Research Article

Fourier Expansions with Polynomial Terms for Random Processes

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Based on calculus of random processes, we present a kind of Fourier expansions with simple polynomial terms via our decomposition method of random processes. Using our method, the expectations and variances of the corresponding coefficients decay fast and partial sum approximations attain the best approximation order. Moreover, since we remove boundary effect in our decomposition of random process, these coefficients can discover the instinct frequency information of this random process. Therefore, our method has an obvious advantage over traditional Fourier expansion. These results are also new for deterministic functions.

1. Introduction

Based on calculus of random processes [1], it is well known that one can expand a random process into Karhunen-Loeve series or Fourier series. For a Karhunen-Loeve series, its coefficients are random variables and its basis consists of deterministic functions which are eigenfunctions of the correlation function of the random process. So, when one analyzes different random processes, one needs to compute different Karhunen-Loeve basis. However, to compute eigenfunctions numerically is very difficult. Therefore, one often expands random processes into Fourier series. Due to the discontinuity on the boundaries, their Fourier coefficients decay slowly and cannot reflect the instinct frequency information of the random process. To solve this problem, we give a new method. First, we give the decompositions of the one-dimensional and two-dimensional random processes. Based on our decompositions, we present a kind of Fourier expansions with simple polynomial terms, in which Fourier coefficients decay fast. Although we discuss these expansions in the setting of random processes, for deterministic functions, our results are also new.

Our main idea is as follows.

For a one-dimensional random process ξ on $[0, 1]$, we present a Fourier expansion with a polynomial term:

$$\xi = g + h, \quad (1)$$

as in (27). The first term g is a random polynomial which is determined by the derivatives of ξ at $t = 0$ and $t = 1$. The second term is a Fourier sine series of $\xi - g$. Theorem 2 shows that the expectations and variances of Fourier coefficients of $\xi - g$ decay fast and Theorem 4 shows that the partial sum approximation of expansion (27) attains the best approximation order.

For a two-dimensional random process ξ on $[0, 1]^2$, we present a Fourier expansion with polynomial terms:

$$\xi = h_1 + h_2 + h_3, \quad (2)$$

as in (49). The first term h_1 is a two-dimensional random polynomial which is determined by the partial derivatives of ξ at the vertices of the unit square $[0, 1]^2$. The second term h_2 contains four one-dimensional Fourier sine series which are determined by the partial derivatives of ξ on the boundary of $[0, 1]^2$. The third term is the two-dimensional Fourier series of $\xi - h_1 - h_2$. Theorems 6 and 7 show that the expectations and variances of all the above Fourier coefficients in expansion (49) decay fast. The partial sum of expansion (49) is a combination of random polynomials and random sine polynomials. This is a good approximation tool. Theorem 9 shows that the partial sum approximation attains the best approximation order. In this combination, degrees of these random polynomials are only determined by smoothness

indexes of the random process while the degrees of sine polynomials are determined by presumed approximation error and smoothness indexes.

These results explain that the Fourier expansion with polynomial terms is better than the Fourier expansion. Our results are also new for deterministic functions.

This paper is organized as follows. In Section 2 we recall some known results in the basic calculus of random processes. In Section 3 we give the decompositions of the one-dimensional and two-dimensional random processes. In Sections 4 and 5 we study Fourier expansions with polynomial terms for one-dimensional and two-dimensional random processes, respectively. In Section 6 we state our conclusions.

2. Calculus of Random Processes

Calculus of random processes is the basis of random differential equations and has a wide application in many fields, such as finance, economy, and geoscience [1–8].

If a real random variable is such that the expectation of its square is finite, then we say that it is a second order random variable. Let $\{\xi_n\}_1^\infty$ be a sequence of second order random variables and let ξ be a second order random variable. If

$$\lim_{n \rightarrow \infty} E[|\xi_n - \xi|^2] = 0, \quad (3)$$

we say $\{\xi_n\}$ converges to ξ in the mean square sense. Based on it, one can derive the concepts of the continuity, the derivatives, and the integrals as well as partial derivatives and double integrals of random process [1, 7, 8]. If the coefficients of a polynomial are random variables, it is said to be a random polynomial. The n -degree derivative of a one-dimensional random polynomial $P_n(t)$ of degree n is a random variable which is independent of t , and its $(n+1)$ -degree derivative vanishes. If a random process is continuous on $[0, 1]$, one says that it belongs to $C([0, 1])$. By $E[\xi]$ and $\text{Var}(\xi)$ denote the expectation and variance of a random process ξ , respectively. If ξ is a continuously differentiable random process on $[a, b]$, then

$$\begin{aligned} E[\xi'(t)] &= (E[\xi(t)])', \\ E\left[\int_a^b \xi(t) dt\right] &= \int_a^b E[\xi(t)] dt. \end{aligned} \quad (4)$$

And the Newton-Leibnitz formula

$$\int_a^b \xi'(t) dt = \xi(b) - \xi(a) \quad (5)$$

holds. For the product of a differentiable random process and a differentiable deterministic function on $[a, b]$, the product differential method holds and the corresponding integration by parts also holds. However, it does not hold for the product of two differentiable random processes. Based on these concepts, theory on Karhunen-Loeve expansions or Fourier expansions of random processes is developed.

Throughout this paper, we denote the vertices of $[0, 1]^2$ by $\{0, 1\}^2 =: \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ and its boundary by $\partial([0, 1]^2)$. The integral part of a positive real λ is denoted

by $[\lambda]$. Denote the differential operators D^{ν_1, ν_2} by $D^{\nu_1, \nu_2} = \partial^{\nu_1 + \nu_2} / \partial t_1^{\nu_1} \partial t_2^{\nu_2}$, where ν_1, ν_2 are nonnegative integers. $y_n = o(x_n)$ means that $y_n/x_n \rightarrow 0$ and $y_n = O(x_n)$ means that y_n/x_n is bounded.

3. Decompositions of Random Processes

We assume that $\xi(t)$ is a random process on $[0, 1]$ and its l -order derivatives $\xi^{(l)} \in C([0, 1])$. If we expand it into Fourier sine series

$$\xi(t) = \sum_{n=1}^{\infty} c_n(\xi) \sin(\pi n t), \quad (6)$$

the expectations of its Fourier sine coefficients $c_n(\xi)$ decay slowly. In fact, for $l \geq 1$,

$$\begin{aligned} c_n(\xi) &= 2 \int_0^1 \xi(t) \sin(\pi n t) dt \\ &= \frac{2}{\pi n} (\xi(0) + (-1)^{n-1} \xi(1)) + \frac{1}{\pi n} \int_0^1 \xi'(t) \cos(\pi n t) dt. \end{aligned} \quad (7)$$

By the Riemann-Lebesgue lemma [9] and (4),

$$E[c_n(\xi)] = \frac{2}{\pi n} (E[\xi(0)] + (-1)^{n-1} E[\xi(1)]) + o\left(\frac{1}{n}\right), \quad (8)$$

so we see that $E[c_n(\xi)] = o(1/n)$ if and only if $E[\xi(0)] = E[\xi(1)] = 0$. This means that the decaying of Fourier sine coefficients depends on the value of the random process at the endpoints. Inspired by this fact, we consider removing a simple random polynomial $g(t)$ from the random process $\xi(t)$ such that the expectations and variances of Fourier sine coefficients of the residual $h(t) = \xi(t) - g(t)$ decay fast. For this purpose, let $a_n(t)$ be a polynomial of degree $2n+1$ and their derivatives satisfy

$$\begin{aligned} a_n^{(2j)}(0) &= 0, & a_n^{(2j)}(1) &= 0 \quad (0 \leq j \leq n-1), \\ a_n^{(2n)}(0) &= 0, & a_n^{(2n)}(1) &= 1. \end{aligned} \quad (9)$$

Then $a_n(t)$ can be represented as follows:

$$\begin{aligned} a_0(t) &= t, & a_1(t) &= \frac{1}{6} t(t^2 - 1), \\ a_n(t) &= \frac{1}{(2n+1)!} t^{2n+1} + \sum_{k=0}^{n-1} b_k t^{2k+1}, \quad (n = 2, 3, \dots), \end{aligned} \quad (10)$$

where the coefficients b_k for $0 \leq k \leq n-1$ satisfy

$$\sum_{k=j}^{n-1} \frac{(2k+1)!}{(2k-2j+1)!} b_k = -\frac{1}{(2n-2j+1)!} \quad (j = 0, \dots, n-1). \quad (11)$$

We construct a random polynomial $g(t)$ as follows:

$$\begin{aligned} g(t) &= \sum_{i=0}^m (\xi^{(2i)}(0) a_i(1-t) + \xi^{(2i)}(1) a_i(t)), \\ &\quad \left(m = \left\lfloor \frac{l-1}{2} \right\rfloor\right). \end{aligned} \quad (12)$$

We get the decomposition formula $\xi = g + h$.

By (9) and (12), we have

$$\begin{aligned} h^{(l)} &\in C([0, 1]), \\ h^{(2i)}(0) &= h^{(2i)}(1) = 0 \quad (i = 0, 1, \dots, m). \end{aligned} \quad (13)$$

In Theorem 2, we will show that the expectations and variances of Fourier sine coefficients of $h(t)$ decay fast.

Let ξ be a two-dimensional random process on $[0, 1]^2$ and $D^{l_1, l_2} \xi \in C([0, 1]^2)$. We decompose it as follows:

$$\xi = h_1 + h_2 + h_3, \quad (14)$$

where

$$\begin{aligned} h_1(t_1, t_2) &= \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} D^{2i_1, 2i_2} \xi(0, 0) a_{i_1}(1-t_1) a_{i_2}(1-t_2) \\ &\quad + \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} D^{2i_1, 2i_2} \xi(0, 1) a_{i_1}(1-t_1) a_{i_2}(t_2) \\ &\quad + \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} D^{2i_1, 2i_2} \xi(1, 0) a_{i_1}(t_1) a_{i_2}(1-t_2) \\ &\quad + \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} D^{2i_1, 2i_2} \xi(1, 1) a_{i_1}(t_1) a_{i_2}(t_2), \end{aligned} \quad (15)$$

$$\begin{aligned} h_2(t_1, t_2) &= \sum_{i_1=0}^{m_1} (A_{i_1}(t_1, t_2) + B_{i_1}(t_1, t_2)) \\ &\quad + \sum_{i_2=0}^{m_2} (\tilde{A}_{i_2}(t_1, t_2) + \tilde{B}_{i_2}(t_1, t_2)), \quad (16) \\ &\quad \left(m_i = \left\lfloor \frac{(l_i - 1)}{2} \right\rfloor, \quad i = 1, 2 \right), \end{aligned}$$

$$h_3(t_1, t_2) = \xi(t_1, t_2) - h_1(t_1, t_2) - h_2(t_1, t_2), \quad (17)$$

$$A_{i_1}(t_1, t_2) = D^{2i_1, 0} \xi_1(0, t_2) a_{i_1}(1-t_1),$$

$$B_{i_1}(t_1, t_2) = D^{2i_1, 0} \xi_1(1, t_2) a_{i_1}(t_1),$$

$$\tilde{A}_{i_2}(t_1, t_2) = D^{0, 2i_2} \xi_1(t_1, 0) a_{i_2}(1-t_2),$$

$$\tilde{B}_{i_2}(t_1, t_2) = D^{0, 2i_2} \xi_1(t_1, 1) a_{i_2}(t_2), \quad (\xi_1 = \xi - h_1). \quad (18)$$

Theorem 1. Let ξ be a random process on $[0, 1]^2$ and $D^{l_1, l_2} \xi \in C([0, 1]^2)$ and let it be decomposed as $\xi = h_1 + h_2 + h_3$, where h_1 , h_2 , and h_3 are stated in (15)–(18). Then, for $0 \leq i_1 \leq m_1$, $0 \leq i_2 \leq m_2$, one gets the following.

(i) $h_1(t_1, t_2)$ is a two-dimensional random polynomial and

$$D^{2i_1, 2i_2} h_1(t_1, t_2) = D^{2i_1, 2i_2} \xi(t_1, t_2), \quad (t_1, t_2) \in \{0, 1\}^2, \quad (19)$$

where $\{0, 1\}^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

(ii) $D^{l_1, l_2} h_3 \in C([0, 1]^2)$ and

$$D^{2i_1, 2i_2} h_3(t_1, t_2) = 0, \quad (t_1, t_2) \in \partial([0, 1]^2), \quad (20)$$

where $m_\nu = [(l_\nu - 1)/2]$ ($\nu = 1, 2$).

Proof. From (9), it follows by (15) that

$$\begin{aligned} D^{2j_1, 2j_2} h_1(0, 0) &= \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} D^{2i_1, 2i_2} \xi(0, 0) a_{i_1}^{(2j_1)}(1) a_{i_2}^{(2j_2)}(1) \\ &= D^{2j_1, 2j_2} \xi(0, 0) \\ &\quad (0 \leq j_1 \leq m_1, \quad 0 \leq j_2 \leq m_2). \end{aligned} \quad (21)$$

Similarly, we discuss all vertices of the unit square $\partial([0, 1]^2)$ and we get (i).

By $\xi_1 = \xi - h_1$ and (i), we get that

$$\begin{aligned} D^{2j_1, 2j_2} A_{i_1}(t_1, 0) &= D^{2j_1, 2j_2} \xi_1(0, 0) a_{i_1}^{(2j_1)}(1-t_1) \\ &= (D^{2i_1, 2j_2} \xi(0, 0) - D^{2i_1, 2j_2} h_1(0, 0)) a_{i_1}^{(2j_1)}(1-t_1) = 0. \end{aligned} \quad (22)$$

Similarly, we have $D^{2j_1, 2j_2} B_{i_1}(t_1, 0) = 0$.

By (9), we get

$$\begin{aligned} D^{2j_1, 2j_2} \tilde{A}_{i_2}(t_1, 0) &= D^{2j_1, 2i_2} \xi_1(t_1, 0) \delta_{i_2, j_2}, \\ D^{2j_1, 2j_2} \tilde{B}_{i_2}(t_1, 0) &= D^{2j_1, 2i_2} \xi(t_1, 0) a_{i_2}^{(2j_2)}(0) = 0, \end{aligned} \quad (23)$$

where $\delta_{m,n} = 0$ ($m \neq n$), $\delta_{m,n} = 1$ ($m = n$). By (16), we have

$$\begin{aligned} D^{2j_1, 2j_2} h_2(t_1, 0) &= \sum_{i_2=0}^{m_2} D^{2j_1, 2j_2} \tilde{A}_{i_2}(t_1, 0) \\ &= \sum_{i_2=0}^{m_2} D^{2j_1, 2j_2} \xi_1(t_1, 0) \delta_{i_2, j_2} = D^{2j_1, 2j_2} \xi_1(t_1, 0). \end{aligned} \quad (24)$$

By $h_3 = \xi - h_1 - h_2 = \xi_1 - h_2$, we have

$$D^{2j_1, 2j_2} h_3(t_1, 0) = D^{2j_1, 2j_2} \xi_1(t_1, 0) - D^{2j_1, 2j_2} h_2(t_1, 0) = 0. \quad (25)$$

Similarly, we discuss each side of the unit square $\partial([0, 1]^2)$ and we get (ii). $\square$

4. One-Dimensional Case

Let ξ be an l -order differentiable real random process on $[0, 1]$. By the decomposition formula, we have $\xi = g + h$, where g is stated in (12). Expanding h into Fourier sine series,

$$\begin{aligned} h(t) &= \sum_{n=1}^{\infty} c_n(h) \sin(\pi n t), \text{ where} \\ c_n(h) &= 2 \int_0^1 h(t) \sin(\pi n t) dt, \end{aligned} \quad (26)$$

we obtain the Fourier expansion of ξ with a polynomial term,

$$\begin{aligned} \xi(t) &= \sum_{i=0}^m \left(\xi^{(2i)}(0) a_i(1-t) + \xi^{(2i)}(1) a_i(t) \right) \\ &+ \sum_{n=1}^{\infty} c_n(h) \sin(\pi n t). \end{aligned} \quad (27)$$

First, we estimate the expectations and variances of the Fourier sine coefficients $c_n(h)$.

Denote

$$\begin{aligned} E_l &= \max_{0 \leq t \leq 1} \left| E \left[\xi^{(l)}(t) \right] \right|, \\ V_l &= \max_{0 \leq t \leq 1} \left(E \left[\left(\xi^{(l)}(t) \right)^2 \right] \right). \end{aligned} \quad (28)$$

(i) *The Case That l Is an Odd Number.* In (26), since $h(t)$ is an l -order differentiable random process and $\sin(\pi n t)$ is a differentiable deterministic function, using the integration by parts l times, we get by (13) that

$$c_n(h) = \frac{2}{(n\pi)^l} \int_0^1 h^{(l)}(t) \cos(\pi n t) dt. \quad (29)$$

Since the expectation and the integral can be exchanged, by (29),

$$E[c_n(h)] = \frac{2}{(n\pi)^l} \int_0^1 E[h^{(l)}(t)] \cos(\pi n t) dt. \quad (30)$$

By $h^{(l)}(t) \in C([0, 1])$, we have $E[h^{(l)}(t)] \in C([0, 1])$. From this and the Riemann-Lebesgue lemma, we get

$$E[c_n(h)] = o\left(\frac{1}{n^l}\right). \quad (31)$$

Noticing that l is an odd number and $m = [(l-1)/2]$, we have $l = 2m + 1$. Again, since g is a random polynomial of degree $2m + 1$, we know that $g^{(l)}(t)$ is a random variable which is independent of t and is denoted by τ . So we deduce that

$$\int_0^1 g^{(l)}(t) \cos(\pi n t) dt = \tau \int_0^1 \cos(\pi n t) dt = 0. \quad (32)$$

From this and $\xi = g + h$, we get by (29) that

$$c_n(h) = \frac{2}{(n\pi)^l} \int_0^1 \xi^{(l)}(t) \cos(\pi n t) dt. \quad (33)$$

From this and (28), we have

$$|E[c_n(h)]| \leq \frac{2}{(n\pi)^l} E_l, \quad (34)$$

$$\begin{aligned} \frac{1}{4} E[c_n^2(h)] &= \frac{1}{(n\pi)^{2l}} \int_0^1 \int_0^1 E[\xi^{(l)}(t) \xi^{(l)}(s)] \cos(\pi n t) \cos(\pi n s) dt ds. \end{aligned} \quad (35)$$

Again, by the Riemann-Lebesgue lemma, $E[c_n^2(h)] = o(1/n^{2l})$. By the Schwarz inequality in the probability theory and (28), we have

$$\begin{aligned} |E[\xi^{(l)}(t) \xi^{(l)}(s)]| &\leq \sqrt{E[(\xi^{(l)}(t))^2]} \sqrt{E[(\xi^{(l)}(s))^2]} \leq V_l, \\ (0 \leq t, s \leq 1). \end{aligned} \quad (36)$$

So we get $E[c_n^2(h)] \leq 4V_l/(n\pi)^{2l}$.

(ii) *The Case That l Is an Even Number.* By $m = [(l-1)/2]$, we have $l = 2m + 2$. Since g is a random polynomial of degree $2m + 1$, we have $g^{(l)}(t) = 0$ ($0 \leq t \leq 1$). Again, use the formula of integration by parts l times to get

$$c_n(h) = -\frac{2}{(n\pi)^l} \int_0^1 \xi^{(l)}(t) \sin(\pi n t) dt. \quad (37)$$

With the help of a similar argument, we also deduce that

$$\begin{aligned} |E[c_n(h)]| &\leq \frac{2E_l}{(n\pi)^l}, \\ E[c_n^2(h)] &\leq \frac{4V_l}{(n\pi)^{2l}}. \end{aligned} \quad (38)$$

From a known formula, we have $\text{Var}(c_n(h)) = E[c_n^2(h)] - (E[c_n(h)])^2$, so $\text{Var}(c_n(h)) \leq E[c_n^2(h)]$.

Summarizing the above results, we get the following.

Theorem 2. *Let ξ be a random process on $[0, 1]$ and $\xi^{(l)} \in C([0, 1])$ and let ξ be expanded as in (27). Then Fourier sine coefficients $c_n(h)$ satisfy*

$$\begin{aligned} E[c_n(h)] &= o\left(\frac{1}{n^l}\right), \quad E[c_n^2(h)] = o\left(\frac{1}{n^{2l}}\right), \\ \text{Var}(c_n(h)) &= o\left(\frac{1}{n^{2l}}\right), \\ |E[c_n(h)]| &\leq \frac{2E_l}{(n\pi)^l}, \quad E[c_n^2(h)] \leq \frac{4V_l}{(n\pi)^{2l}}, \end{aligned} \quad (39)$$

$$\text{Var}(c_n(h)) \leq \frac{4V_l}{(n\pi)^{2l}},$$

where E_l, V_l are stated as in (28).

Remark 3. Under the condition of Theorem 2, if we directly expand ξ into the traditional Fourier series or Fourier sine series, it is easy to prove that the expectations and variances of coefficients $c_n(\xi)$ satisfy

$$\begin{aligned} E[c_n(\xi)] &= O\left(\frac{1}{n}\right), \\ \text{Var}(c_n(\xi)) &= O\left(\frac{1}{n^2}\right). \end{aligned} \quad (40)$$

Next, we consider the partial sum approximation of expansion (27). We define its partial sum as

$$\begin{aligned} \varphi_N(\xi; t) &= \sum_{i=0}^m \left(\xi^{(2i)}(0) a_i(1-t) + \xi^{(2i)}(1) a_i(t) \right) \\ &\quad + \sum_{n=1}^N c_n(h) \sin(\pi n t), \end{aligned} \quad (41)$$

where $m = [(l-1)/2]$ and the second term of the right-hand side is the partial sum $S_N(h)$ of the Fourier sine series (26) of h .

The partial sum $\varphi_N(\xi; t)$ is a combination of an l -degree random polynomial and an N -order random sine polynomial. This is a good approximation tool and can attain the best approximation order for l -degree continuously differentiable random processes on $[0, 1]$.

In fact, by (27) and (41), $\varphi_N(\xi) - \xi = S_N(h) - h$. Denote $\|f\|_2^2 = \int_0^1 f^2(t) dt$. By the Parseval identity,

$$2E[\|\varphi_N(\xi) - \xi\|_2^2] = 2E[\|S_N(h) - h\|_2^2] = \sum_{n=N+1}^{\infty} E[c_n^2(h)]. \quad (42)$$

By Theorem 2, we have $E[c_n^2(h)] \leq 4V_l/(n\pi)^{2l}$, so

$$\begin{aligned} E[\|\varphi_N - \xi\|_2^2] &\leq 2V_l \sum_{n=N+1}^{\infty} \frac{1}{(n\pi)^{2l}} \leq \frac{2V_l}{\pi^{2l}} \int_N^{\infty} \frac{dt}{t^{2l}} \\ &= \frac{2V_l}{(2l-1)\pi(N\pi)^{2l-1}}. \end{aligned} \quad (43)$$

So we get the following.

Theorem 4. Let ξ be an l -order continuously differentiable random process on $[0, 1]$ and let φ_N be the partial sum of its Fourier expansion with polynomial terms which is stated in (41). Then the mean square error

$$MSE^2 = E[\|\varphi_N - \xi\|_2^2] \leq \frac{2V_l}{(2l-1)\pi(N\pi)^{2l-1}}, \quad (44)$$

where V_l is stated in (28).

Theorem 4 gives the best approximation order for l -order continuously differentiable random processes on $[0, 1]$ even if it can be smoothly extended to a 2-periodic random process with the same smoothness, its approximation order is just $MSE^2 = O(1/N^{2l-1})$.

Remark 5. Under the condition of Theorem 4, it is easy to prove that the partial sum $S_N(\xi)$ of the Fourier expansion or Fourier sine expansion of ξ only satisfies $MSE^2 = E[\|S_N(\xi) - \xi\|_2^2] = O(1/N)$.

Comparing Theorems 2 and 4 with Remarks 3 and 5, we see that the Fourier expansion with a polynomial term is better than the Fourier expansion.

5. Two-Dimensional Case

We will extend the one-dimensional results to the two-dimensional case.

5.1. Fourier Sine Expansion with Polynomial Terms. Let ξ be a two-dimensional real random process and $D^{l_1, l_2} \xi \in C([0, 1]^2)$. By the decomposition formula of two-dimensional random processes, we have $\xi(t_1, t_2) = h_1(t_1, t_2) + h_2(t_1, t_2) + h_3(t_1, t_2)$, where h_1 , h_2 , and h_3 are stated as in (15)–(18). For convenience, in this decomposition, we denote

$$\begin{aligned} g_{i,1} &= D^{2i_1, 0} \xi_1(0, \cdot), & g_{i,2} &= D^{2i_1, 0} \xi_1(1, \cdot), \\ \varphi_{i,1} &= D^{0, 2i_2} \xi_1(\cdot, 0), & \varphi_{i,2} &= D^{0, 2i_2} \xi_1(\cdot, 1), \end{aligned} \quad (45)$$

where $\xi_1 = \xi - h_1$. We expand the one-dimensional processes $g_{i,1}(t_2)$, $g_{i,2}(t_1)$, $\varphi_{i,1}(t_2)$, and $\varphi_{i,2}(t_1)$ into the one-dimensional Fourier sine series,

$$\begin{aligned} g_{i,1}(t_2) &= \sum_{n=1}^{\infty} c_n(g_{i,1}) \sin(\pi n t_2), \quad \text{where} \\ c_n(g_{i,1}) &= 2 \int_0^1 g_{i,1}(t_2) \sin(\pi n t_2) dt_2, \quad (\nu = 1, 2), \\ \varphi_{i,1}(t_2) &= \sum_{n=1}^{\infty} c_n(\varphi_{i,1}) \sin(\pi n t_2), \quad \text{where} \\ c_n(\varphi_{i,1}) &= 2 \int_0^1 \varphi_{i,1}(t_2) \sin(\pi n t_2) dt_2, \quad (\nu = 1, 2), \end{aligned} \quad (46)$$

and expand the two-dimensional process $h_3(t_1, t_2)$ into the two-dimensional Fourier sine series,

$$h_3(t_1, t_2) = \sum_{n_1, n_2=1}^{\infty} c_{n_1, n_2}(h_3) \sin(\pi n_1 t_1) \sin(\pi n_2 t_2), \quad (47)$$

where

$$c_n(h_3) = 4 \int_0^1 \int_0^1 h_3(t_1, t_2) \sin(\pi n t_1) \sin(\pi n t_2) dt_1 dt_2. \quad (48)$$

We finally expand ξ into the Fourier sine series with polynomial terms as follows:

$$\begin{aligned}
& \xi(t_1, t_2) \\
&= \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} D^{2i_1, 2i_2} \xi(0, 0) a_{i_1}(1-t_1) a_{i_2}(1-t_2) \\
&+ \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} D^{2i_1, 2i_2} \xi(0, 1) a_{i_1}(1-t_1) a_{i_2}(t_2) \\
&+ \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} D^{2i_1, 2i_2} \xi(1, 0) a_{i_1}(t_1) a_{i_2}(1-t_2) \\
&+ \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} D^{2i_1, 2i_2} \xi(1, 1) a_{i_1}(t_1) a_{i_2}(t_2) \\
&+ \sum_{i_1=0}^{m_1} \left(a_{i_1}(1-t_1) \sum_{n=1}^{\infty} c_n(g_{i_1,1}) \sin(\pi n t_2) + a_{i_1}(t_1) \right. \\
&\quad \cdot \left. \sum_{n=1}^{\infty} c_n(g_{i_1,2}) \sin(\pi n t_2) \right) \\
&+ \sum_{i_2=0}^{m_2} \left(a_{i_2}(1-t_2) \sum_{n=1}^{\infty} c_n(\varphi_{i_2,1}) \sin(\pi n t_1) + a_{i_2}(t_2) \right. \\
&\quad \cdot \left. \sum_{n=1}^{\infty} c_n(\varphi_{i_2,2}) \sin(\pi n t_1) \right) \\
&+ \sum_{n_1, n_2=1}^{\infty} c_{n_1, n_2}(h_3) \sin(\pi n_1 t_1) \sin(\pi n_2 t_2), \\
&\quad \left(m_\nu = \left\lfloor \frac{l_\nu - 1}{2} \right\rfloor, \quad \nu = 1, 2 \right),
\end{aligned} \tag{49}$$

where a_i are the fundamental polynomials stated in Section 3.

Now we estimate the expectations and variances of these Fourier sine coefficients: $c_n(g_{i_1, \nu})$, $c_n(\varphi_{i_2, \nu})$ ($\nu = 1, 2$), and $c_{n_1, n_2}(h_3)$. Denote

$$\begin{aligned}
E_{l_1, l_2} &= \max_{\substack{0 \leq t_1, t_2 \leq 1 \\ 1 \leq i_1 \leq l_1 \\ 1 \leq i_2 \leq l_2}} \left| E \left[D^{i_1, i_2} \xi(t_1, t_2) \right] \right|, \\
V_{l_1, l_2} &= \max_{\substack{0 \leq t_1, t_2 \leq 1 \\ 1 \leq i_1 \leq l_1 \\ 1 \leq i_2 \leq l_2}} \left(E \left[\left(D^{i_1, i_2} \xi(t_1, t_2) \right)^2 \right] \right).
\end{aligned} \tag{50}$$

Note that

$$g_{i_1, 1}(t_2) = D^{2i_1, 0} \xi_1(0, t_2), \quad (\xi_1 = \xi - h_1). \tag{51}$$

Thus $g_{i_1, 1}^{(l_2)} \in C([0, 1])$. By Theorem 1(i), we obtain that, for $0 \leq i_2 \leq [(l_2 - 1)/2]$,

$$\begin{aligned}
g_{i_1, 1}^{(2i_2)}(0) &= D^{2i_1, 2i_2} \xi_1(0, 0) \\
&= D^{2i_1, 2i_2} \xi(0, 0) - D^{2i_1, 2i_2} h_1(0, 0) = 0.
\end{aligned} \tag{52}$$

Similarly, $g_{i_1, 1}^{(2i_2)}(1) = 0$.

First, we assume that the index l_2 is an odd number. By the integration by parts,

$$c_n(g_{i_1, 1}) = \frac{2}{(n\pi)^{l_2}} \int_0^1 g_{i_1, 1}^{(l_2)}(t_2) \cos(\pi n t_2) dt_2. \tag{53}$$

Since $\xi_1 = \xi - h_1$, we have by (45) that

$$g_{i_1, 1}^{(l_2)}(t_2) = D^{2i_1, l_2} \xi_1(0, t_2) = D^{2i_1, l_2} \xi(0, t_2) - D^{2i_1, l_2} h_1(0, t_2). \tag{54}$$

Again, by (10) and (15), we know that $h_1(t_1, t_2)$ is a random polynomial of degree l_2 with respect to t_2 . Therefore, $D^{2i_1, l_2} h_1(0, t_2) = L$ is a random variable which is independent of t_2 , so we have

$$\int_0^1 D^{2i_1, l_2} h_1(0, t_2) \cos(\pi n t_2) dt_2 = L \int_0^1 \cos(\pi n t_2) dt_2 = 0. \tag{55}$$

From this and (53), we deduce that, for $0 \leq i_1 \leq [(l_1 - 1)/2]$,

$$c_n(g_{i_1, 1}) = \frac{2}{(n\pi)^{l_2}} \int_0^1 D^{2i_1, l_2} \xi(0, t_2) \cos(\pi n t_2) dt_2. \tag{56}$$

So, by (50), $|E[c_n(g_{i_1, 1})]| \leq 2E_{l_1, l_2}/(n\pi)^{l_2}$ and

$$\begin{aligned}
E[c_n^2(g_{i_1, 1})] &= \frac{4}{(n\pi)^{2l_2}} \int_0^1 \int_0^1 R_{i_1}(t_2, s_2) \cos(\pi n t_2) \\
&\quad \cdot \cos(\pi n s_2) dt_2 ds_2,
\end{aligned} \tag{57}$$

where $R_{i_1}(t_2, s_2) = E[D^{2i_1, l_2} \xi(0, t_2) D^{2i_1, l_2} \xi(0, s_2)]$. By the Schwarz inequality and (50), we have

$$\begin{aligned}
& |R_{i_1}(t_2, s_2)| \\
&\leq \left(E \left[\left(D^{2i_1, l_2} \xi(0, t_2) \right)^2 \right] \right)^{1/2} \left(E \left[\left(D^{2i_1, l_2} \xi(0, s_2) \right)^2 \right] \right)^{1/2} \\
&\leq V_{l_1, l_2}, \quad (0 \leq t_2, s_2 \leq 1).
\end{aligned} \tag{58}$$

So $E[c_n^2(g_{i_1, 1})] \leq 4V_{l_1, l_2}/(n\pi)^{2l_2}$ ($0 \leq i_1 \leq [(l_1 - 1)/2]$).

Now we assume that the index l_2 is an even number. Then

$$c_n(g_{i_1,1}) = -\frac{2}{(n\pi)^{l_2}} \int_0^1 D^{2i_1, l_2} \xi(0, t_2) \sin(\pi n t_2) dt_2. \quad (59)$$

Similarly, we also can get that, for $0 \leq i_1 \leq [(l_1 - 1)/2]$,

$$\begin{aligned} |E[c_n(g_{i_1,1})]| &\leq \frac{2E_{l_1, l_2}}{(n\pi)^{l_2}}, \\ E[c_n^2(g_{i_1,1})] &\leq \frac{4V_{l_1, l_2}}{(n\pi)^{2l_2}}. \end{aligned} \quad (60)$$

Again, since $\text{Var}(c_n(g_{i_1,1})) = E[c_n^2(g_{i_1,1})] - (E[c_n(g_{i_1,1})])^2$, we deduce that

$$\text{Var}(c_n(g_{i_1,1})) \leq E[c_n^2(g_{i_1,1})]. \quad (61)$$

Similarly, we discuss the coefficients $c_n(g_{i_1,2})$, $c_n(\varphi_{i_2,1})$, and $c_n(\varphi_{i_2,2})$. Finally, we deduce the following.

Theorem 6. Let ξ be a random process on $[0, 1]^2$ and $D^{l_1, l_2} \xi \in C([0, 1]^2)$ and let $g_{i_1, \nu}$, $\varphi_{i_2, \nu}$ be stated as in (45). Then, for $0 \leq i_1 \leq [(l_1 - 1)/2]$ and $0 \leq i_2 \leq [(l_2 - 1)/2]$,

$$|E[c_n(g_{i_1, \nu})]| \leq \frac{2E_{l_1, l_2}}{(n\pi)^{l_2}}, \quad E[c_n^2(g_{i_1, \nu})] \leq \frac{4V_{l_1, l_2}}{(n\pi)^{2l_2}},$$

$$\text{Var}(c_n(g_{i_1, \nu})) \leq \frac{4V_{l_1, l_2}}{(n\pi)^{2l_2}},$$

$$|E[c_n(\varphi_{i_2, \nu})]| \leq \frac{2E_{l_1, l_2}}{(n\pi)^{l_1}}, \quad E[c_n^2(\varphi_{i_2, \nu})] \leq \frac{4V_{l_1, l_2}}{(n\pi)^{2l_1}},$$

$$\text{Var}(c_n(\varphi_{i_2, \nu})) \leq \frac{4V_{l_1, l_2}}{(n\pi)^{2l_1}} \quad (\nu = 1, 2), \quad (62)$$

where E_{l_1, l_2} and V_{l_1, l_2} are stated in (50).

By Theorem 1(ii), we know that $h_3(t_1, t_2)$ satisfies

$$D^{2i_1, 2i_2} h_3(t_1, t_2) = 0,$$

$$(t_1, t_2) \in \partial([0, 1]^2), \quad 0 \leq i_\nu \leq \left\lfloor \frac{(l_\nu - 1)}{2} \right\rfloor, \quad (\nu = 1, 2) \quad (63)$$

and $D^{l_1, l_2} h_3 \in C([0, 1]^2)$. If l_1 and l_2 are both odd numbers, using the integration by parts, we get

$$\begin{aligned} c_{n_1, n_2}(h_3) &= \frac{4}{(n_1\pi)^{l_1} (n_2\pi)^{l_2}} \int_0^1 \int_0^1 D^{l_1, l_2} h_3(t_1, t_2) \cos(\pi n_1 t_1) \\ &\quad \cdot \cos(\pi n_2 t_2) dt_1 dt_2 \\ &= \frac{\tilde{c}_{n_1, n_2}(D^{l_1, l_2} h_3)}{(n_1\pi)^{l_1} (n_2\pi)^{l_2}}, \end{aligned} \quad (64)$$

where

$$\tilde{c}_{n_1, n_2}(f) = 4 \int_0^1 \int_0^1 f(t_1, t_2) \cos(\pi n_1 t_1) \cos(\pi n_2 t_2) dt_1 dt_2, \quad (65)$$

which is the Fourier cosine coefficients of f on $[0, 1]^2$.

From $\xi = h_1 + h_2 + h_3$, it follows that

$$\begin{aligned} \tilde{c}_{n_1, n_2}(D^{l_1, l_2} h_3) &= \tilde{c}_{n_1, n_2}(D^{l_1, l_2} \xi) - \tilde{c}_{n_1, n_2}(D^{l_1, l_2} h_1) \\ &\quad - \tilde{c}_{n_1, n_2}(D^{l_1, l_2} h_2). \end{aligned} \quad (66)$$

First, we consider the Fourier coefficients of $D^{l_1, l_2} h_1$. Noticing that $a_i(t)$ is a polynomial of degree $2i + 1$, we know by (15) that $D^{l_1, l_2} h_1(t_1, t_2)$ is a random variable which is independent of t_1, t_2 . So we have

$$\begin{aligned} \tilde{c}_{n_1, n_2}(D^{l_1, l_2} h_1) \\ = 4 \int_0^1 \int_0^1 D^{l_1, l_2} h_1 \cos(\pi n_1 t_1) \cos(\pi n_2 t_2) dt_1 dt_2 = 0. \end{aligned} \quad (67)$$

By (16), we have

$$\begin{aligned} \tilde{c}_{n_1, n_2}(D^{l_1, l_2} h_2) &= \sum_{i_1=0}^{m_1} \tilde{c}_{n_1, n_2}(D^{l_1, l_2} A_{i_1} + D^{l_1, l_2} B_{i_1}) \\ &\quad + \sum_{i_2=0}^{m_2} \tilde{c}_{n_1, n_2}(D^{l_1, l_2} \tilde{A}_{i_2} + D^{l_1, l_2} \tilde{B}_{i_2}). \end{aligned} \quad (68)$$

By (18) and (10), we deduce that $\tilde{c}_{n_1}(a_{i_1}^{(l_2)}(1 - \cdot))$ vanishes and

$$\tilde{c}_{n_1, n_2}(D^{l_1, l_2} A_{i_1}) = \tilde{c}_{n_2}(g_{i_1, 1}^{(l_2)}) \tilde{c}_{n_1}(a_{i_1}^{(l_2)}(1 - \cdot)) = 0, \quad (69)$$

where $\tilde{c}_n(f) = 2 \int_0^1 f(t) \cos(\pi n t) dt$. Similarly, we have

$$\tilde{c}_{n_1, n_2}(D^{l_1, l_2} B_{i_1}) = \tilde{c}_{n_1, n_2}(D^{l_1, l_2} \tilde{A}_{i_2}) = \tilde{c}_{n_1, n_2}(D^{l_1, l_2} \tilde{B}_{i_2}) = 0. \quad (70)$$

So we get $\tilde{c}_{n_1, n_2}(D^{l_1, l_2} h_2) = 0$. From this, (66), and (67),

$$\tilde{c}_{n_1, n_2}(D^{l_1, l_2} h_3) = \tilde{c}_{n_1, n_2}(D^{l_1, l_2} \xi). \quad (71)$$

So, by (64), we get

$$\begin{aligned} c_{n_1, n_2}(h_3) &= \frac{\tilde{c}_{n_1, n_2}(D^{l_1, l_2} \xi)}{(n_1 \pi)^{l_1} (n_2 \pi)^{l_2}} \\ &= \frac{4}{(\pi n_1)^{l_1} (\pi n_2)^{l_2}} \int_0^1 \int_0^1 D^{l_1, l_2} \xi(t_1, t_2) \cos(\pi n_1 t_1) \\ &\quad \cdot \cos(\pi n_2 t_2) dt_1 dt_2. \end{aligned} \quad (72)$$

So

$$\begin{aligned} E[c_{n_1, n_2}(h_3)] &\leq \frac{4E_{l_1, l_2}}{(\pi n_1)^{l_1} (\pi n_2)^{l_2}} \quad (\text{by (50)}), \\ E[c_{n_1, n_2}^2(h_3)] &= \frac{16}{(\pi n_1)^{2l_1} (\pi n_2)^{2l_2}} \\ &\quad \cdot \int_{[0,1]^4} E[D^{l_1, l_2} \xi(t_1, t_2) D^{l_1, l_2} \xi(s_1, s_2)] \\ &\quad \cdot \cos(\pi n_1 t_1) \cos(\pi n_2 t_2) \cos(\pi n_1 s_1) \\ &\quad \cdot \cos(\pi n_2 s_2) dt_1 dt_2 ds_1 ds_2. \end{aligned} \quad (73)$$

By the Schwarz inequality and (50), we get

$$\begin{aligned} &|E[D^{l_1, l_2} \xi(t_1, t_2) D^{l_1, l_2} \xi(s_1, s_2)]| \\ &\leq \left(E[(D^{l_1, l_2} \xi(t_1, t_2))^2]\right)^{1/2} \left(E[(D^{l_1, l_2} \xi(s_1, s_2))^2]\right)^{1/2} \\ &\leq V_{l_1, l_2}, \end{aligned} \quad (74)$$

so

$$E[c_{n_1, n_2}^2(h_3)] \leq \frac{16V_{l_1, l_2}}{(\pi n_1)^{2l_1} (\pi n_2)^{2l_2}}. \quad (75)$$

From this and $\text{Var}(c_{n_1, n_2}(h_3)) \leq E[c_{n_1, n_2}^2(h_3)]$, we have

$$\text{Var}(c_{n_1, n_2}(h_3)) \leq \frac{16V_{l_1, l_2}}{(n_1 \pi)^{2l_1} (n_2 \pi)^{2l_2}}. \quad (76)$$

We easily check that whether l_1 and l_2 are odd or even, this formula and (73)–(75) always hold. Again, by the Riemann-Lebesgue lemma, we have the following.

Theorem 7. Let ξ be a random process on $[0, 1]^2$ and $D^{l_1, l_2} \xi \in C([0, 1]^2)$ and let ξ be expanded as in (49). Then the Fourier coefficients $c_{n_1, n_2}(h_3)$ satisfy

$$\begin{aligned} |E[c_{n_1, n_2}(h_3)]| &= o\left(\frac{1}{n_1^{l_1} n_2^{l_2}}\right), \\ E[c_{n_1, n_2}^2(h_3)] &= o\left(\frac{1}{n_1^{2l_1} n_2^{2l_2}}\right), \\ \text{Var}(c_{n_1, n_2}(h_3)) &= o\left(\frac{1}{n_1^{2l_1} n_2^{2l_2}}\right), \\ |E[c_{n_1, n_2}(h_3)]| &\leq \frac{4E_{l_1, l_2}}{(\pi n_1)^{l_1} (\pi n_2)^{l_2}}, \\ E[c_{n_1, n_2}^2(h_3)] &\leq \frac{16V_{l_1, l_2}}{(\pi n_1)^{2l_1} (\pi n_2)^{2l_2}}, \\ \text{Var}(c_{n_1, n_2}(h_3)) &\leq \frac{16E_{l_1, l_2}^2}{(n_1 \pi)^{2l_1} (n_2 \pi)^{2l_2}}, \end{aligned} \quad (77)$$

where E_{l_1, l_2} and V_{l_1, l_2} are stated as in (50).

Remark 8. Under the condition of Theorem 7, if we directly expand ξ into the traditional Fourier series or Fourier sine series, it is easy to prove that the expectations and variances of coefficients $c_{n_1, n_2}(\xi)$ only satisfy

$$\begin{aligned} E[c_{n_1, n_2}(\xi)] &= O\left(\frac{1}{n_1 n_2}\right), \\ \text{Var}(c_{n_1, n_2}(\xi)) &= O\left(\frac{1}{n_1^2 n_2^2}\right). \end{aligned} \quad (78)$$

5.2. A Good Approximation Tool of Random Processes. We assume that ξ is a real random process on $[0, 1]^2$ and $D^{l_1, l_2} \xi \in C([0, 1]^2)$. We expand it into Fourier sine series with polynomial terms (49). Replace “ ∞ ” on the right-hand side of (49) by N to obtain partial sums of expansion (49), denoted by R_N ,

$$R_N(\xi; t_1, t_2) = P_{m_1, m_2}(t_1, t_2) + T_{m_1, m_2, N}(t_1, t_2), \quad (79)$$

where $P_{m_1, m_2}(t_1, t_2)$ is a two-dimensional random polynomial and

$$\begin{aligned} P_{m_1, m_2}(t_1, t_2) &= \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} D^{2i_1, 2i_2} \xi(0, 0) a_{i_1}(1-t_1) a_{i_2}(1-t_2) \\ &\quad + \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} D^{2i_1, 2i_2} \xi(0, 1) a_{i_1}(1-t_1) a_{i_2}(t_2) \\ &\quad + \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} D^{2i_1, 2i_2} \xi(1, 0) a_{i_1}(t_1) a_{i_2}(1-t_2) \\ &\quad + \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} D^{2i_1, 2i_2} \xi(1, 1) a_{i_1}(t_1) a_{i_2}(t_2), \end{aligned} \quad (80)$$

and $T_{m_1, m_2, N}(t_1, t_2)$ is a sum of four one-dimensional random sine polynomials with polynomial factors and a two-dimensional random sine polynomial, and

$$\begin{aligned} T_{m_1, m_2, N}(t_1, t_2) &= \sum_{i_1=0}^{m_1} (S_N(g_{i_1,1}; t_2) a_{i_1}(1-t_1) + S_N(g_{i_1,2}; t_2) a_{i_1}(t_1)) \\ &\quad + \sum_{i_2=0}^{m_2} (S_N(\varphi_{i_2,1}; t_1) a_{i_2}(1-t_2) + S_N(\varphi_{i_2,2}; t_1) a_{i_2}(t_2)) \\ &\quad + S_N(h_3; t_1, t_2) \left(m_\nu = \left[\frac{l_\nu - 1}{2} \right], \nu = 1, 2 \right), \end{aligned} \quad (81)$$

where $S_N(\psi)$ is the N th partial sum of the Fourier sine series of ψ and

$$S_N(h_3, t_1, t_2) = \sum_{n_1, n_2=1}^N c_{n_1, n_2}(h_3) \sin(\pi n_1 t_1) \sin(\pi n_2 t_2). \quad (82)$$

The partial sum $R_N(\xi; t_1, t_2)$ is a combination of a random polynomial $P_{m_1, m_2}(t_1, t_2)$ and random sine polynomials with polynomial factors $T_{m_1, m_2, N}(t_1, t_2)$. It is a good approximation tool for an l -order differentiable random process and can attain the best approximation order. In this combination, the degree of the random polynomial is determined by smoothness index of random process, while the degrees of random sine polynomials are determined by the presumed approximation error.

In fact, from this and (49), we get

$$\begin{aligned} R_N(\xi; t_1, t_2) - \xi(t_1, t_2) &= \sum_{i_1=0}^{m_1} ((S_N(g_{i_1,1}; t_2) - g_{i_1,1}(t_2)) a_{i_1}(1-t_1) \\ &\quad + (S_N(g_{i_1,2}; t_2) - g_{i_1,2}(t_2)) a_{i_1}(t_1)) \\ &\quad + \sum_{i_2=0}^{m_2} ((S_N(\varphi_{i_2,1}; t_1) - \varphi_{i_2,1}(t_1)) a_{i_2}(1-t_2) \\ &\quad + (S_N(\varphi_{i_2,2}; t_1) - \varphi_{i_2,2}(t_1)) a_{i_2}(t_2)) \\ &\quad + (h_3(t_1, t_2) - S_N(h_3; t_1, t_2)). \end{aligned} \quad (83)$$

Denote $\|f\|_2^2 = \int_0^1 \int_0^1 f^2(t_1, t_2) dt_1 dt_2$. By the inequality $(\alpha_1 + \dots + \alpha_l)^2 \leq l^2(\alpha_1^2 + \dots + \alpha_l^2)$, we have

$$\begin{aligned} &\frac{1}{(2m_1 + 2m_2 + 5)^2} E[\|R_N(\xi; t_1, t_2) - \xi(t_1, t_2)\|_2^2] \\ &\leq \sum_{i_1=0}^{m_1} E[\|(S_N(g_{i_1,1}; t_2) - g_{i_1,1}(t_2)) a_{i_1}(1-t_1)\|_2^2] \\ &\quad + \sum_{i_1=0}^{m_1} E[\|(S_N(g_{i_1,2}; t_2) - g_{i_1,2}(t_2)) a_{i_1}(t_1)\|_2^2] \end{aligned}$$

$$\begin{aligned} &+ \sum_{i_2=0}^{m_2} E[\|(S_N(\varphi_{i_2,1}; t_1) - \varphi_{i_2,1}(t_1)) a_{i_2}(1-t_2)\|_2^2] \\ &+ \sum_{i_2=0}^{m_2} E[\|(S_N(\varphi_{i_2,2}; t_1) - \varphi_{i_2,2}(t_1)) a_{i_2}(t_2)\|_2^2] \\ &+ E[\|S_N(h_3; t_1, t_2) - h_3(t_1, t_2)\|_2^2] \\ &= J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned} \quad (84)$$

Noticing that $(S_N(g_{i_1,1}; t_2) - g_{i_1,1}(t_2)) a_{i_1}(1-t_1)$ is a product of separation variables, we deduce that

$$\begin{aligned} E[\|(S_N(g_{i_1,1}; t_2) - g_{i_1,1}(t_2)) a_{i_1}(1-t_1)\|_2^2] \\ = E[\|S_N(g_{i_1,1}; t_2) - g_{i_1,1}(t_2)\|_2^2] \|a_{i_1}(1-t_1)\|_2^2. \end{aligned} \quad (85)$$

Since $a_{i_1}(t)$ is a polynomial of degree $2i_1 + 1$, by (9), we have

$$\begin{aligned} c_n(a_{i_1}(1-\cdot)) &= 2(-1)^{n-1} \int_0^1 a_{i_1}(t) \sin(\pi n t) dt \\ &= \frac{2(-1)^{n+i_1-1}}{(\pi n)^{2i_1}} \int_0^1 a_{i_1}^{(2i_1)}(t) \sin(\pi n t) dt. \end{aligned} \quad (86)$$

By (10), $a_{i_1}^{(2i_1)}(t) = t$, so

$$c_n(a_{i_1}(1-\cdot)) = \frac{2(-1)^{i_1}}{(\pi n)^{2i_1+1}}. \quad (87)$$

By the Parseval identity in the space $L^2([0, 1]^2)$, we have

$$\begin{aligned} \|a_{i_1}(1-\cdot)\|_2^2 &= \frac{1}{2} \sum_{n=1}^{\infty} |c_n(a_{i_1}(1-\cdot))|^2 \\ &= \frac{2}{\pi^{4i_1+2}} \sum_{n=1}^{\infty} \frac{1}{n^{4i_1+2}} \leq \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{3} \\ &\quad (0 \leq i_1 \leq m_1). \end{aligned} \quad (88)$$

By Theorem 6 and the Parseval identity, we have

$$\begin{aligned} E[\|S_N(g_{i_1,1} - g_{i_1,1})\|_2^2] \\ = \frac{1}{2} \sum_{n=N+1}^{\infty} E[c_n^2(g_{i_1,1})] \leq 2V_{l_1, l_2} \sum_{n=N+1}^{\infty} \frac{1}{(\pi n)^{2l_2}} \\ \leq 2V_{l_1, l_2} \int_N^{\infty} \frac{dt}{(\pi t)^{2l_2}} = \frac{2V_{l_1, l_2}}{\pi(2l_2-1)(\pi N)^{2l_2-1}}. \end{aligned} \quad (89)$$

From this, (84), and (85), we have

$$J_1 \leq \frac{(l_1 + 1) V_{l_1, l_2}}{3\pi(2l_2-1)(\pi N)^{2l_2-1}}. \quad (90)$$

Similarly, we get

$$\sum_{k=1}^4 J_k \leq \frac{4(l_M + 1)V_{l_1, l_2}}{3\pi(2l_m - 1)(\pi N)^{2l_m - 1}}, \quad (91)$$

where $l_M = \max\{l_1, l_2\}$ and $l_m = \min\{l_1, l_2\}$.

Using the two-dimensional Parseval identity and Theorem 7 and $\sum_{n_1=1}^N (1/(\pi n_1)^{2l_1}) \leq 1/6$, we get

$$\begin{aligned} J_5 &\leq 4V_{l_1, l_2} \sum_{n_1=1}^N \frac{1}{(\pi n_1)^{2l_1}} \sum_{n_2=N+1}^{\infty} \frac{1}{(\pi n_2)^{2l_2}} \\ &\quad + 4V_{l_1, l_2} \sum_{n_2=1}^N \frac{1}{(\pi n_2)^{2l_2}} \sum_{n_1=N+1}^{\infty} \frac{1}{(\pi n_1)^{2l_1}} \\ &\quad + 4V_{l_1, l_2} \sum_{n_1=N+1}^{\infty} \frac{1}{(\pi n_1)^{2l_1}} \sum_{n_2=N+1}^{\infty} \frac{1}{(\pi n_2)^{2l_2}} \\ &\leq \frac{2V_{l_1, l_2}}{\pi(2l_m - 1)(\pi N)^{2l_m - 1}}. \end{aligned} \quad (92)$$

From this and (84), we have the following.

Theorem 9. Let ξ be a random process on $[0, 1]^2$ and $D^{l_1, l_2} \xi \in C([0, 1]^2)$ and let $R_N(\xi)$ be the partial sum of its Fourier sine series with polynomial terms stated as in (79). Then the mean square error

$$MSE^2 = E[\|R_N(\xi) - \xi\|_2^2] \leq \frac{K}{\pi^2(\pi N)^{2l_m - 1}}, \quad (93)$$

where $K = ((5l_M + 11)V_{l_1, l_2})/(2l_m - 1)$ and $l_M = \max\{l_1, l_2\}$, $l_m = \min\{l_1, l_2\}$, and V_{l_1, l_2} is stated in (50).

Theorem 9 gives the best approximation order even if ξ can be smoothly extended to an l -order continuously differentiable, 2-periodic random process with the same smoothness index, its approximation order is just $MSE^2 = O(1/N^{2l_m - 1})$.

Remark 10. Under the condition of Theorem 9, it is easy to prove that the partial sum $S_N(\xi)$ of either traditional Fourier expansion or Fourier sine expansion of ξ only satisfies

$$MSE^2 = E[\|S_N(\xi) - \xi\|_2^2] = O\left(\frac{1}{N}\right). \quad (94)$$

Comparing Theorems 6, 7, and 9 with Remarks 8 and 10, we see that the Fourier expansion with polynomial terms is better than the Fourier expansion.

6. Conclusions

In this paper, we give Fourier sine expansions with polynomial terms of random processes and estimate expectations and variances of Fourier sine coefficients as well as the mean square error for their partial sums.

6.1. One-Dimensional Case. Let $\xi(t)$ be an l -order continuously differentiable random process on $[0, 1]$. If $l \geq 1$ and we expand ξ into Fourier sine series,

$$\xi(t) = \sum_{n=1}^{\infty} c_n(\xi) \sin(\pi n t), \quad (95)$$

then

$$\begin{aligned} E[c_n(\xi)] &= \frac{2}{\pi n} (E[\xi(0)] + (-1)^{n-1} E[\xi(1)]) + o\left(\frac{1}{n}\right), \\ E[\|\xi - S_N\|_2^2] &= O\left(\frac{1}{N}\right). \end{aligned} \quad (96)$$

Now we consider that its Fourier sine expansions with polynomial terms are

$$\xi(t) = g(t) + \sum_{n=1}^{\infty} c_n \sin(\pi n t), \quad (97)$$

where $c_n = 2 \int_0^1 h(t) \sin(\pi n t) dt$ ($h = \xi - g$) and $g(t)$ is a random polynomial of degree $2m + 1$:

$$\begin{aligned} g(t) &= \sum_{i=0}^m (\xi^{(2i)}(0) a_i (1-t) + \xi^{(2i)}(1) a_i(t)), \\ &\quad \left(m = \left\lfloor \frac{l-1}{2} \right\rfloor\right), \end{aligned} \quad (98)$$

$$a_0(t) = t, \quad (99)$$

$$a_1(t) = \frac{1}{6}t(t^2 - 1),$$

$$a_n(t) = \frac{1}{(2n+1)!} t^{2n+1} + \sum_{k=0}^{n-1} b_k t^{2k+1}, \quad (n = 2, 3, \dots),$$

and the coefficients b_k for $0 \leq k \leq n-1$ satisfy

$$\sum_{k=j}^{n-1} \frac{(2k+1)!}{(2k-2j+1)!} b_k = -\frac{1}{(2n-2j+1)!}, \quad (j = 0, \dots, n-1). \quad (99)$$

The following results hold.

(i) The expectations and variances of c_n satisfy

$$\begin{aligned} |E[c_n(h)]| &\leq \frac{2E_l}{(n\pi)^l}, \\ \text{Var}(c_n(h)) &\leq \frac{4V_l}{(n\pi)^{2l}}, \end{aligned} \quad (100)$$

where

$$\begin{aligned} E_l &= \max_{0 \leq t \leq 1} |E[\xi^{(l)}(t)]|, \\ V_l &= \max_{0 \leq t \leq 1} (E[(\xi^{(l)}(t))^2]). \end{aligned} \quad (101)$$

(ii) If $\varphi_N(\xi, t)$ is the N th partial sum of (97),

$$\begin{aligned} \varphi_N(\xi, t) = & \sum_{i=0}^m \left(\xi^{(2i)}(0) a_i(1-t) + \xi^{(2i)}(1) a_i(t) \right) \\ & + \sum_{n=1}^N c_n(h) \sin(\pi n t), \quad \left(m = \left\lfloor \frac{l-1}{2} \right\rfloor \right), \end{aligned} \quad (102)$$

then the mean square error

$$E \left[\|\varphi_N - \xi\|_2^2 \right] \leq \frac{2V_l}{(2l-1)\pi(N\pi)^{2l-1}}. \quad (103)$$

From this and (96), we see that the Fourier expansion with polynomial terms has an obvious advantage over the traditional Fourier expansion.

6.2. Two-Dimensional Case. Let ξ be a two-dimensional random process on $[0, 1]^2$ and $D^{l_1, l_2} \xi \in C([0, 1]^2)$. Its Fourier sine expansion with polynomial terms is

$$\xi(t_1, t_2) = P_{m_1, m_2}(t_1, t_2) + T_{m_1, m_2}(t_1, t_2), \quad (104)$$

where

$$\begin{aligned} P_{m_1, m_2}(t_1, t_2) = & \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} D^{2i_1, 2i_2} \xi(0, 0) a_{i_1}(1-t_1) a_{i_2}(1-t_2) \\ & + \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} D^{2i_1, 2i_2} \xi(0, 1) a_{i_1}(1-t_1) a_{i_2}(t_2) \\ & + \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} D^{2i_1, 2i_2} \xi(1, 0) a_{i_1}(t_1) a_{i_2}(1-t_2) \\ & + \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} D^{2i_1, 2i_2} \xi(1, 1) a_{i_1}(t_1) a_{i_2}(t_2) \end{aligned} \quad (105)$$

is a two-dimensional random polynomial and

$$\begin{aligned} T_{m_1, m_2}(t_1, t_2) = & \sum_{i_1=0}^{m_1} \left(a_{i_1}(1-t_1) \sum_{n=1}^{\infty} c_n(g_{i_1, 1}) \sin(\pi n t_2) \right. \\ & \left. + a_{i_1}(t_1) \sum_{n=1}^{\infty} c_n(g_{i_1, 2}) \sin(\pi n t_2) \right) \\ & + \sum_{i_2=0}^{m_2} \left(a_{i_2}(1-t_2) \sum_{n=1}^{\infty} c_n(\varphi_{i_2, 1}) \sin(\pi n t_1) \right. \\ & \left. + a_{i_2}(t_2) \sum_{n=1}^{\infty} c_n(\varphi_{i_2, 2}) \sin(\pi n t_1) \right) \\ & + \sum_{n_1, n_2=1}^{\infty} c_{n_1, n_2}(h) \sin(\pi n_1 t_1) \sin(\pi n_2 t_2) \\ & \left(m_1 = \left\lfloor \frac{l_1-1}{2} \right\rfloor, m_2 = \left\lfloor \frac{l_2-1}{2} \right\rfloor \right). \end{aligned} \quad (106)$$

For $0 \leq i_1 \leq m$ and $0 \leq i_2 \leq m_2$, the Fourier sine coefficients are as follows:

$$\begin{aligned} c_n(g_{i_1, \nu}) &= 2 \int_0^1 g_{i_1, \nu}(t_2) \sin(\pi n t_2) dt_2, \\ c_n(\varphi_{i_2, \nu}) &= 2 \int_0^1 \varphi_{i_2, \nu}(t_2) \sin(\pi n t_2) dt_2, \end{aligned} \quad (107)$$

$(\nu = 1, 2),$

where

$$\begin{aligned} g_{i_1, 1} &= D^{2i_1, 0} \xi_1(0, \cdot), & g_{i_1, 2} &= D^{2i_1, 0} \xi_1(1, \cdot), \\ \varphi_{i_2, 1} &= D^{0, 2i_2} \xi_1(\cdot, 0), & \varphi_{i_2, 2} &= D^{0, 2i_2} \xi_1(\cdot, 1), \\ & & (\xi_1 &= \xi - P_{m_1, m_2}(t_1, t_2)), \end{aligned}$$

$$c_n(h) = 4 \int_0^1 \int_0^1 h(t_1, t_2) \sin(\pi n t_1) \sin(\pi n t_2) dt_1 dt_2, \quad (108)$$

where

$$\begin{aligned} h(t_1, t_2) &= \xi(t_1, t_2) - P_{m_1, m_2}(t_1, t_2) \\ &= \sum_{i_1=0}^{m_1} \left(D^{2i_1, 0} \xi_1(0, t_2) a_{i_1}(1-t_1) \right. \\ &\quad \left. + D^{2i_1, 0} \xi_1(1, t_2) a_{i_1}(t_1) \right) \\ &\quad - \sum_{i_2=0}^{m_2} \left(D^{0, 2i_2} \xi_1(t_1, 0) a_{i_2}(1-t_2) \right. \\ &\quad \left. + D^{0, 2i_2} \xi_1(t_1, 1) a_{i_2}(t_2) \right), \\ &\quad (\xi_1 = \xi - P_{m_1, m_2}). \end{aligned} \quad (109)$$

For $0 \leq i_1 \leq [(l_1 - 1)/2]$ and $0 \leq i_2 \leq [(l_2 - 1)/2]$, the following estimates hold:

(i)

$$\begin{aligned} |E[c_n(g_{i_1, \nu})]| &\leq \frac{2E_{l_1, l_2}}{(\pi n)^{l_2}}, & \text{Var}(c_n(g_{i_1, \nu})) &\leq \frac{4V_{l_1, l_2}}{(\pi n)^{2l_2}}, \\ |E[c_n(\varphi_{i_2, \nu})]| &\leq \frac{2E_{l_1, l_2}}{(\pi n)^{l_1}}, & \text{Var}(c_n(\varphi_{i_2, \nu})) &\leq \frac{4V_{l_1, l_2}}{(\pi n)^{2l_1}} \end{aligned}$$

$(\nu = 1, 2),$

$$\begin{aligned} |E[c_{n_1, n_2}(h)]| &\leq \frac{4E_{l_1, l_2}}{(\pi n_1)^{l_1} (\pi n_2)^{l_2}}, \\ \text{Var}(c_{n_1, n_2}(h)) &\leq \frac{16V_{l_1, l_2}}{(n_1 \pi)^{2l_1} (n_2 \pi)^{2l_2}}. \end{aligned} \quad (110)$$

(ii) Let $R_N(\xi)$ be the N th partial sum of (104):

$$R_N(\xi; t_1, t_2) = P_{m_1, m_2}(t_1, t_2) + T_{m_1, m_2, N}(t_1, t_2), \quad (111)$$

where $T_{m_1, m_2, N}$ is obtained by replacing “ ∞ ” on the right-hand side of (106) by N :

$$\begin{aligned}
 T_{m_1, m_2, N}(t_1, t_2) &= \sum_{i_1=0}^{m_1} \left(a_{i_1} (1 - t_1) \sum_{n=1}^N c_n (g_{i_1,1}) \sin(\pi n t_2) \right. \\
 &\quad \left. + a_{i_1} (t_1) \sum_{n=1}^N c_n (g_{i_1,2}) \sin(\pi n t_2) \right) \\
 &\quad + \sum_{i_2=0}^{m_2} \left(a_{i_2} (1 - t_2) \sum_{n=1}^N c_n (\varphi_{i_2,1}) \sin(\pi n t_1) \right. \\
 &\quad \left. + a_{i_2} (t_2) \sum_{n=1}^N c_n (\varphi_{i_2,2}) \sin(\pi n t_1) \right) \\
 &\quad + \sum_{n_1, n_2=1}^N c_{n_1, n_2}(h) \sin(\pi n_1 t_1) \sin(\pi n_2 t_2) \\
 &\quad \left(m_1 = \left\lfloor \frac{l_1 - 1}{2} \right\rfloor, m_2 = \left\lfloor \frac{l_2 - 1}{2} \right\rfloor \right). \tag{112}
 \end{aligned}$$

Then the mean square error

$$E \left[\|R_N(\xi) - \xi\|_2^2 \right] \leq \frac{K}{\pi^2 (\pi N)^{2l_m - 1}}, \tag{113}$$

where $K = ((5l_M + 11)V_{l_1, l_2})/(2l_m - 1)$, $l_M = \max\{l_1, l_2\}$, and $l_m = \min\{l_1, l_2\}$.

$R_N(\xi; t_1, t_2)$ is a combination of random polynomials and random sine polynomials. This is a good approximation tool which can attain the best approximation order.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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