 \in H$. It is easy to see that equality (44) implies $m_1 = m_2$. Thus, $(x_1, x_2) \in (H \times H)^*$ if and only if $x_1, x_2 \in H^* := \{x \in H : m_1(x) \neq 0\}$. On the other hand, the additive function a can be written in the form $a(x_1, x_2) = a_1(x_1) + a_2(x_2)$ for all $x_1, x_2 \in H^*$, where $a_1, a_2 : H^* \rightarrow \mathbb{C}$ are additive functions. Now, the equality $a(x_1, x_2) = -a(x_2, x_1)$ implies $a_2(x) = -a_1(x)$ for all $x \in H^*$. Thus we have $m(x_1, x_2) = m_1(x_1 + x_2)$ for all $x_1, x_2 \in H$ and $a(x_1, x_2) = a_1(x_1) - a_1(x_2)$ for all $x_1, x_2 \in H^*$. Thus, by letting $m_1 = m$, $a_1 = a$ we get (40) and (39) from (42) and (43), respectively, since if $x_1 \notin H^*$ or $x_2 \notin H^*$, then $m(x_1, x_2) = 0$. This completes the proof. $\square$

3. Stability of (5)

Throughout this section we denote by G a 2-divisible commutative group and $f, g : G \rightarrow \mathbb{C}$. In this section we consider the stability of functional equation (5); that is, we deal with the functional inequality

$$|f(x + \sigma y) - f(x)g(y) + g(x)f(y)| \leq \psi(y) \quad (45)$$

for all $x, y \in G$. For the proof of the stability of (45) we need the following.

Lemma 6. *Let $f, g : G \rightarrow \mathbb{C}$ satisfy the functional inequality (45) for all $x, y \in G$. Then there exist $\mu_1, \mu_2 \in \mathbb{C}$ (not both zero) and $M > 0$ such that*

$$|\mu_1 f(x) - \mu_2 g(x)| \leq M \quad (46)$$

for all $x \in G$, or else

$$f(x + \sigma y) - f(x)g(y) + g(x)f(y) = 0 \quad (47)$$

for all $x, y \in G$.

Proof. Suppose that $\mu_1 f(x) - \mu_2 g(x)$ is bounded only when $\mu_1 = \mu_2 = 0$. Let

$$k(x, y) = f(x + y) - f(x)g(\sigma y) + g(x)f(\sigma y) \quad (48)$$

for all $x, y \in G$. Choose y_1 satisfying $f(\sigma y_1) \neq 0$. Then we have

$$g(x) = \omega_1 f(x) + \omega_2 f(x + y_1) - \omega_2 k(x, y_1) \quad (49)$$

for all $x \in G$, where $\omega_1 = g(\sigma y_1)/f(\sigma y_1)$ and $\omega_2 = -1/f(\sigma y_1)$. From (48) and (49) we have

$$\begin{aligned} f(x + y + z) &= f(x + y)g(\sigma z) - g(x + y)f(\sigma z) \\ &\quad + k(x + y, z) = (f(x)g(\sigma y) - g(x)f(\sigma y) \\ &\quad + k(x, y))g(\sigma z) - (\omega_1 f(x + y) \\ &\quad + \omega_2 f(x + y + y_1) - \omega_2 k(x + y, y_1))f(\sigma z) \\ &\quad + k(x + y, z) = (f(x)g(\sigma y) - g(x)f(\sigma y) \\ &\quad + k(x, y))g(\sigma z) - \omega_1 (f(x)g(\sigma y) \\ &\quad - g(x)f(\sigma y) + k(x, y))f(\sigma z) \\ &\quad - \omega_2 (f(x)g(\sigma(y + y_1)) - g(x)f(\sigma(y + y_1)) \\ &\quad + k(x, y + y_1))f(\sigma z) + \omega_2 k(x + y, y_1)f(\sigma z) \\ &\quad + k(x + y, z) \end{aligned} \quad (50)$$

for all $x, y, z \in G$. Also, from (48) we have

$$\begin{aligned} f(x + y + z) &= f(x)g(\sigma(y + z)) \\ &\quad - g(x)f(\sigma(y + z)) + k(x, y + z) \end{aligned} \quad (51)$$

for all $x, y, z \in G$. Equating (50) and (51) we have

$$\begin{aligned} &(g(\sigma y)g(\sigma z) - \omega_1 g(\sigma y)f(\sigma z) \\ &\quad - \omega_2 g(\sigma(y + y_1))f(\sigma z) - g(\sigma(y + z)))f(x) \\ &\quad + (-f(\sigma y)g(\sigma z) + \omega_1 f(\sigma y)f(\sigma z) \\ &\quad + \omega_2 f(\sigma(y + y_1))f(\sigma z) + f(\sigma(y + z)))g(x) \\ &\quad = -k(x, y)g(\sigma z) + \omega_1 k(x, y)f(\sigma z) + \omega_2 k(x, y \\ &\quad + y_1)f(\sigma z) - \omega_2 k(x + y, y_1)f(\sigma z) - k(x \\ &\quad + y, z) + k(x, y + z) \end{aligned} \quad (52)$$

for all $x, y, z \in G$. Fixing y, z in (52) we have

$$\begin{aligned} &|-k(x, y)g(\sigma z) + \omega_1 k(x, y)f(\sigma z) \\ &\quad + \omega_2 k(x, y + y_1)f(\sigma z) - \omega_2 k(x + y, y_1)f(\sigma z) \\ &\quad - k(x + y, z) + k(x, y + z)| \leq |k(x, y)||g(\sigma z)| \\ &\quad + |k(x, y)||\omega_1 f(\sigma z)| + |k(x, y + y_1)||\omega_2 f(\sigma z)| \\ &\quad + |k(x + y, y_1)||\omega_2 f(\sigma z)| + |k(x + y, z)| \\ &\quad + |k(x, y + z)| \leq \psi(\sigma y)|g(\sigma z)| + \psi(\sigma y) \\ &\quad \cdot |\omega_1 f(\sigma z)| + \psi(\sigma(y + y_1))|\omega_2 f(\sigma z)| + \psi(\sigma y_1) \\ &\quad \cdot |\omega_2 f(\sigma z)| + \psi(\sigma z) + \psi(\sigma(y + z)) \end{aligned} \quad (53)$$

for all $x, y, z \in G$. So by our assumption, the left hand side of (52) vanishes, so does its right hand side; that is,

$$\begin{aligned} & -k(x, y)g(\sigma z) \\ & + (\omega_1 k(x, y) + \omega_2 k(x, y + y_1) - \omega_2 k(x + y, y_1)) \quad (54) \\ & \cdot f(\sigma z) = k(x + y, z) - k(x, y + z) \end{aligned}$$

for all $x, y, z \in G$. From (48) we can write

$$\begin{aligned} & k(x + y, z) - k(x, y + z) \\ & = f(x + y + z) - f(x + y)g(\sigma z) \\ & \quad + g(x + y)f(\sigma z) - f(x + y + z) \\ & \quad + f(x)g(\sigma(y + z)) - g(x)f(\sigma(y + z)) \\ & = -f(\sigma(x + y + z)) + f(\sigma z)g(x + y) \\ & \quad - g(\sigma z)f(x + y) + f(\sigma(x + y + z)) \quad (55) \\ & \quad - f(\sigma(y + z))g(x) + g(\sigma(y + z))f(x) \\ & = -f(\sigma z + \sigma(x + y)) + f(\sigma z)g(x + y) \\ & \quad - g(\sigma z)f(x + y) + f(\sigma(y + z) + \sigma x) \\ & \quad - f(\sigma(y + z))g(x) + g(\sigma(y + z))f(x) \\ & = -k(\sigma z, \sigma(x + y)) + k(\sigma(y + z), \sigma x) \end{aligned}$$

for all $x, y, z \in G$. Using (55) and the triangle inequality we have

$$\begin{aligned} & |k(x + y, z) - k(x, y + z)| \\ & = |k(\sigma z, \sigma(x + y)) - k(\sigma(y + z), \sigma x)| \quad (56) \\ & \leq |k(\sigma z, \sigma(x + y))| + |k(\sigma(y + z), \sigma x)| \\ & \leq \psi(x + y) + \psi(x) \end{aligned}$$

for all $x, y, z \in G$. Thus, if we fix x, y in (54) the left hand side of (54) is a bounded function of z . Hence by our assumption, we have $k(x, y) \equiv 0$. This completes the proof. $\square$

Lemma 7 (see [16, p. 104]). Assume that $f, g : G \rightarrow \mathbb{C}$ satisfy the functional inequality; for each $y \in G$, there exists a positive constant M_y such that

$$|f(x + y) - f(x)g(y)| \leq M_y \quad (57)$$

for all $x, y \in G$; then either f is a bounded function or g is an exponential function.

In the following, we assume that

$$\Phi_1(x) := \sum_{k=0}^{\infty} 2^{-k-1} \psi(2^k x) < \infty \quad (58)$$

for all $x \in G$, or

$$\Phi_2(x) := \sum_{k=0}^{\infty} 2^k \psi(2^{-k-1} x) < \infty \quad (59)$$

for all $x \in G$.

Lemma 8 (see [17]). Assume that $f : G \rightarrow \mathbb{C}$ satisfies the functional inequality

$$|f(x + y) - f(x) - f(y)| \leq \psi(y) \quad (60)$$

for all $x, y \in G$. Then there exists a unique additive function a_1 given by

$$a_1(x) = \lim_{n \rightarrow \infty} 2^{-n} f(2^n x) \quad (61)$$

such that

$$|f(x) - a_1(x)| \leq \Phi_1(x) \quad (62)$$

for all $x \in G$ provided that (58) holds, and there exists a unique additive function a_2 given by

$$a_2(x) = \lim_{n \rightarrow \infty} 2^n f(2^{-n} x) \quad (63)$$

such that

$$|f(x) - a_2(x)| \leq \Phi_2(x) \quad (64)$$

for all $x \in G$ provided that (59) holds.

Theorem 9. Let $f, g : G \rightarrow \mathbb{C}$ satisfy the functional inequality

$$|f(x + \sigma y) - f(x)g(y) + g(x)f(y)| \leq \psi(y) \quad (65)$$

for all $x, y \in G$; then (f, g) satisfies one of the following:

- (i) $f = 0$, g is arbitrary,
- (ii) both f and g are bounded functions,
- (iii) $f(x) = \lambda(m(x) - m(\sigma x))$, $g(x) = \gamma m(x) + (1 - \gamma)m(\sigma x)$ for all $x \in G$, where $m : G \rightarrow \mathbb{C}$ is an exponential function satisfying $m \neq m \circ \sigma$ and $\lambda, \gamma \in \mathbb{C}$,
- (iv) $f(x) = a(x)m(x)$, $g(x) = (1 + \beta a(x))m(x)$ for all $x \in G$, where $m : G \rightarrow \mathbb{C}$ is an exponential function satisfying $m = m \circ \sigma$, and $a : G \rightarrow \mathbb{C}$ is an additive function satisfying $a(x) = -a(\sigma x)$ for all $x \in G$, and $\beta \in \mathbb{C}$.
- (v) there exist $\beta \in \mathbb{C}$ and a bounded exponential function m satisfying $m \neq m \circ \sigma$ such that

$$g(x) = \beta f(x) + m(x) \quad (66)$$

for all $x \in G$, and f satisfies the condition that there exist $C, d \geq 0$ satisfying

$$|f(x)| \leq C(\psi(x) + d) \quad (67)$$

for all $x \in G$,

- (vi) there exist $\beta \in \mathbb{C}$ and a bounded exponential function m satisfying $m = m \circ \sigma$ such that

$$g(x) = \beta f(x) + m(x) \quad (68)$$

for all $x \in G$, and f satisfies one of the following; there exists an additive function $a_1 : G \rightarrow \mathbb{C}$ such that

$$|f(x) - (a_1(x) + f(0))m(x)| \leq 2\Phi_1(x) \quad (69)$$

for all $x \in G$, or there exists an additive function $a_2 : G \rightarrow \mathbb{C}$ such that

$$|f(x) - (a_2(x) + f(0))m(x)| \leq 2\Phi_2(x) \quad (70)$$

for all $x \in G$, where Φ_1 and Φ_2 are the functions given in (58) and (59).

Proof. In view of Lemma 6, if f, g satisfy (47), then by Theorem 2 we get (iii) or (iv). Now, we consider the case when f, g satisfy (46). If $f = 0$, g is arbitrary which is the case (i). If f is a nontrivial bounded function, in view of (65), g is also bounded which gives the case (ii). If f is unbounded, it follows from (46) that $\mu_2 \neq 0$ and g can be written as

$$g(x) = \beta f(x) + m(x) \quad (71)$$

for all $x \in G$, where $\beta \in \mathbb{C}$ and m is a bounded function. Putting (71) in (65) we have

$$|f(x + \sigma y) - f(x)m(y) + m(x)f(y)| \leq \psi(y) \quad (72)$$

for all $x, y \in G$. Replacing y by σy in (72) we have

$$|f(x + y) - f(x)m(\sigma y) + m(x)f(\sigma y)| \leq \psi(\sigma y) \quad (73)$$

for all $x, y \in G$. Using the triangle inequality with (73) we have

$$|f(x + y) - f(x)m(\sigma y)| \leq L|f(\sigma y)| + \psi(\sigma y) \quad (74)$$

for all $x, y \in G$, where $L = \sup_{x \in G} |m(x)|$. By Lemma 7, $m \circ \sigma$ is an exponential function and so is m . If $m = 0$, putting $y = 0$ in (72) we have

$$|f(x)| \leq \psi(0) \quad (75)$$

for all $x \in G$. Thus we have $m \neq 0$, since f is unbounded. Since m is a nonzero bounded exponential function, it follows from the equalities

$$\begin{aligned} m(x) &= m(x - y)m(y), \\ m(kx) &= m(x)^k, \end{aligned} \quad (76)$$

$$x, y \in G, k = 0, \pm 1, \pm 2, \dots$$

that $m(x) \neq 0$ for all $x \in G$ and $|m(x)| = 1$ for all $x \in G$. Putting $x = 0$ in (72) and multiplying $|m(x)|$ in the result we have

$$\begin{aligned} |m(x)f(\sigma y) + m(x)f(y) - f(0)m(x)m(y)| \\ \leq \psi(y) \end{aligned} \quad (77)$$

for all $y \in G$. From (73) and (77) we have

$$\begin{aligned} |f(x + y) - f(x)m(\sigma y) - m(x)f(y) \\ + f(0)m(x)m(y)| \leq 2\psi(y) \end{aligned} \quad (78)$$

for all $x, y \in G$. First, we consider the case $m(y_0) \neq m(\sigma y_0)$ for some $y_0 \in G$. Replacing x by y and y by x in (78) we have

$$\begin{aligned} |f(y + x) - f(y)m(\sigma x) - m(y)f(x) \\ + f(0)m(y)m(x)| \leq 2\psi(x) \end{aligned} \quad (79)$$

for all $x, y \in G$. From (78) and (79), using the triangle inequality and putting $y = y_0$ we have

$$|f(x)| \leq C(\psi(x) + d) \quad (80)$$

for all $x \in G$, where $C = 2/|m(y_0) - m(\sigma y_0)|$ and $d = \psi(y_0) + |f(y_0)|$, which gives (v). Now, we consider the case when $m(x) = m(\sigma x)$ for all $x \in G$. Dividing both the sides (78) by $|m(x)m(y)|$ we have

$$|F(x + y) - F(x) - F(y)| \leq 2\psi(y) \quad (81)$$

for all $x, y \in G$, where $F(x) = f(x)/m(x) - f(0)$. Using Lemma 8 with (81) and multiplying $|m(x)|$ in the result we get (69) and (70). This completes the proof. $\square$

In particular, let $G = X \times X$ with X a normed space, $\psi(x_1, x_2) = \epsilon(\|x_1\|^p + \|x_2\|^p)$, $p \geq 0$, and $\sigma(x_1, x_2) = (x_2, x_1)$ for all $x_1, x_2 \in X$. Then by Theorems 5 and 9 we have the following.

Corollary 10. Let $f, g : X \times X \rightarrow \mathbb{C}$ satisfy the functional inequality

$$\begin{aligned} |f(x_1 + y_2, x_2 + y_1) - f(x_1, x_2)g(y_1, y_2) \\ + g(x_1, x_2)f(y_1, y_2)| \leq \epsilon(\|y_1\|^p + \|y_2\|^p) \end{aligned} \quad (82)$$

for all $x_1, x_2, y_1, y_2 \in X$. Then (f, g) satisfies one of the following:

- (i) $f = 0$, g is arbitrary,
- (ii) both f and g are bounded functions,
- (iii) there exist exponential functions $m_1, m_2 : X \rightarrow \mathbb{C}$ satisfying $m_1 \neq m_2$ and $\lambda, \gamma \in \mathbb{C}$ such that

$$\begin{aligned} f(x_1, x_2) &= \lambda(m_1(x_1)m_2(x_2) - m_1(x_2)m_2(x_1)), \\ g(x_1, x_2) &= \gamma m_1(x_1)m_2(x_2) \\ &\quad + (1 - \gamma)m_1(x_2)m_2(x_1) \end{aligned} \quad (83)$$

for all $x_1, x_2 \in X$,

- (iv) there exist an exponential function $m : X \rightarrow \mathbb{C}$, an additive function $a : X \rightarrow \mathbb{C}$, and $\beta \in \mathbb{C}$ such that

$$\begin{aligned} f(x_1, x_2) &= a(x_1 - x_2)m(x_1 + x_2), \\ g(x_1, x_2) &= (1 + \beta a(x_1 - x_2))m(x_1 + x_2) \end{aligned} \quad (84)$$

for all $x_1, x_2 \in X$,

- (v) there exist $\beta \in \mathbb{C}$ and bounded exponential functions $m_1, m_2 : X \rightarrow \mathbb{C}$ satisfying $m_1 \neq m_2$ such that

$$g(x_1, x_2) = \beta f(x_1, x_2) + m_1(x_1)m_2(x_2) \quad (85)$$

for all $x_1, x_2 \in X$, and f satisfies the condition that there exist $C, d \geq 0$ satisfying

$$|f(x_1, x_2)| \leq C(\psi(x_1, x_2) + d) \quad (86)$$

for all $x_1, x_2 \in X$,

(vi) there exist $\beta \in \mathbb{C}$ and a bounded exponential function $m : X \rightarrow \mathbb{C}$ such that

$$g(x_1, x_2) = \beta f(x_1, x_2) + m(x_1 + x_2) \quad (87)$$

for all $x_1, x_2 \in X$, and f satisfies one of the following; there exist additive functions $a_1, a_2 : X \rightarrow \mathbb{C}$ such that

$$\begin{aligned} &|f(x_1, x_2) \\ &\quad - (a_1(x_1) + a_2(x_2) + f(0, 0))m(x_1 + x_2)| \\ &\leq \frac{2\epsilon(\|x_1\|^p + \|x_2\|^p)}{2 - 2^p} \end{aligned} \quad (88)$$

for all $x_1, x_2 \in X$ provided that $0 \leq p < 1$, or there exist additive functions $b_1, b_2 : X \rightarrow \mathbb{C}$ such that

$$\begin{aligned} &|f(x_1, x_2) \\ &\quad - (b_1(x_1) + b_2(x_2) + f(0, 0))m(x_1 + x_2)| \\ &\leq \frac{2\epsilon(\|x_1\|^p + \|x_2\|^p)}{2^p - 2} \end{aligned} \quad (89)$$

for all $x_1, x_2 \in X$ provided that $p > 1$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (no. 2015R1D1A3A01019573). The authors are very thankful to the referee for valuable suggestions that improved the presentation of the paper.

References

- [1] J. Aczél, *Lectures on Functional Equations in Several Variables*, Academic Press, New York, NY, USA, 1966.
- [2] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, New York, NY, USA, 1989.
- [3] A. M. Perkins and P. K. Sahoo, "On two functional equations with involution on groups related to sine and cosine functions," *Aequationes Mathematicae*, vol. 89, no. 5, pp. 1251–1263, 2015.
- [4] H. Stetkær, "D'Alembert and Wilson's functional equations for vector and 2×2 matrix valued functions," *Mathematica Scandinavica*, vol. 87, pp. 115–132, 2000.
- [5] H. Stetkær, "Functional equations on abelian groups with involution," *Aequationes Mathematicae*, vol. 54, no. 1-2, pp. 144–172, 1997.
- [6] J. d'Alembert, "Addition au Mémoire sur la courbe que forme une corde tendue mise en vibration," *Histoire de l'Académie Royale, Berlin*, vol. 6, pp. 355–360, 1750.
- [7] W. H. Wilson, "Two general functional equations," *Bulletin of the American Mathematical Society*, vol. 31, no. 7, pp. 330–334, 1925.
- [8] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, NY, USA, 1960.
- [9] L. Székelyhidi, "The stability of the sine and cosine functional equations," *Proceedings of the American Mathematical Society*, vol. 110, no. 1, pp. 109–115, 1990.
- [10] J. Chung and J. Chang, "On a generalized Hyers-Ulam stability of trigonometric functional equations," *Journal of Applied Mathematics*, vol. 2012, Article ID 610714, 14 pages, 2012.
- [11] J. Chang and J. Chung, "Hyers-Ulam stability of trigonometric functional equations," *Korean Mathematical Society. Communications*, vol. 23, no. 4, pp. 567–575, 2008.
- [12] S. Czerwik, *Stability of Functional Equations of Ulam-Hyers-Rassias Type*, Hadronic Press, Palm Harbor, Fla, USA, 2003.
- [13] W. Fechner, J. Brzdek, M. S. Moslehian, and J. Sikorska, "Recent developments of the conditional stability of the homomorphism equation," *Banach Journal of Mathematical Analysis*, vol. 9, no. 3, pp. 278–326, 2015.
- [14] G. H. Kim, "On the stability of mixed trigonometric functional equations," *Banach Journal of Mathematical Analysis*, vol. 1, no. 2, pp. 227–236, 2007.
- [15] J. Chung and P. K. Sahoo, "Solution of several functional equations on nonunital semigroups using Wilson's functional equations with involution," *Abstract and Applied Analysis*, vol. 2014, Article ID 463918, 9 pages, 2014.
- [16] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.
- [17] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, no. 1-2, pp. 64–66, 1950.