Research Article

An Approximation Method for Solving Volterra Integrodifferential Equations with a Weighted Integral Condition

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We apply the reproducing kernel Hilbert space (RKHS) method for getting analytical and approximate solutions for second-order hyperbolic integrodifferential equations with a weighted integral condition. The analytical solution is represented in the form of series; thus, the n -terms approximate solutions are obtained. The results of the numerical examples are compared with the exact solutions to illustrate the accuracy and the effectivity of this method.

1. Introduction

In functional analysis, a reproducing kernel Hilbert space is a Hilbert space of functions in which pointwise evaluation is a continuous linear functional. The subject was originally developed by Aronszajn in 1950 (see [1]).

Reproducing kernel theory has important applications in numerical analysis, differential equations, integral equations, probability, and statistics [2–4]. Recently, using the RKHS method, the authors in [5–21] have discussed singular linear two-point boundary value problems, singular nonlinear two-point periodic boundary value problems, nonlinear system of boundary value problems, initial value problems, singular integral equations, nonlinear partial differential equations, operator equations, and fourth-order integrodifferential equations (IDEs), differential equations with integral condition.

The purpose of this paper is to extend the application of the reproducing kernel Hilbert space method to solve a second-order boundary value problem of IDEs of the following Volterra type:

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = f(x, t) + \int_0^t a(t-s)K(s, U(x, s))ds, \quad (1)$$

for all $(x, t) \in Q = (0, 1) \times (0, 1)$, subject to the initial conditions

$$U(x, 0) = \psi_1(x), \quad (2)$$

$$U_t(x, 0) = \psi_2(x),$$

$$U_x(0, t) = \psi_3(t) \quad (3)$$

and the weighted integral condition

$$\int_0^1 xU(x, t)dx = 0, \quad (4)$$

where f , ψ_1 , ψ_2 , ψ_3 , a , and K are given functions, such that

$$\psi_1'(0) = \psi_2'(0) = 0, \quad (5)$$

$$\psi_3(0) = \psi_3'(0) = 0.$$

The operator $K(t, u)$ is linear with respect to u and continuous according to both variables t and u . The function a is continuous and denote $l = \max_{0 \leq t \leq 1} |a(t)|$.

In order to put initial conditions and integral boundary conditions of (1) into the reproducing kernel space $H(Q)$ constructed in the following sections, we have to homogenize

these conditions. For this, let $u(x, t) = U(x, t) - \psi_1(x) - t\psi_2(x) - (x - 2/3)\psi_3(t)$; then the problem can be converted into the following form:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = F(x, t) + \int_0^t a(t-s) K(s, u(x, s)) ds, \quad (6)$$

for all $(x, t) \in Q$, subject to the initial conditions

$$\begin{aligned} u(x, 0) &= 0, \\ u_t(x, 0) &= 0, \\ u_x(0, t) &= 0, \end{aligned} \quad (7)$$

and the weighted integral condition

$$\int_0^1 xu(x, t) dx = 0. \quad (8)$$

The theoretical aspects of (1) with conditions (2)–(4) have been studied by Galerkin method in [22]; however, no numerical method was presented. In fact, by this procedure, the authors have found several difficulties to find the discretization of this problem due to the presence of the integral condition. Therefore, to overcome these difficulties, in the present work, we propose the RKHS method which is a simple and effective method for obtaining the numerical solution that converges rapidly to the exact solution with a small error and with a good precision. Moreover, the partial derivatives of $u_n(x, t)$ are also convergent to the partial of $u(x, t)$. So, from this work, we show the applicability of the RKHS method to solve this kind of problem without discretization such as Galerkin and Rothe's method. Furthermore, the RKHS method has an advantage that it is the possibility to pick any point in the domain of integration as well as in the approximate solutions and all its partial derivatives up to order two that will be applicable.

In recent years, a growing interest has been devoted to the study of IDEs which are a combination of differential and Volterra-Fredholm integral equations. IDEs appear in various fields of science such as physics, biology, and engineering. We should mention also that integrodifferential equations are usually difficult to solve analytically.

Recently, various kinds of numerical methods have been used for efficient approximation solution. In [23], authors have presented a practical matrix method for solving nonlinear Volterra-Fredholm integrodifferential equations. In [24], authors have given a numerical solution of nonlinear Volterra-Fredholm integrodifferential equations using the spectral homotopy analysis method. In [25–28], authors have obtained the representation of the exact solution for the nonlinear Volterra-Fredholm integral equations by using the reproducing kernel method.

To sum up, in this paper, we define several reproducing kernel spaces in Section 2, whereas, in Section 3, we introduce a linear operator, a complete normal orthogonal system, and some essential results. In Section 4, we provide the main results and the exact and approximate solution of (1)–(4) and we also develop an iterative method for this kind of problems.

Finally, in Section 5, the results of some numerical examples are presented and are compared with the analytical solution to confirm the good accuracy of the presented method.

2. Reproducing Kernel Spaces

In this section, we define some useful reproducing kernel spaces.

Definition 1. Let

$$H_3[0, 1] = \left\{ u(x) \mid u, u', u'' \text{ are absolutely continuous real value functions in } [0, 1], \right. \quad (9)$$

$$\left. u^{(3)}(x) \in L^2[0, 1], u'(0) = \int_0^1 xu(x) dx = 0 \right\}.$$

Define, respectively, the inner product and the norm in $H_3[0, 1]$ by

$$\begin{aligned} \langle u(x), v(x) \rangle_{H_3} &= \sum_{i=0}^2 u^{(i)}(0) v^{(i)}(0) \\ &\quad + \int_0^1 u^{(3)}(x) v^{(3)}(x) dx, \end{aligned} \quad (10)$$

$$\|u\|_{H_3} = \sqrt{\langle u(x), u(x) \rangle_{H_3}},$$

$$u, v \in H_3[0, 1].$$

We have the following result.

Theorem 2. *The space $H_3[0, 1]$ is a complete reproducing kernel space; that is, for each fixed $x \in [0, 1]$, there exists $\widehat{G}_3(y, x) \in H_3[0, 1]$ (simply $\widehat{G}_x(y)$) and $y \in [0, 1]$ such that $\langle u(y), \widehat{G}_x(y) \rangle_{H_3} = u(x)$ for any $u(y) \in H_3[0, 1]$. The reproducing kernel $\widehat{G}_x(y)$ is given by*

$$\widehat{G}_x(y) = \begin{cases} \sum_{i=1}^6 c_i(x) y^{i-1} - \frac{y^7 C(x)}{5040}, & y \leq x, \\ \sum_{i=1}^6 d_i(x) y^{i-1} - \frac{y^7 C(x)}{5040}, & x < y, \end{cases} \quad (11)$$

where

$$\begin{aligned} c_1(x) &= \frac{655}{9727} - \frac{1134}{9727x^2} - \frac{378}{9727x^3} + \frac{252}{9727x^4} \\ &\quad + \frac{131}{233448x^5} + \frac{18}{48635x^7}, \\ c_2(x) &= -\frac{x^4}{24}, \\ c_3(x) &= -\frac{1134}{9727} + \frac{2290}{9727x^2} + \frac{2290}{29181x^3} + \frac{63}{19454x^4} \\ &\quad - \frac{189}{194540x^5} + \frac{9}{194540x^7}, \end{aligned}$$

$$\begin{aligned}
c_4(x) &= -\frac{378}{9727} - \frac{189}{38908x^2} - \frac{63}{38908x^3} + \frac{21}{19454x^4} \\
&\quad - \frac{63}{194540x^5} + \frac{3}{194540x^7}, \\
c_5(x) &= 2520 + 315x^2 + 105x^3 - 70x^4 + 21x^5 \\
&\quad - \frac{x^7}{97270}, \\
c_6(x) &= -\frac{378}{9727} - \frac{189}{194540x^2} - \frac{63}{194540x^3} + \frac{21}{97270x^4} \\
&\quad - \frac{63}{972700x^5} + \frac{3}{972700x^7}, \\
d_1(x) &= \frac{655}{9727} - \frac{1134}{9727x^2} - \frac{378}{9727x^3} + \frac{252}{9727x^4} \\
&\quad + \frac{131}{233448x^5} + \frac{18}{48635x^7}, \\
d_2(x) &= 0, \\
d_3(x) &= -\frac{1134}{9727} + \frac{2290}{9727x^2} + \frac{2290}{29181x^3} + \frac{63}{19454x^4} \\
&\quad - \frac{189}{194540x^5} + \frac{9}{194540x^7}, \\
d_4(x) &= -\frac{378}{9727} - \frac{189}{38908x^2} - \frac{63}{38908x^3} + \frac{21}{19454x^4} \\
&\quad - \frac{63}{194540x^5} + \frac{3}{194540x^7}, \\
d_5(x) &= \frac{252}{9727} - \frac{x}{24} + \frac{63}{19454x^2} + \frac{21}{19454x^3} \\
&\quad - \frac{7}{9727x^4} + \frac{21}{97270x^5} - \frac{1}{97270x^7}, \\
d_6(x) &= \frac{131}{233448} - \frac{189}{194540x^2} - \frac{63}{194540x^3} \\
&\quad + \frac{21}{97270x^4} - \frac{63}{972700x^5} + \frac{3}{972700x^7}, \\
C(x) &= -\frac{18144}{9727} - \frac{2268}{9727x^2} - \frac{756}{9727x^3} + \frac{504}{9727x^4} \\
&\quad - \frac{756}{48635x^5} + \frac{36}{48635x^7}.
\end{aligned} \tag{12}$$

Proof. Through several integrations by parts of (10), we obtain

$$\begin{aligned}
&\langle u(y), \widehat{G}_x(y) \rangle_{H_3} \\
&= \sum_{i=0}^2 u^{(i)}(0) [\partial_y^i \widehat{G}_x(0) - (-1)^{2-i} \partial_y^{5-i} \widehat{G}_x(0)] \\
&\quad + \sum_{i=0}^2 (-1)^{2-i} u^{(i)}(1) \partial_y^{5-i} \widehat{G}_x(1) \\
&\quad - \int_0^1 u(y) \partial_y^6 \widehat{G}_x(y) dy.
\end{aligned} \tag{13}$$

Since $u(x) \in H_3[0, 1]$, we have

$$u'(0) = \int_0^1 xu(x) dx = 0. \tag{14}$$

Then

$$\begin{aligned}
\langle u(y), \widehat{G}_x(y) \rangle_{H_3} &= u(0) [\widehat{G}_x(0) - \partial_y^5 \widehat{G}_x(0)] \\
&\quad + u^{(2)}(0) [\partial_y^2 \widehat{G}_x(0) - \partial_y^3 \widehat{G}_x(0)] \\
&\quad + \sum_{i=0}^2 (-1)^{2-i} u^{(i)}(1) \partial_y^{5-i} \widehat{G}_x(1) \\
&\quad - \int_0^1 u(y) \partial_y^6 \widehat{G}_x(y) dy \\
&\quad - C(x) \int_0^1 yu(y) dy.
\end{aligned} \tag{15}$$

Note that the property of the reproducing kernel $\langle u(y), \widehat{G}_x(y) \rangle_{H_3} = u(x)$; then $\widehat{G}_x(y)$ is the solution of the following generalized differential equation:

$$\partial_y^6 \widehat{G}_x(y) + C(x)y = -\delta(y-x), \tag{16}$$

with the boundary conditions

$$\begin{aligned}
\widehat{G}_x(0) - \partial_y^5 \widehat{G}_x(0) &= 0, \\
\partial_y^2 \widehat{G}_x(0) - \partial_y^3 \widehat{G}_x(0) &= 0, \\
\partial_y^{5-i} \widehat{G}_x(1) &= 0, \quad i = 0, 1, 2.
\end{aligned} \tag{17}$$

As $x \neq y$, it follows that

$$\widehat{G}_x(y) = \begin{cases} \sum_{i=1}^6 c_i(x) y^{i-1} - \frac{y^7 C(x)}{5040}, & y \leq x, \\ \sum_{i=1}^6 d_i(x) y^{i-1} - \frac{y^7 C(x)}{5040}, & x < y. \end{cases} \tag{18}$$

Let us find the coefficients $c_i(x)$, $d_i(x)$, and $C(x)$. Since $\partial_y^6 \widehat{G}_x(y) + C(x)y = -\delta(y-x)$, we have

$$\begin{aligned}
\partial_y^5 \widehat{G}_x(x-0) - \partial_y^5 \widehat{G}_x(x+0) &= 1, \\
\partial_y^i \widehat{G}_x(x+0) &= \partial_y^i \widehat{G}_x(x-0), \\
i &= 0, 1, 2, 3, 4.
\end{aligned} \tag{19}$$

Since $\widehat{G}_x(y) \in H_3[0, 1]$, it follows that

$$\begin{aligned} \partial_y^1 \widehat{G}_x(0) &= 0, \\ \int_0^1 y \widehat{G}_x(y) dy &= 0. \end{aligned} \quad (20)$$

Through (17)–(20), the unknown coefficients of (11) can be obtained. $\square$

Definition 3. Define the space

$$\begin{aligned} H'_3[0, 1] \\ = \left\{ u(t) \mid u, u', u'' \text{ are absolutely continuous functions in } [0, 1] \right\} \\ \left[\begin{array}{l} u^{(3)}(t) \in L^2[0, 1], \quad u'(0) = u(0) = 0 \end{array} \right\}. \end{aligned} \quad (21)$$

The inner product and the norm in $H'_3[0, 1]$ are defined, respectively, by

$$\begin{aligned} \langle u(t), v(t) \rangle_{H'_3} &= \sum_{i=0}^2 u^{(i)}(0) v^{(i)}(0) \\ &\quad + \int_0^1 u^{(3)}(x) v^{(3)}(x) dx, \\ \|u\|_{H'_3} &= \sqrt{\langle u(t), u(t) \rangle_{H'_3}}. \end{aligned} \quad (22)$$

The space $H'_3[0, 1]$ is a complete reproducing kernel space, and its reproducing kernel $\check{G}_s(t)$ is given by

$$\check{G}_s(t) = \begin{cases} \frac{1}{120}s^5 - \frac{1}{24}s^4t + \frac{1}{4}s^2t^2 + \frac{1}{12}s^3t^2, & s \leq t, \\ -\frac{1}{24}s^4t + \frac{1}{120}t^5 + \frac{1}{12}s^2t^2(3+t), & t < s. \end{cases} \quad (23)$$

Definition 4. Define

$$H(Q) = \left\{ u \mid \frac{\partial^4 u}{\partial^2 x \partial^2 t} \text{ is absolutely continuous real-valued function in } Q, \right. \\ \left. \frac{\partial^6 u}{\partial^3 x \partial^3 t} \in L^2(Q), \quad u(x, 0) = \frac{\partial u(x, 0)}{\partial t} = 0, \quad \frac{\partial u(0, t)}{\partial x} = \int_0^1 x u(x, t) dx = 0 \right\}. \quad (24)$$

The inner product and the norm in $H(Q)$ are defined, respectively, by

$$\begin{aligned} \langle u(x, t), v(x, t) \rangle_H \\ = \sum_{i=0}^2 \int_0^1 \left[\frac{\partial^3}{\partial t^3} \frac{\partial^i}{\partial t^i} u(0, t) \frac{\partial^3}{\partial t^3} \frac{\partial^i}{\partial t^i} v(0, t) \right] dt \\ + \sum_{j=0}^2 \left\langle \frac{\partial^j}{\partial t^j} u(x, 0), \frac{\partial^j}{\partial t^j} v(x, 0) \right\rangle_{H_3} \\ + \int_0^1 \int_0^1 \left[\frac{\partial^6}{\partial x^3 \partial t^3} u(x, t) \frac{\partial^6}{\partial x^3 \partial t^3} v(x, t) \right] dt dx, \\ \|u\|_{H(Q)} = \sqrt{\langle u(x, t), u(x, t) \rangle_H}. \end{aligned} \quad (25)$$

$H(Q)$ is a reproducing kernel space and its reproducing kernel function is

$$G_{(y,s)}(x, t) = \widehat{G}_x(y) \check{G}_s(t) \quad (26)$$

such that, for any $u(x, t) \in H(Q)$,

$$\begin{aligned} u(y, s) &= \langle u(x, t), G_{(y,s)}(x, t) \rangle_H, \\ G_{(x,t)}(y, s) &= G_{(y,s)}(x, t). \end{aligned} \quad (27)$$

Definition 5. Define

$$\begin{aligned} H_1[0, 1] \\ = \{ u(x) \mid u \text{ is absolutely continuous in } [0, 1], u' \\ \in L^2[0, 1] \}. \end{aligned} \quad (28)$$

The inner product and the norm in $H_1[0, 1]$ are defined, respectively, by

$$\begin{aligned} \langle u(x), v(x) \rangle_{H_1} &= u(0) v(0) + \int_0^1 u'(x) v'(x) dx, \\ \|u\|_{H_1} &= \sqrt{\langle u(x), u(x) \rangle_{H_1}}. \end{aligned} \quad (29)$$

Definition 6. Define

$$\begin{aligned} \widehat{H}(Q) &= \left\{ u(x, t) \mid u \text{ is absolutely continuous in } Q, \right. \\ &\quad \left. \frac{\partial^2 u}{\partial x \partial t} \in L^2(Q) \right\}. \end{aligned} \quad (30)$$

The inner product and the norm in $\widehat{H}(Q)$ are defined, respectively, by

$$\begin{aligned} \langle u(x, t), v(x, t) \rangle_{\widehat{H}} &= \int_0^1 \left[\frac{\partial}{\partial t} u(0, t) \frac{\partial}{\partial t} v(0, t) \right] dt \\ &\quad + \langle u(x, 0), v(x, 0) \rangle_{H_1} \\ &\quad + \int_0^1 \int_0^1 \left[\frac{\partial^2}{\partial x \partial t} u(x, t) \frac{\partial^2}{\partial x \partial t} v(x, t) \right] dt dx, \end{aligned} \quad (31)$$

$$\|u\|_{\widehat{H}} = \sqrt{\langle u(x, t), u(x, t) \rangle_{\widehat{H}}}.$$

$\widehat{H}(Q)$ is a complete reproducing kernel space; thus, its reproducing kernel function is (see [11])

$$R_{(y,s)}(x, t) = \dot{R}_y(x) \dot{R}_s(t), \quad (32)$$

where

$$\dot{R}_y(x) = \begin{cases} 1 + y, & y \leq x, \\ 1 + x, & x < y. \end{cases} \quad (33)$$

3. A Bounded Linear Operator on $H(Q)$

Define differential operator $L: H(Q) \rightarrow \widehat{H}(Q)$ by

$$Lu(x, t) = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}; \quad (34)$$

therefore, (1) can be converted into an equivalent form as follows:

$$Lu(x, t) = \widetilde{F}(x, t, u(x, t)), \quad (35)$$

where $u(x, t) \in H(Q)$, $\widetilde{F}(x, t, u(x, t)) \in \widehat{H}(Q)$, and

$$\begin{aligned} \widetilde{F}(x, t, u(x, t)) &= f(x, t) \\ &\quad + \int_0^t a(t-s) K(s, u(x, s)) ds. \end{aligned} \quad (36)$$

It is easy to show that L is a bounded linear operator from $H(Q)$ to $\widehat{H}(Q)$. Now, we choose a countable subset $S = \{(x_1, t_1), (x_2, t_2), \dots\}$ in Q and we define

$$\begin{aligned} \varphi_i(x, t) &= R_{(x_i, t_i)}(x, t), \\ \omega_i(x, t) &= L^* \varphi_i(x, t). \end{aligned} \quad (37)$$

Lemma 7. Assume that S is dense in Q ; then, $\{\omega_i(x, t)\}_{i=1}^\infty$ is a complete system in $H(Q)$, and

$$\omega_i(x, t) = (L_{(y,s)} G_{(y,s)}(x, t)), \quad \text{for } (y, s) = (x_i, t_i). \quad (38)$$

Applying Gram-Schmidt process, we obtain an orthogonal basis $\{\bar{\omega}_i(x, t)\}_{i=1}^\infty$ of $H(Q)$, such that

$$\bar{\omega}_i(x, t) = \sum_{j=1}^i \beta_{ij} \omega_j(x, t), \quad (39)$$

where β_{ij} are orthogonal coefficients.

4. The Exact and Approximate Solution

Lemma 8. If S is dense in Q , then the exact solution of (35) is

$$u(x, t) = \sum_{i=1}^\infty \sum_{j=1}^i \beta_{ij} \widetilde{F}(x_j, t_j, u(x_j, t_j)) \bar{\omega}_i(x, t). \quad (40)$$

Proof. The exact solution $u(x, t)$ can be expanded in the Fourier series in terms of normal orthogonal basis $\{\bar{\omega}_i(x, t)\}_{i=1}^\infty$ in $H(Q)$:

$$\begin{aligned} u(x, t) &= \sum_{i=1}^\infty \langle u(x, t), \bar{\omega}_i(x, t) \rangle_H \bar{\omega}_i(x, t) \\ &= \sum_{i=1}^\infty \sum_{j=1}^i \beta_{ij} \langle u(x, t), \omega_j(x, t) \rangle_H \bar{\omega}_i(x, t) \\ &= \sum_{i=1}^\infty \sum_{j=1}^i \beta_{ij} \langle u(x, t), L^* \varphi_i(x, t) \rangle_H \bar{\omega}_i(x, t) \\ &= \sum_{i=1}^\infty \sum_{j=1}^i \beta_{ij} \langle Lu(x, t), \varphi_i(x, t) \rangle_{\widehat{H}(Q)} \bar{\omega}_i(x, t) \\ &= \sum_{i=1}^\infty \sum_{j=1}^i \beta_{ij} \langle Lu(x, t), R_{(x_i, t_i)}(x, t) \rangle_{\widehat{H}(Q)} \bar{\omega}_i(x, t) \\ &= \sum_{i=1}^\infty \sum_{j=1}^i \beta_{ij} (Lu)(x_j, t_j) \bar{\omega}_i(x, t) \\ &= \sum_{i=1}^\infty \sum_{j=1}^i \beta_{ij} \widetilde{F}(x_j, t_j, u(x_j, t_j)) \bar{\omega}_i(x, t). \end{aligned} \quad (41)$$

□

4.1. Iterative Method. We construct an iterative method to obtain the approximate solutions of (1) in the reproducing kernel space $H(Q)$. Let

$$A_i = \sum_{j=1}^i \beta_{ij} \widetilde{F}(x_j, t_j, u(x_j, t_j)). \quad (42)$$

Then

$$u(x, t) = \sum_{i=1}^\infty A_i \bar{\omega}_i(x, t). \quad (43)$$

Define the initial function $u_0(x, t) = 0$ and the n -term approximation to $u(x, t)$ by

$$u_n(x, t) = \sum_{i=1}^n \bar{A}_i \bar{\omega}_i(x, t), \quad (44)$$

where the coefficients $\tilde{A}_i$ are given as

$$\begin{aligned}\tilde{A}_1 &= \beta_{11} \tilde{F}(x_1, t_1, u_0(x_1, t_1)), \\ \tilde{A}_2 &= \sum_{j=1}^2 \beta_{2j} \tilde{F}(x_j, t_j, u_{j-1}(x_j, t_j)), \\ &\vdots \\ \tilde{A}_n &= \sum_{j=1}^n \beta_{nj} \tilde{F}(x_j, t_j, u_{j-1}(x_j, t_j)).\end{aligned}\quad (45)$$

Theorem 9. If S is dense in Q , then

$$(Lu_n)(x, t) = \tilde{F}(x, t, u_{n-1}(x, t)), \quad (46)$$

where $u_n(x, t)$ is given by (44).

Theorem 10. Suppose that $\| \int_0^t K(s, u(x, s)) ds \|_{\widehat{H}(Q)} \leq C \|u\|_{H(Q)}$ and $0 < lC \|L^{-1}\| < 1$; then

$$u_n(x, t) \xrightarrow{\|\cdot\|_H} u(x, t) \quad \text{as } (n \rightarrow \infty). \quad (47)$$

Proof. From the preceding theorem, we have

$$(Lu_n)(x, t) = \tilde{F}(x, t, u_{n-1}(x, t)) \quad (48)$$

so

$$\begin{aligned}\|u_n(x, t) - u(x, t)\|_H &= \|L^{-1}(\tilde{F}(x, t, u_{n-1}(x, t)) \\ &\quad - \tilde{F}(x, t, u(x, t)))\|_H \leq \|L^{-1}\| \|\tilde{F}(x, t, u_{n-1}(x, t)) \\ &\quad - \tilde{F}(x, t, u(x, t))\|_{\widehat{H}(Q)} \leq \|L^{-1}\| \left\| \left(\int_0^t a(t-s) \right. \right. \\ &\quad \cdot K(s, u_{n-1}(x, s) - u(x, s)) ds \Big) \Big\|_{\widehat{H}(Q)} \leq C \\ &\quad \cdot \max_{0 \leq t \leq 1} |a(t)| \|L^{-1}\| \|u_{n-1}(x, t) - u(x, t)\|_H.\end{aligned}\quad (49)$$

From $lC \|L^{-1}\| < 1$ and (49), we get

$$u_n(x, t) \xrightarrow{\|\cdot\|_H} u(x, t) \quad \text{as } n \rightarrow \infty. \quad (50)$$

□

Proposition 11. Suppose that $u_n(x, t) \xrightarrow{\|\cdot\|_H} u(x, t)$ as $(n \rightarrow \infty)$; then

$$\left| \frac{\partial^{i+j}}{\partial x^i \partial t^j} u(x, t) - \frac{\partial^{i+j}}{\partial x^i \partial t^j} u_n(x, t) \right| \rightarrow 0, \quad (51)$$

$$n \rightarrow \infty, \quad i, j = 0, 1, 2.$$

Now, the approximate solution $u_n^m(x, t)$ can be obtained by taking finitely many terms in the series representation of $u_n(x, t)$ and

$$u_n^m(x, t) = \sum_{i=1}^m \sum_{j=1}^i \beta_{ij} \tilde{F}(x_j, t_j, u_{n-1}(x_j, t_j)) \bar{\omega}_i(x, t). \quad (52)$$

5. Numerical Examples

In order to calculate the approximate solution $u_n^m(x, t)$, we take

$$\begin{aligned}(x_j, t_j) &= \left(\frac{i}{N}, \frac{k}{M} \right), \\ i &= 0, 1, \dots, N, \quad k = 0, 1, \dots, M, \quad j = 0, 1, 2, \dots \\ m &= (N+1)(M+1), \\ e_n^m(x, t) &= |u(x, t) - u_n^m(x, t)|, \\ (e_n^m)_{xx}(x, t) &= |u_{xx}(x, t) - (u_n^m)_{xx}(x, t)|, \\ (e_n^m)_t(x, t) &= |u_t(x, t) - (u_n^m)_t(x, t)|, \\ (e_n^m)_{xt}(x, t) &= |u_{xt}(x, t) - (u_n^m)_{xt}(x, t)|, \\ (e_n^m)_{tt}(x, t) &= |u_{tt}(x, t) - (u_n^m)_{tt}(x, t)|, \\ (e_n^m)_x(x, t) &= |u_x(x, t) - (u_n^m)_x(x, t)|.\end{aligned}\quad (53)$$

Example 1. Consider the following hyperbolic integrodifferential equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f(x, t) + \int_0^t (t-s)^4 u(x, s) ds, \quad (54)$$

for all $(x, t) \in Q$, subject to the initial conditions

$$\begin{aligned}u(x, 0) &= \frac{\partial u(x, 0)}{\partial t} = 0, \\ \frac{\partial u(0, t)}{\partial x} &= t^2,\end{aligned}\quad (55)$$

and the weighted integral condition

$$\int_0^1 xu(x, t) dx = 0, \quad (56)$$

where

$$f(x, t) = -4 + \frac{(2t^7)}{105} + \exp x \left(2 - t^2 - \frac{t^7}{105} \right). \quad (57)$$

The exact solution is given by $u(x, t) = (\exp x - 2)t^2$. In order to homogenize the initial conditions, we put $v(x, t) = u(x, t) - (x - 2/3)t^2$; then the problem can be converted into the following form:

$$\begin{aligned}\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} &= F(x, t) + \int_0^t (t-s)^4 v(x, s) ds, \\ v(x, 0) &= \frac{\partial v(x, 0)}{\partial t} = 0, \\ \frac{\partial v(0, t)}{\partial x} &= 0,\end{aligned}\quad (58)$$

$$\int_0^1 xv(x, t) dx = 0,$$

TABLE 1: Comparison of results for u ($n = 2, m = 25, 121$).

(x_i, t_i)	u	u_2^{25}	e_2^{25}	u_2^{121}	e_2^{121}
(0.00, 0.01)	-0.0000333333	-0.0000333637	3.033825×10^{-8}	-0.0000333333	1.27745×10^{-8}
(0.10, 0.05)	-0.000820406	-0.00082375	3.34428×10^{-6}	-0.000820406	1.2258×10^{-6}
(0.15, 0.10)	-0.00321499	-0.00323799	0.000022998	-0.00321499	6.31116×10^{-6}
(0.35, 0.20)	-0.0105706	-0.0106773	0.000106649	-0.0105706	0.0000198708
(0.49, 0.31)	-0.0183567	-0.0185277	0.000170909	-0.0183885	0.0000317253
(0.57, 0.45)	-0.0273509	-0.0275692	0.000218296	-0.0273509	0.0000385202
(0.69, 0.52)	-0.00800865	-0.00804727	0.0000386212	-0.00800874	8.8295×10^{-8}
(0.73, 0.71)	0.0059218	0.00598197	0.0000601728	0.00594941	0.0000276115
(0.81, 0.73)	0.0557278	0.0560604	0.000332616	0.0558007	0.0000728933
(0.99, 0.99)	0.36058	0.362213	0.00163333	0.360756	0.000176542

TABLE 2: Comparison of results for u_x ($n = 2, m = 121$).

(x_i, t_i)	u_x	$(u_2^{121})_x$	$(e_2^{121})_x$	u_t	$(u_2^{121})_t$	$(e_2^{121})_t$
(0.00, 0.01)	0.00000	0.0000	0.00000	-0.00666667	-0.00667043	3.76×10^{-6}
(0.10, 0.05)	0.000262927	0.000263362	4.34808×10^{-7}	-0.0328162	-0.0328162	0.000065
(0.15, 0.10)	0.00161834	0.0016219	3.5546×10^{-6}	-0.0642998	-0.0644248	0.000124
(0.35, 0.20)	0.0167627	0.0167986	0.0000359083	-0.105706	-0.105901	0.000194
(0.49, 0.31)	0.0607656	0.0608914	0.000125813	-0.118431	-0.118615	0.00018
(0.57, 0.45)	0.155574	0.155866	0.000291519	-0.12156	-0.121689	0.00012
(0.69, 0.52)	0.268701	0.269116	0.000415711	-0.0308025	-0.0307829	0.000019
(0.73, 0.71)	0.541948	0.54252	0.00057215	0.0166811	0.0167237	0.000042
(0.81, 0.73)	0.66501	0.665538	0.000527674	0.152679	0.152725	0.000045
(0.99, 0.99)	1.65758	1.659	0.00142569	0.728444	0.727913	0.00053

TABLE 3: Comparison of results for u_{xx} ($n = 2, m = 121$).

(x_i, t_i)	u_{xx}	$(u_2^{121})_{xx}$	$(e_2^{121})_{xx}$	u_{tt}	$(u_2^{121})_{tt}$	$(e_2^{121})_{tt}$
(0.00, 0.01)	0.0001000	0.000100171	1.71×10^{-7}	-0.666667	-0.667391	0.000724
(0.10, 0.05)	0.00276293	0.00277046	7.52×10^{-6}	-0.656325	-0.658253	0.001928
(0.15, 0.10)	0.0116183	0.0116374	0.0000190	-0.642998	-0.64296	0.0000379
(0.35, 0.20)	0.0567627	0.0568428	0.0000801	-0.528532	-0.528362	0.000169
(0.49, 0.31)	0.156866	0.157201	0.0003349	-0.382034	-0.382073	0.000038
(0.57, 0.45)	0.358074	0.35823	0.0001562	-0.270133	-0.270318	0.0001851
(0.69, 0.52)	0.539101	0.539354	0.0002531	-0.0592356	-0.0059133	0.0001026
(0.73, 0.71)	1.04605	1.04597	0.0000733	0.0234945	0.0232241	0.00027045
(0.81, 0.73)	1.19791016	1.19798	0.00006967	0.209149	0.208289	0.00085993
(0.99, 0.99)	2.63768	2.64076	0.003076	0.735802	0.735802	0.00426859

where

$$F(x, t) = -\frac{3}{315} (t^7 + 105t^2 - 210) \exp x - \frac{3}{315} ((4 + 3x) + (-210 + t^7)). \quad (59)$$

The numerical results are presented in Tables 1, 2, 3, and 4 and Figures 1 and 2.

Example 2. Consider the following hyperbolic integrodifferential equation:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= f(x, t) + \int_0^t \left(\int_0^x u(r, s) dr \right) ds, \\ u(x, 0) &= \frac{\partial u(x, 0)}{\partial t} = 0, \\ \frac{\partial u(0, t)}{\partial x} &= 0, \end{aligned} \quad (60)$$

$$\int_0^1 xu(x, t) dx = 0,$$

TABLE 4: Comparison of results for u_{xt} ($n = 2, m = 121$).

(x_i, t_i)	u_{xt}	$(u_2^{121})_{xt}$	$(e_2^{121})_{xt}$
(0.00, 0.01)	0.00000	0.0000	0.00000
(0.10, 0.05)	0.0105171	0.0105402	0.0000231397
(0.15, 0.10)	0.0323668	0.0324379	0.0000710837
(0.35, 0.20)	0.167627	0.167991	0.000364067
(0.49, 0.31)	0.392036	0.392836	0.000799951
(0.57, 0.45)	0.69144	0.692595	0.00115446
(0.69, 0.52)	1.03346	1.03456	0.00109998
(0.73, 0.71)	1.52661	1.52694	0.000326263
(0.81, 0.73)	1.82195	1.82199	0.000041932
(0.99, 0.99)	3.34864	3.34924	0.000598899

FIGURE 1: Consider $e_2^{25}(x, t) = |u(x, t) - u_2^{25}(x, t)|$. FIGURE 2: Consider $e_2^{121}(x, t) = |u(x, t) - u_2^{121}(x, t)|$.

where

$$f(x, t) = -2t^2 + \frac{xt^3}{6} - \frac{x^3t^3}{9} + \left(-\left(\frac{1}{2}\right) + x^2\right). \quad (61)$$

The exact solution is given by $u(x, t) = (x^2 - 1/2)t^2$.

FIGURE 3: Consider $e_2^{25}(x, t) = |u(x, t) - u_2^{25}(x, t)|$. FIGURE 4: Consider $|u(x, t) - u_2^{25}(x, t)|/|u(x, t)|$.

The numerical results are presented in Tables 5, 6, 7, and 8 and Figures 3 and 4.

6. Conclusion

In this paper, we have applied RKHS method to solve a hyperbolic integrodifferential equation with a weighted integral condition. The numeric results have demonstrated that only a small number of iterations can be used to obtain numeric results with a good precision. From this work, we see that the absolute errors are monotonically decreasing if m increase. The numerical examples show that the approximate solution and its partial derivatives of a superior order or equal to two converge to the exact solution and its partial derivatives, respectively. Consequently, we confirm that this method is effective for the integrodifferential equation with weighted integral conditions. All the computations are performed using Mathematica 9.0.

TABLE 5: Comparison of results for u ($n = 2, m = 25$).

(x_i, t_i)	u	u_2^{25}	e_2^{25}	e_2^{25}/u
(0.00, 0.01)	-0.000050000	-0.0000500038	3.81237×10^{-9}	0.0000762474
(0.10, 0.05)	-0.001225000	-0.00122559	5.90641×10^{-7}	0.000482156
(0.15, 0.10)	-0.004775	-0.0047785	3.49902×10^{-6}	0.000732779
(0.35, 0.20)	-0.01510000	-0.0151099	9.9186×10^{-6}	0.000656861
(0.49, 0.31)	-0.0249764	-0.0249918	0.0000153856	0.000616005
(0.57, 0.45)	-0.0354578	-0.0354835	0.0000257761	0.000726954
(0.69, 0.52)	-0.00646256	-0.00645257	9.99319×10^{-6}	0.00154632
(0.73, 0.71)	0.0165849	0.0166669	0.0000819828	0.00494322
(0.81, 0.73)	0.0831857	0.083387	0.000201347	0.00242045
(0.99, 0.99)	0.470546	0.471707	0.00116079	0.00246689

TABLE 6: Comparison of results for u_x ($n = 2, m = 25$).

(x_i, t_i)	u_x	$(u_2^{25})_x$	$(e_2^{25})_x$	u_t	$(u_2^{25})_t$	$(e_2^{25})_t$
(0.00, 0.01)	0.0000	0.0000	0.000	-0.010	-0.0100011	1.13×10^{-6}
(0.10, 0.05)	0.0005	0.000499228	7.71×10^{-7}	-0.049	-0.049031	0.00003
(0.15, 0.10)	0.003	0.00300832	8.32×10^{-6}	-0.0955	-0.0955913	0.00009
(0.35, 0.20)	0.028	0.0280282	0.000028	-0.151	-0.151122	0.00012
(0.49, 0.31)	0.094178	0.0942695	0.00009	-0.161138	-0.161259	0.00012
(0.57, 0.45)	0.23085	0.231107	0.00025	-0.15759	-0.157779	0.00018
(0.69, 0.52)	0.373152	0.37359	0.00043	-0.024856	-0.0247779	0.00007
(0.73, 0.71)	0.735986	0.737218	0.0012	0.046718	0.0471139	0.00039
(0.81, 0.73)	0.863298	0.864699	0.0014	0.227906	0.228805	0.00089
(0.99, 0.99)	1.9406	1.94427	0.0036	0.950598	0.954343	0.0037

TABLE 7: Comparison of results for u_{xx} ($n = 2, m = 25$).

(x_i, t_i)	u_{xx}	$(u_2^{25})_{xx}$	$(e_2^{25})_{xx}$	u_{tt}	$(u_2^{25})_{tt}$	$(e_2^{25})_{tt}$
(0.00, 0.01)	0.0002	0.000196032	3.96×10^{-6}	-1.0000	-1.00022	0.0002
(0.10, 0.05)	0.005	0.00504296	0.00004	-0.98	-0.981028	0.00102
(0.15, 0.10)	0.02	0.0201771	0.0001	-0.955	-0.956444	0.0014
(0.35, 0.20)	0.08	0.0799329	0.00006	-0.755	0.0007	0.0001
(0.49, 0.31)	0.1922	0.191932	0.00026	-0.5198	-0.520859	0.0010
(0.57, 0.45)	0.405	0.404939	0.00006	-0.3502	-0.351163	0.0009
(0.69, 0.52)	0.5408	0.541206	0.0004	-0.0478	-0.0473431	0.0004
(0.73, 0.71)	1.0082	1.00936	0.0011	0.0658	0.0670897	0.0012
(0.81, 0.73)	1.0658	1.06519	0.0006	0.3122	0.314928	0.0027
(0.99, 0.99)	1.9406	1.94427	0.0036	0.950598	0.954343	0.0037

TABLE 8: Comparison of results for u_{xt} ($n = 2, m = 25$).

(x_i, t_i)	u_{xt}	$(u_2^{25})_{xt}$	$(e_2^{25})_{xt}$
(0.00, 0.01)	0.000	0.000	0.000
(0.10, 0.05)	0.02	0.0199786	0.00002
(0.15, 0.10)	0.06	0.0602084	0.0002
(0.35, 0.20)	0.28	0.280396	0.00039
(0.49, 0.31)	0.6076	0.608402	0.0008
(0.57, 0.45)	1.026	1.02799	0.0019
(0.69, 0.52)	1.4352	1.43788	0.002
(0.73, 0.71)	2.0732	2.07878	0.005
(0.81, 0.73)	2.3652	2.37112	0.005
(0.99, 0.99)	3.9204	3.93184	0.011

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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