Research Article

Multiplication Operator with BMO Symbols and Berezin Transform

Xue Feng, Kan Zhang, Jianguo Dong, Xianmin Liu, and Chi Guan

College of Sciences, Shenyang Agricultural University, Shenyang 110866, China

Correspondence should be addressed to Kan Zhang; kanzhang2004@163.com

Received 26 May 2014; Accepted 16 November 2014

Academic Editor: Janusz Matkowski

Copyright © 2015 Xue Feng et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We discuss multiplication operator with a special symbol on the weighted Bergman space of the unit ball. We give the necessary and sufficient conditions for the compactness of multiplication operator on the weighted Bergman space of the unit ball.

1. Introduction

Let B_n denote the unit ball in $\mathbb{C}^n$, and let ν be the normalized Lebesgue volume measure on B_n . For $-1 < \alpha < \infty$, we denote by ν_α the measure on B_n defined by $d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha d\nu(z)$, where $c_\alpha = \Gamma(n+\alpha+1)/n!\Gamma(\alpha+1)$ is a normalizing constant such that $\nu_\alpha(B_n) = 1$. For $1 \leq p < \infty$, we write $\|\cdot\|_{\alpha,p}$ for the norm on $L^p(B_n, d\nu_\alpha)$ and $\langle \cdot, \cdot \rangle_\alpha$ for the inner product on $L^2(B_n, d\nu_\alpha)$. The Bergman space $A_\alpha^2(B_n)$ is the space of holomorphic functions which are square-integrable with respect to measure $d\nu_\alpha$ on B_n . Reproducing kernels K_w^α and normalized reproducing kernels k_w^α in $A_\alpha^2(B_n)$ are given by

$$\begin{aligned} K_w^\alpha(z) &= \frac{1}{(1 - \langle z, w \rangle)^{n+\alpha+1}}, \\ k_w^\alpha(z) &= \frac{(1 - |w|^2)^{(n+\alpha+1)/2}}{(1 - \langle z, w \rangle)^{n+\alpha+1}}, \end{aligned} \quad (1)$$

respectively, for $z, w \in B_n$. For every $h \in A_\alpha^2(B_n)$ we have $\langle h, K_w^\alpha \rangle_\alpha = h(w)$, for all $w \in B_n$. The orthogonal projection P_α of $L^2(B_n, d\nu_\alpha)$ onto $A_\alpha^2(B_n)$ is given by

$$(P_\alpha g)(w) = \langle g, K_w^\alpha \rangle_\alpha = \int_{B_n} g(z) \frac{1}{(1 - \langle w, z \rangle)^{n+\alpha+1}} d\nu_\alpha(z), \quad (2)$$

for $g \in L^2(B_n, d\nu_\alpha)$ and $w \in B_n$.

Given $f \in L^1(B_n, d\nu_\alpha)$, the Toeplitz operator $T_f : L^2(B_n, d\nu_\alpha) \rightarrow A_\alpha^2(B_n)$, the Hankel operator $H_f : L^2(B_n, d\nu_\alpha) \rightarrow (A_\alpha^2(B_n))^\perp$, and the multiplication operator $M_f : A_\alpha^2(B_n) \rightarrow L^2(B_n, d\nu_\alpha)$ are given by

$$\begin{aligned} (T_f h)(z) &= \int_{B_n} \frac{f(w) h(w)}{(1 - \langle z, w \rangle)^{n+\alpha+1}} d\nu_\alpha(w), \\ (H_f h)(z) &= f(z) g(z) - (T_f g)(z), \\ M_f(h) &= fh, \end{aligned} \quad (3)$$

respectively. For $f \in L^1(B_n, d\nu_\alpha)$, we define the Berezin transform of f to be the function $\tilde{f}$; that is,

$$\tilde{f}(z) = \int_{B_n} f(w) |k_z^\alpha(w)|^2 d\nu_\alpha(w). \quad (4)$$

If f is bounded, then $\tilde{f}$ is a bounded function on B_n . Since the kernels k_z^α converge weakly to zero as z tends ∂B_n , we have that if f is compact, then $\tilde{f} \rightarrow 0$ as $z \rightarrow \partial B_n$. The converse (in both cases) is not necessarily true. According to the definition of Berezin transform, the mean oscillation of f in the Bergman metric is the function $\text{MO}(f)(z)$ defined on B_n by

$$\text{MO}(f)(z) = |\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2|. \quad (5)$$

For $z \in B_n$, let ψ_z be the automorphism of B_n such that $\psi_z(0) = z$ and $\psi_z = (\psi_z)^{-1}$. Thus, we have the change-of-variable formula

$$\int_{B_n} h(\psi_z(w)) |k_z^\alpha(w)|^2 dv_\alpha(w) = \int_{B_n} h(w) dv_\alpha(w), \quad (6)$$

for every $h \in L^1(B_n, dv_\alpha)$.

Multiplication operators are one of the most widely studied classes of concrete operators. The study of their behavior on the Hardy and Bergman spaces has generated an extensive list of results in the operator theory and in the theory of function spaces [1–6]. One of the useful approaches is the use of the Berezin transform [7–11]. This method is motivated by its connections with quantum physics and noncommutative geometry.

In general, Berezin transform $\tilde{f}$ plays important role in giving necessary and sufficient conditions for the boundedness and compactness of the Toeplitz operator [12, 13]. However Berezin transform $|\tilde{f}|^2$ or the mean oscillation $\text{MO}(f)$ is used to obtain the necessary and sufficient conditions for the boundedness and compactness of the Hankel operator or multiplication operator [14, 15]. This work is partially motivated by using Berezin transform $\tilde{f}$ to obtain necessary and sufficient conditions for the compactness of multiplication operator on the weighted Bergman space of the unit ball.

Throughout the paper, we will use the letter c to denote a generic positive constant that can change its value at each occurrence.

2. Main Results

In this section, we give the necessary and sufficient conditions for the compactness of multiplication operator on the weighted Bergman space of the unit ball. We furthermore obtain the necessary and sufficient conditions for the compactness of Toeplitz operator and Hankel operator.

Theorem 1. Suppose $|f|/(1 - |z|)^{4n+4\alpha+4}$ is bounded on B_n . Then M_f is compact operator on $A_\alpha^2(B_n)$ if and only if $|\tilde{f}|(z) \rightarrow 0$ as $z \rightarrow \partial B_n$.

Proof. Suppose $|\tilde{f}|(z) \rightarrow 0$ as $z \rightarrow \partial B_n$.

Since

$$\langle M_f g, h \rangle_\alpha = \langle fg, h \rangle_\alpha = \langle g, \tilde{f}h \rangle_\alpha, \quad (7)$$

it is clear that $(M_f)^* = M_{\tilde{f}}$. It suffices to prove that the operator $(M_f)^*$ is compact by showing that $(M_f)^*$ can be approximated by compact operators in the operator norm.

Let $g \in L^2(B_n, dv_\alpha)$. Then $(M_f)^* g \in A_\alpha^2(B_n)$, so we have

$$\begin{aligned} ((M_f)^* g)(z) &= \langle (M_f)^* g, K_z^\alpha \rangle_\alpha \\ &= \int_{B_n} g(w) \overline{f(w) K_z^\alpha(w)} dv_\alpha(w), \end{aligned} \quad (8)$$

for $z \in B_n$.

We define for $0 < r < 1$ an operator S_r by

$$(S_r g)(z) = \int_{B_n} \chi_{rB_n}(z) g(w) \overline{f(w) K_z^\alpha(w)} dv_\alpha(w). \quad (9)$$

Since $|f|/(1 - |z|)^{4n+4\alpha+4}$ is bounded on B_n , we prove that

$$\begin{aligned} &\int_{B_n} \int_{B_n} |\chi_{rB_n}(z) \overline{f(w) K_z^\alpha(w)}|^2 dv_\alpha(w) dv_\alpha(z) \\ &= \int_{rB_n} \int_{B_n} |\overline{f(w) K_z^\alpha(w)}|^2 dv_\alpha(w) dv_\alpha(z) \\ &< +\infty. \end{aligned} \quad (10)$$

Thus, the operator S_r is a Hilbert-Schmidt operator. Since

$$\begin{aligned} &((M_f)^* - S_r)g(z) \\ &= \int_{B_n} g(w) \chi_{B_n \setminus rB_n}(z) \overline{f(w) K_z^\alpha(w)} dv_\alpha(w), \end{aligned} \quad (11)$$

$(M_f)^* - S_r$ is an integral operator with kernel $K_r^f(z, w) = \chi_{B_n \setminus rB_n}(z) \overline{f(w) K_z^\alpha(w)}$.

By Schur's test, whenever there exists a positive measurable function h on B_n and constants c_1 and c_2 such that

$$\int_{B_n} |K_r^f(z, w)| h(z) dv_\alpha(z) \leq c_1 h(w), \quad (12)$$

for all w in B_n , and

$$\int_{B_n} |K_r^f(z, w)| h(w) dv_\alpha(w) \leq c_2 h(z), \quad (13)$$

for all z in B_n , we have

$$\|(M_f)^* - S_r\| \leq c_1 c_2. \quad (14)$$

Let $h(z) = (1 - |z|^2)^{(n+\alpha+1)/2}$. Since

$$\begin{aligned} K_z^\alpha(\psi_z(v)) k_z^\alpha(v) &= \frac{1}{(1 - |z|^2)^{(n+1+\alpha)/2}}, \\ 1 - |\psi_z(v)|^2 &= \frac{(1 - |z|^2)(1 - |v|^2)}{|1 - \langle v, z \rangle|^2}, \end{aligned} \quad (15)$$

and Hölder inequality, it is easy to prove that

$$\begin{aligned}
& \int_{B_n} \chi_{B_n \setminus rB_n}(z) |f(w) K_z^\alpha(w)| \frac{(1-|w|^2)^{(n+\alpha+1)/2}}{(1-|z|^2)^{(n+\alpha+1)/2}} dv_\alpha(w) \\
&= \int_{B_n} \chi_{B_n \setminus rB_n}(z) |f(w) k_z^\alpha(w)| \frac{(1-|w|^2)^{(n+\alpha+1)/2}}{(1-|z|^2)^{n+\alpha+1}} dv_\alpha(w) \\
&= \int_{B_n} \chi_{B_n \setminus rB_n}(z) |f(\psi_z(w)) k_z^\alpha(\psi_z(w))| \\
&\quad \times \frac{(1-|\psi_z(w)|^2)^{(n+\alpha+1)/2}}{(1-|z|^2)^{n+\alpha+1}} |k_z^\alpha(w)|^2 dv_\alpha(w) \\
&= \int_{B_n} \chi_{B_n \setminus rB_n}(z) |f(\psi_z(w))| \frac{(1-|w|^2)^{(n+\alpha+1)/2}}{|1-\langle z, w \rangle|^{2n+2\alpha+2}} dv_\alpha(w) \\
&= \int_{B_n} \chi_{B_n \setminus rB_n}(z) \sqrt{|f(\psi_z(w))|} \sqrt{|f(\psi_z(w))|} \\
&\quad \times \frac{(1-|w|^2)^{(n+\alpha+1)/2}}{|1-\langle z, w \rangle|^{2n+2\alpha+2}} dv_\alpha(w) \\
&\leq \chi_{B_n \setminus rB_n}(z) \left[\int_{B_n} |f(\psi_z(w))| dv_\alpha(w) \right]^{1/2} \\
&\quad \times \left[\int_{B_n} |f(\psi_z(w))| \frac{(1-|w|^2)^{n+\alpha+1}}{|1-\langle z, w \rangle|^{4n+4\alpha+4}} dv_\alpha(w) \right]^{1/2} \\
&\leq c \chi_{B_n \setminus rB_n}(z) \left[\int_{B_n} |f(\psi_z(w))| dv_\alpha(w) \right]^{1/2} \\
&\quad \times \left[\int_{B_n} |f(\psi_z(w))| \frac{1}{(1-|z|)^{3n+3\alpha+3}} dv_\alpha(w) \right]^{1/2} \\
&\leq c \left[\widetilde{|f|}(z) \right]^{1/2} \\
&= c_1,
\end{aligned} \tag{16}$$

where $c_1 = c[\widetilde{|f|}(z)]^{1/2}$, $r < |z| < 1$.

Since

$$K_z^\alpha(\psi_z(v)) k_z^\alpha(v) = \frac{1}{(1-|z|^2)^{(n+1+\alpha)/2}}, \tag{17}$$

$$1 - \langle \psi_z(w), z \rangle = \frac{1-|z|^2}{1-\langle w, z \rangle},$$

then we obtain

$$\begin{aligned}
& \int_{B_n} \chi_{B_n \setminus rB_n}(z) |f(w) K_z^\alpha(w)| \frac{(1-|z|^2)^{(n+\alpha+1)/2}}{(1-|w|^2)^{(n+\alpha+1)/2}} dv_\alpha(z) \\
&= \int_{B_n} \chi_{B_n \setminus rB_n}(w) |f(z) K_z^\alpha(w)| \frac{(1-|w|^2)^{(n+\alpha+1)/2}}{(1-|z|^2)^{(n+\alpha+1)/2}} dv_\alpha(w) \\
&= \int_{B_n} \chi_{B_n \setminus rB_n}(\psi_z(w)) |f(z) K_z^\alpha(\psi_z(w))| \\
&\quad \times \frac{(1-|\psi_z(w)|^2)^{(n+\alpha+1)/2}}{(1-|z|^2)^{(n+\alpha+1)/2}} |k_z^\alpha(w)|^2 dv_\alpha(w) \\
&= \int_{B_n} \chi_{B_n \setminus rB_n}(\psi_z(w)) |f(z)| \\
&\quad \times \frac{(1-|\psi_z(w)|^2)^{(n+\alpha+1)/2}}{|1-\langle z, w \rangle|^{4n+4\alpha+4}} dv_\alpha(w) \\
&\leq c_2,
\end{aligned} \tag{18}$$

where c_2 is positive number.

By the above analysis, we get (12) and (13). By Schur's test we get $\|(M_f)^* - S_r\| \leq c_1 c_2$, where $c_1 \rightarrow 0$ as $|z| \rightarrow \partial B_n$ and c_2 does not depend on r . So $\widetilde{|f|}(z) \rightarrow 0$ as $z \rightarrow \partial B_n$ implies that M_f is compact on $A_\alpha^2(B_n)$.

Suppose M_f is compact on $A_\alpha^2(B_n)$.

Since the kernels k_z^α converge weakly to zero as z tends ∂B_n , then we have $\|M_f k_z^\alpha\|_{\alpha,2}$ converges to zero as z tends ∂B_n . So we obtain

$$\left[\widetilde{|f|}(z) \right]^{1/2} = \|M_f k_z^\alpha\|_{\alpha,2} \rightarrow 0, \tag{19}$$

as $z \rightarrow \partial B_n$. $\square$

Let $f \in L^1(B_n, dv_\alpha)$ and let $p \geq 1$; we say that $f \in \text{BMO}_\alpha^p(B_n)$ whenever

$$\|f\|_{\text{BMO}_\alpha^p} = \sup_{z \in B_n} \|f \circ \psi_z - \widetilde{f}(z)\|_{\alpha,p} < \infty. \tag{20}$$

Note that $\|\cdot\|_{\text{BMO}_\alpha^p}$ does not distinguish constants, while $\|f\|_{\alpha,p} = \|f\|_{\text{BMO}_\alpha^p} + |\widetilde{f}(0)|$ is a norm in $\text{BMO}_\alpha^p(B_n)$. By Theorem 5 in [16], we know that $\text{BMO}_\alpha^p(B_n)$ is equivalent to BMO_α^p (see the definition in [16]).

For any $p \geq 1$, let VMO_α^p denote the subspace of BMO_α^p consisting of functions f such that

$$\lim_{|z| \rightarrow 1^-} \|f \circ \psi_z - \widetilde{f}(z)\|_{\alpha,p} = 0. \tag{21}$$

Theorem 2. Suppose $f \in \text{VMO}_\alpha^1$ and $|f|/(1-|z|)^{4n+4\alpha+4}$ is bounded on B_n . Then the following are equivalent:

- (a) $\tilde{f}(z) \rightarrow 0$ as $z \rightarrow \partial B_n$;
- (b) M_f is compact operator on $A_\alpha^2(B_n)$;
- (c) T_f is compact operator on $A_\alpha^2(B_n)$.

Proof. It suffices to prove that (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c).

(a) $\Leftrightarrow$ (b). Since

$$\begin{aligned}
 |\tilde{f}(z)| &\leq |\widetilde{|f|}(z)|, \\
 |\widetilde{|f|}(z) - |\tilde{f}(z)| &= \int_{B_n} (|f(w)| - |\tilde{f}(z)|) \frac{(1 - |z|^2)^{n+\alpha+1}}{|1 - \langle z, w \rangle|^{2n+2\alpha+2}} d\nu_\alpha(w) \\
 &\leq \int_{B_n} |f(w) - \tilde{f}(z)| \frac{(1 - |z|^2)^{n+\alpha+1}}{|1 - \langle z, w \rangle|^{2n+2\alpha+2}} d\nu_\alpha(w) \\
 &= \int_{B_n} |f \circ \psi_z(w) - \tilde{f}(z)| d\nu_\alpha(w) \\
 &= \|f \circ \psi_z - \tilde{f}(z)\|_{\alpha,1},
 \end{aligned} \tag{22}$$

then we obtain that $\tilde{f}(z) \rightarrow 0$ as $z \rightarrow \partial B_n$ if and only if $|\widetilde{|f|}(z) \rightarrow 0$ as $z \rightarrow \partial B_n$. By Theorem 1, we obtain that M_f is compact operator on $A_\alpha^2(B_n)$ if and only if $\tilde{f}(z) \rightarrow 0$ as $z \rightarrow \partial B_n$.

(a) $\Leftrightarrow$ (c). It is clear that $\tilde{f}(z) \rightarrow 0$ as $z \rightarrow \partial B_n$ if and only if T_f with BMO_α^1 symbol is compact operator on $A_\alpha^2(B_n)$ in [12]. Since $\text{VMO}_\alpha^1 \subset \text{BMO}_\alpha^1$, then it is clear that $\tilde{f}(z) \rightarrow 0$ as $z \rightarrow \partial B_n$ if and only if T_f is compact operator on $A_\alpha^2(B_n)$. $\square$

Corollary 3. Suppose $f \in \text{VMO}_\alpha^1$, $|f|/(1 - |z|)^{4n+4\alpha+4}$ is bounded on B_n , and H_f is compact operator on $A_\alpha^2(B_n)$. Then $M_{f-\tilde{f}}$ is compact operator on $A_\alpha^2(B_n)$.

Proof. Suppose H_f is compact operator on $A_\alpha^2(B_n)$. So we obtain $H_{f-\tilde{f}}$ is compact operator on $A_\alpha^2(B_n)$. Since $\widetilde{f - \tilde{f}}(z) \rightarrow 0$ as $z \rightarrow \partial B_n$, then $T_{f-\tilde{f}}$ is compact operator on $A_\alpha^2(B_n)$. Since

$$M_{f-\tilde{f}}g = T_{f-\tilde{f}}g + H_{f-\tilde{f}}g, \tag{23}$$

we obtain $M_{f-\tilde{f}}$ is compact operator on $A_\alpha^2(B_n)$. $\square$

By Lemma 17 and Theorem 19 in [16] and Theorem 2, we obtain the following theorem.

Theorem 4. Suppose $f \in \text{VMO}_\alpha^1$, $\tilde{f}(0) = 0$, and $|f|/(1 - |z|)^{4n+4\alpha+4}$ is bounded on B_n . Then the following are equivalent:

- (a) M_f is compact operator on $A_\alpha^2(B_n)$;
- (b) T_f is compact operator on $A_\alpha^2(B_n)$;
- (c) H_f is compact operator on $A_\alpha^2(B_n)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

Kan Zhang is partly supported by NSFC (no. 113711821).

References

- [1] X. Wang, G. Cao, and J. Xia, "Toeplitz operators on Fock-Sobolev spaces with positive measure symbols," *Science China Mathematics*, vol. 57, no. 7, pp. 1443–1462, 2014.
- [2] K. Hedayatian and L. Karimi, "On convexity of composition and multiplication operators on weighted Hardy spaces," *Abstract and Applied Analysis*, vol. 2009, Article ID 931020, 9 pages, 2009.
- [3] P. S. Bourdon, "Fredholm multiplication and composition operators on the Hardy space," *Integral Equations and Operator Theory*, vol. 13, no. 4, pp. 607–610, 1990.
- [4] K. Stroethoff and D. C. Zheng, "Toeplitz and Hankel operators on Bergman spaces," *Transactions of the American Mathematical Society*, vol. 329, no. 2, pp. 773–794, 1992.
- [5] K. H. Zhu, "Multipliers of BMO in the Bergman metric with applications to Toeplitz operators," *Journal of Functional Analysis*, vol. 87, no. 1, pp. 31–50, 1989.
- [6] K. H. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer, New York, NY, USA, 2005.
- [7] B. Yousefi, "Unicellularity of the multiplication operator on Banach spaces of formal power series," *Studia Mathematica*, vol. 147, no. 3, pp. 201–209, 2001.
- [8] B. S. Komal and S. Gupta, "Multiplication operators between Orlicz spaces," *Integral Equations and Operator Theory*, vol. 41, no. 3, pp. 324–330, 2001.
- [9] R. K. Singh and J. S. Manhas, "Multiplication operators on weighted spaces of vector-valued continuous functions," *Journal of the Australian Mathematical Society (Series A)*, vol. 50, no. 1, pp. 98–107, 1991.
- [10] R. Yoneda, "Multiplication operators, integration operators and companion operators on weighted Bloch space," *Hokkaido Mathematical Journal*, vol. 34, no. 1, pp. 135–147, 2005.
- [11] R. F. Allen and F. Colonna, "Isometries and spectra of multiplication operators on the Bloch space," *Bulletin of the Australian Mathematical Society*, vol. 79, no. 1, pp. 147–160, 2009.
- [12] K. Zhang, C. M. Liu, and Y. F. Lu, "Toeplitz operators with BMO symbols on the weighted Bergman space of the unit ball," *Acta Mathematica Sinica (English Series)*, vol. 27, no. 11, pp. 2129–2142, 2011.
- [13] A. Dieudonne and E. Tchoundja, "Toeplitz operators with L^1 symbols on Bergman spaces in the unit ball of C^n ," *Advances in Pure and Applied Mathematics*, vol. 2, no. 1, pp. 65–88, 2011.
- [14] M. Jovovic and D. Zheng, "Compact operators and Toeplitz algebras on multiply-connected domains," *Journal of Functional Analysis*, vol. 261, no. 1, pp. 25–50, 2011.
- [15] J. L. Wang and Z. Wu, "Multipliers between BMO spaces on open unit ball," *Integral Equations and Operator Theory*, vol. 45, no. 2, pp. 231–249, 2003.
- [16] K. H. Zhu, "BMO and Hankel operators on Bergman spaces," *Pacific Journal of Mathematics*, vol. 155, no. 2, pp. 377–395, 1992.