Research Article

On the Geometry of Müntz Spaces

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Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ satisfy $0 < \lambda_1 < \lambda_2 < \dots$, $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$ and $\inf_k (\lambda_{k+1} - \lambda_k) > 0$. We investigate the Müntz spaces $M_p(\Lambda) = \overline{\text{span}}\{t^{\lambda_k} : k = 1, 2, \dots\} \subset L_p(0, 1)$ for $1 \leq p \leq \infty$. We show that, for each p , there is a Müntz space $\mathcal{F}_p$ which contains isomorphic copies of all Müntz spaces as complemented subspaces. $\mathcal{F}_p$ is uniquely determined up to isomorphisms by this maximality property. We discuss explicit descriptions of $\mathcal{F}_p$. In particular $\mathcal{F}_p$ is isomorphic to a Müntz space $M_p(\widehat{\Lambda})$ where $\widehat{\Lambda}$ consists of positive integers. Finally we show that the Banach spaces $(\sum_n \oplus F_n)_p$ for $1 \leq p < \infty$ and $(\sum_n \oplus F_n)_0$ for $p = \infty$ are always isomorphic to suitable Müntz spaces $M_p(\Lambda)$ if the F_n are the spans of arbitrary finitely many monomials over $[0, 1]$.

1. Introduction

Our paper is concerned with the Banach space geometry of Müntz spaces which is largely unexplored.

Let $\Lambda = \{\lambda_k : k = 1, 2, \dots\}$ be a sequence of real numbers with $0 < \lambda_1 < \lambda_2 < \dots$ and

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty, \quad (1)$$

$$\inf_k (\lambda_{k+1} - \lambda_k) > 0.$$

For $1 \leq p < \infty$ and $f : [0, 1] \rightarrow \mathbb{R}$, put

$$\|f\|_p = \left(\int_0^1 |f(t)|^p dt \right)^{1/p}, \quad (2)$$

$$\|f\|_{\infty} = \sup_{t \in [0, 1]} |f(t)|.$$

For $\lambda > 0$, let t^{λ} be the function $f : [0, 1] \rightarrow \mathbb{R}$ with $f(t) = t^{\lambda}$. In this paper, we consider the Müntz spaces

$$M_p(\Lambda) = \overline{\text{span}}\{t^{\lambda_k} : k = 1, 2, \dots\} \subset L_p(0, 1) \quad (3)$$

for $1 \leq p \leq \infty$.

By the Müntz theorem [1, 2], it is clear that $M_p(\Lambda) \neq L_p(0, 1)$ if $1 \leq p < \infty$ and $M_{\infty}(\Lambda) \neq C_0(0, 1) := \{f : [0, 1] \rightarrow \mathbb{R} : f \text{ continuous}, f(0) = 0\}$. It is known that $M_p(\Lambda)$ is isomorphic to l_p for $1 \leq p < \infty$ and $M_{\infty}(\Lambda)$ is isomorphic to c_0 provided that Λ satisfies in addition the lacunary condition $\inf_k (\lambda_{k+1}/\lambda_k) > 1$ [3]. On the other hand, there are examples of Müntz spaces $M_{\infty}(\Lambda)$ which are not isomorphic to c_0 ([4] or [5, Corollary 10.2.4.]).

We want to show that, for fixed p , all Müntz spaces are isomorphic to complemented subspaces of one special function space which itself is a Müntz space $M_p(\widehat{\Lambda})$ where $\widehat{\Lambda}$ consists of positive integers.

For two Banach spaces X and Y , we write $X \sim Y$ if X is isomorphic to Y .

For a sequence of Banach spaces E_n , we put

$$\left(\sum_{n=1}^{\infty} \oplus E_n \right)_p = \left\{ (e_n) : e_n \in E_n, n = 1, 2, \dots, \left(\sum_{n=1}^{\infty} \|e_n\|^p \right)^{1/p} < \infty \right\}, \quad (4)$$

if $1 \leq p < \infty$, and

$$\left(\sum_{n=1}^{\infty} \oplus E_n \right)_0 = \left\{ (e_n) : e_n \in E_n, n=1, 2, \dots, \lim_{n \rightarrow \infty} \|e_n\| = 0 \right\} \quad (5)$$

with the norm $\|(e_n)\| = \sup_n \|e_n\|$.

Theorem 1. Let (Ω_k) be a sequence of finite subsets of $\mathbb{N}$ containing each finite subset of $\mathbb{N}$ infinitely many times. Moreover let $F_p(\Omega_k)$ be the linear span of the functions t^λ , $\lambda \in \Omega_k$, in $L_p(0, 1)$. Put

$$\mathcal{F}_p = \left(\sum_{k \in \mathbb{N}} \oplus F_p(\Omega_k) \right)_p \quad \text{if } 1 \leq p < \infty, \quad (6)$$

$$\mathcal{F}_\infty = \left(\sum_{k \in \mathbb{N}} \oplus F_\infty(\Omega_k) \right)_0.$$

- (a) For each $p \in [1, \infty]$, there is a set $\widehat{\Lambda} \subset \mathbb{N}$ satisfying (1) such that $\mathcal{F}_p$ is isomorphic to $M_p(\widehat{\Lambda})$.
- (b) If Λ is any set of positive real numbers satisfying (1), then $M_p(\Lambda)$ is isomorphic to a complemented subspace of $\mathcal{F}_p$.

On the other hand, we do not know if any complemented subspace of $\mathcal{F}_p$ is isomorphic to a Müntz space.

We prove Theorem 1 in Section 4. It turns out that the “maximality” condition of Theorem 1(b) implies uniqueness up to isomorphism. Indeed we have the following.

Theorem 2. Let $1 \leq p \leq \infty$ and assume that Λ is a set of real positive numbers satisfying (1). If $M_p(\Lambda)$ contains a complemented isomorphic copy of any Müntz space $M_p(\widehat{\Lambda})$ then $M_p(\Lambda)$ is isomorphic to the space $\mathcal{F}_p$ of Theorem 1.

Proof. Consider the spaces $F_p(\Omega_k)$ of Theorem 1. By the choice of the Ω_k and the definition of $\mathcal{F}_p$, we have $\mathcal{F}_p \sim \mathcal{F}_p \oplus \mathcal{F}_p \oplus \dots$.

Now let Λ satisfy the assumptions of Theorem 2. Since $\mathcal{F}_p$ is isomorphic to a Müntz space, by Theorem 1 there is a complemented subspace $A \subset M_p(\Lambda)$ with $M_p(\Lambda) \sim A \oplus \mathcal{F}_p$. Moreover, by Theorem 1(b) there is a complemented subspace $B \subset \mathcal{F}_p$ with $\mathcal{F}_p \sim B \oplus M_p(\Lambda)$. Using Pelczynski's decomposition method [6, Theorem 2.2.3.], we obtain $\mathcal{F}_p \sim M_p(\Lambda)$. $\square$

Corollary 3. $\mathcal{F}_\infty$ is not isomorphic to c_0 .

Proof. There is a Müntz space $M_\infty(\Lambda)$ which is not isomorphic to c_0 ([4] or [5, Corollary 10.2.4.]). But, according to Theorem 1, $M_\infty(\Lambda)$ is isomorphic to a complemented subspace of $\mathcal{F}_\infty$. Since all infinite dimensional complemented subspaces of c_0 are isomorphic to c_0 [7, Theorem I.2.a.3.], the space $\mathcal{F}_\infty$ cannot be isomorphic to c_0 . $\square$

In connection with the proof of Theorem 1, we also get the following.

Theorem 4. Let Ω_n , $n = 1, 2, \dots$, be arbitrary finite subsets of positive real numbers and put $F_p(\Omega_n) = \text{span}\{t^\lambda : \lambda \in \Omega_n\} \subset L_p(0, 1)$. Then, for each $p \in [1, \infty]$, there is a subset $\Lambda \subset \mathbb{N}$ satisfying (1) such that $(\sum_n \oplus F_p(\Omega_n))_p \sim M_p(\Lambda)$, if $1 \leq p < \infty$, and $(\sum_n \oplus F_\infty(\Omega_n))_0 \sim M_\infty(\Lambda)$, if $p = \infty$.

We do not know if, conversely, any Müntz space $M_p(\Lambda)$ is isomorphic to the direct sum of the spaces $F_p(\Omega_n)$ for suitable Ω_n .

2. Finite Dimensional Subspaces of $L_p(0, 1)$

Put $C_{0,0}(0, 1) = \{f \in C_0(0, 1) : f(1) = 0\}$. For a finite dimensional subspace $E \subset L_p(0, 1)$, $1 \leq p < \infty$, or $E \subset C_0(0, 1)$, $p = \infty$, and $\epsilon > 0$, put

$$\alpha(E, p, \epsilon) = \sup \left\{ \alpha > 0 : \|e \cdot 1_{[0, \alpha]}\|_p \leq \epsilon \|e\|_p \quad \forall e \in E \right\}. \quad (7)$$

If $E \subset C_0(0, 1)$, $p = \infty$, also put

$$\gamma(E, \epsilon) = \inf \left\{ \gamma < 1 : \|(e(1) - e) \cdot 1_{[\gamma, 1]}\|_\infty \leq \epsilon \|e\|_\infty \quad \forall e \in E \right\}. \quad (8)$$

Similarly, for $E \subset L_p(0, 1)$, $1 \leq p < \infty$, or $E \subset C_{0,0}(0, 1)$, $p = \infty$, define

$$\beta(E, p, \epsilon) = \inf \left\{ \beta > 0 : \|e \cdot 1_{[\beta, 1]}\|_p \leq \epsilon \|e\|_p \quad \forall e \in E \right\}. \quad (9)$$

We collect a few properties of these parameters in the following.

Lemma 5. One has

- (a) $\alpha(E, p, \epsilon) \leq \beta(E, p, \epsilon)$ for all $p < \infty$ if $\epsilon < 1/2$,
- (b) $\alpha(E, \infty, \epsilon) \leq \beta(E, \infty, \epsilon)$ if $\epsilon < 1$,
- (c) $\alpha(E, \infty, \epsilon) \leq \gamma(E, \epsilon)$ if $\epsilon < 1/3$.

Proof. We prove (c). The proofs of (a) and (b) are similar. Let $e \in E$ and let $\gamma < 1$ be such that $\|(e(1) - e) \cdot 1_{[\gamma, 1]}\|_\infty \leq \epsilon \|e\|_\infty$. If $t \in [\gamma, 1]$, we obtain $|e(t) - e(\gamma)| \leq 2\epsilon \|e\|_\infty$. This yields

$$\|e\|_\infty = \max \left(\|e \cdot 1_{[0, \gamma]}\|_\infty, \|e \cdot 1_{[\gamma, 1]}\|_\infty \right) \leq \max \left(\|e \cdot 1_{[0, \gamma]}\|_\infty, |e(\gamma)| + 2\epsilon \|e\|_\infty \right). \quad (10)$$

Hence, $(1 - 2\epsilon)\|e\|_\infty \leq \|e \cdot 1_{[0, \gamma]}\|_\infty$. For $\epsilon < 1/3$, we obtain $(1 - 2\epsilon) > \epsilon$ and therefore $\epsilon \|e\|_\infty < \|e \cdot 1_{[0, \gamma]}\|_\infty$. This implies $\alpha(E, \infty, \epsilon) \leq \gamma(E, \epsilon)$. $\square$

Lemma 6. Consider $E \subset L_p(0, 1)$, $1 \leq p < \infty$, or $E \subset C_{0,0}(0, 1)$, $p = \infty$. Fix $0 < \epsilon < 1/2$ and let $\alpha = \alpha(E, p, \epsilon)$ and $\beta = \beta(E, p, \epsilon)$. Then one has

$$(1 - 2\epsilon) \|e\|_p \leq \|e \cdot 1_{[\alpha, \beta]}\|_p \leq \|e\|_p \quad \forall e \in E. \quad (11)$$

For $p = \infty$, one even has $\|e \cdot 1_{[\alpha, \beta]}\|_\infty = \|e\|_\infty$,

Proof. We obtain

$$\begin{aligned} \|e\|_p &\leq \|e \cdot 1_{[\alpha, \beta]}\|_p + \|e \cdot 1_{[0, \alpha]}\|_p + \|e \cdot 1_{[\beta, 1]}\|_p \\ &\leq \|e \cdot 1_{[\alpha, \beta]}\|_p + 2\epsilon \|e\|_p. \end{aligned} \quad (12)$$

which yields the first inequality. The second inequality is trivial. The last assertion of Lemma 6 follows directly from the definitions of α and β . $\square$

Proposition 7. Fix $\epsilon_n > 0$ such that $\sum_{n=1}^\infty \epsilon_n < 1/8$. Let $E_n \subset L_p(0, 1)$, $1 \leq p < \infty$, or $E_n \subset C_{0,0}(0, 1)$, $p = \infty$, be finite dimensional subspaces with

$$\beta_n := \beta(E_n, p, \epsilon_n) < \alpha(E_{n+1}, p, \epsilon_{n+1}) =: \alpha_{n+1}. \quad (13)$$

Then, for any $e_n \in E_n$ and $N \in \mathbb{N}$, one has

$$\begin{aligned} \frac{1}{2} \left(\sum_{n=1}^N \|e_n\|_p^p \right)^{1/p} &\leq \left\| \sum_{n=1}^N e_n \right\|_p \leq \frac{4}{3} \left\| \sum_{n=1}^N e_n \cdot 1_{[\alpha_n, \beta_n]} \right\|_p \\ &\leq \frac{4}{3} \left(\sum_{n=1}^N \|e_n\|_p^p \right)^{1/p}, \end{aligned} \quad (14)$$

if $1 \leq p < \infty$, and

$$\begin{aligned} \frac{7}{8} \sup_{n \leq N} \|e_n\|_\infty &\leq \left\| \sum_{n=1}^N e_n \right\|_\infty \leq \frac{9}{8} \left\| \sum_{n=1}^N e_n \cdot 1_{[\alpha_n, \beta_n]} \right\|_\infty \\ &\leq \frac{9}{8} \sup_{n \leq N} \|e_n\|_\infty, \end{aligned} \quad (15)$$

if $p = \infty$.

Proof. Let $1 \leq p < \infty$. Put $e = \sum_{n=1}^N e_n$. We obtain with Lemma 6 the following:

$$\begin{aligned} \left\| e - \sum_{n=1}^N e_n \cdot 1_{[\alpha_n, \beta_n]} \right\|_p &= \left\| \sum_{n=1}^N (e_n - e_n \cdot 1_{[\alpha_n, \beta_n]}) \right\|_p \\ &\leq \sum_{n=1}^N 2\epsilon_n \|e_n\|_p \\ &\leq \sum_{n=1}^N \frac{2\epsilon_n}{1 - 2\epsilon_n} \|e_n \cdot 1_{[\alpha_n, \beta_n]}\|_p \\ &\leq \frac{1}{3} \sup_{n \leq N} \|e_n \cdot 1_{[\alpha_n, \beta_n]}\|_p \\ &\leq \frac{1}{3} \left(\sum_{n=1}^N \|e_n \cdot 1_{[\alpha_n, \beta_n]}\|_p^p \right)^{1/p}. \end{aligned} \quad (16)$$

Here we used $\sum_{n=1}^\infty \epsilon_n < 1/8$. Hence, $1 - 2\epsilon_n \geq 3/4$ for all n and $\sum_{n=1}^\infty 2\epsilon_n / (1 - 2\epsilon_n) < 1/3$.

This yields, in view of (13),

$$\begin{aligned} \|e\|_p &\leq \left\| \sum_{n=1}^N e_n \cdot 1_{[\alpha_n, \beta_n]} \right\|_p + \frac{1}{3} \left(\sum_{n=1}^N \|e_n \cdot 1_{[\alpha_n, \beta_n]}\|_p^p \right)^{1/p} \\ &= \frac{4}{3} \left(\sum_{n=1}^N \|e_n \cdot 1_{[\alpha_n, \beta_n]}\|_p^p \right)^{1/p} \leq \frac{4}{3} \left(\sum_{n=1}^N \|e_n\|_p^p \right)^{1/p}, \\ \|e\|_p &\geq \left\| \sum_{n=1}^N e_n \cdot 1_{[\alpha_n, \beta_n]} \right\|_p - \frac{1}{3} \left(\sum_{n=1}^N \|e_n \cdot 1_{[\alpha_n, \beta_n]}\|_p^p \right)^{1/p} \\ &= \frac{2}{3} \left(\sum_{n=1}^N \|e_n \cdot 1_{[\alpha_n, \beta_n]}\|_p^p \right)^{1/p} \geq \frac{1}{2} \left(\sum_{n=1}^N \|e_n\|_p^p \right)^{1/p}. \end{aligned} \quad (17)$$

Here we used Lemma 6 and $1 - 2\epsilon_n \geq 3/4$. The proof for $p = \infty$ is similar. $\square$

Proposition 8. Fix $\epsilon_n > 0$ such that $\sum_{n=1}^\infty \epsilon_n < 1/8$. Let $E_n \subset C_0(0, 1)$ be finite dimensional subspaces with

$$\gamma_n := \gamma(E_n, \epsilon_n) < \alpha(E_{n+1}, \infty, \epsilon_{n+1}) =: \alpha_{n+1}. \quad (18)$$

Then, for any $e_k \in E_k$ and $N \in \mathbb{N}$, one has

$$\frac{1}{2} \sup_{n \leq N} \left\| \sum_{k=1}^n e_k \right\|_\infty \leq \left\| \sum_{k=1}^N e_k \right\|_\infty \leq \sup_{n \leq N} \left\| \sum_{k=1}^n e_k \right\|_\infty. \quad (19)$$

Proof. By (18) we have, for any $n \leq N$,

$$\begin{aligned} \left| \left\| \sum_{k=1}^N e_k \cdot 1_{[0, \gamma_n]} \right\|_\infty - \left\| \sum_{k=1}^n e_k \cdot 1_{[0, \gamma_n]} \right\|_\infty \right| \\ \leq \sum_{k=n+1}^N \|e_k \cdot 1_{[0, \alpha_k]}\|_\infty \leq \sum_{k=n+1}^N \epsilon_k \|e_k\|_\infty. \end{aligned} \quad (20)$$

Equation (18) and Lemma 5 also imply that γ_n are monotonically increasing. Hence, if $t \in [\gamma_n, 1]$, we obtain

$$\begin{aligned} \left| \sum_{k=1}^n e_k(t) - \sum_{k=1}^n e_k(\gamma_n) \right| \\ \leq \sum_{k=1}^n (|e_k(t) - e_k(1)| + |e_k(1) - e_k(\gamma_n)|) \\ \leq \sum_{k=1}^n 2 \|(e_k(1) - e_k) \cdot 1_{[\gamma_n, 1]}\|_\infty \leq 2 \sum_{k=1}^n \epsilon_k \|e_k\|_\infty. \end{aligned} \quad (21)$$

Here we used that $\gamma_k \leq \gamma_n$ and hence $[\gamma_n, 1] \subset [\gamma_k, 1]$ for all $k = 1, \dots, n$. The preceding estimate implies

$$\begin{aligned} \left\| \sum_{k=1}^n e_k \right\|_\infty &\leq \max \left(\left\| \sum_{k=1}^n e_k \cdot 1_{[0, \gamma_n]} \right\|_\infty, \left| \sum_{k=1}^n e_k(\gamma_n) \right| \right) \\ &\quad + 2 \sum_{k=1}^n \epsilon_k \|e_k\|_\infty \leq \left\| \sum_{k=1}^n e_k \cdot 1_{[0, \gamma_n]} \right\|_\infty \\ &\quad + 2 \sum_{k=1}^n \epsilon_k \|e_k\|_\infty. \end{aligned} \quad (22)$$

We conclude

$$0 \leq \left\| \sum_{k=1}^n e_k \right\|_{\infty} - \left\| \sum_{k=1}^n e_k \cdot 1_{[0, \gamma_n]} \right\|_{\infty} \quad (23)$$

$$\leq 2 \sum_{k=1}^n \epsilon_k \|e_k\|_{\infty},$$

$$\begin{aligned} \left\| \sum_{k=1}^N e_k \right\|_{\infty} &\geq \left\| \sum_{k=1}^N e_k \cdot 1_{[0, \gamma_n]} \right\|_{\infty} \\ &\geq \left\| \sum_{k=1}^n e_k \cdot 1_{[0, \gamma_n]} \right\|_{\infty} - \sum_{k=n+1}^N \epsilon_k \|e_k\|_{\infty} \\ &\geq \left\| \sum_{k=1}^n e_k \right\|_{\infty} - \sum_{k=1}^N 2\epsilon_k \|e_k\|_{\infty} \\ &\geq \left\| \sum_{k=1}^n e_k \right\|_{\infty} - \frac{1}{4} \sup_{k \leq N} \|e_k\|_{\infty}. \end{aligned} \quad (24)$$

Applying this estimate for n and $n-1$ yields

$$\begin{aligned} 2 \left\| \sum_{k=1}^N e_k \right\|_{\infty} &\geq \left\| \sum_{k=1}^n e_k \right\|_{\infty} + \left\| \sum_{k=1}^{n-1} e_k \right\|_{\infty} - \frac{1}{2} \sup_{k \leq N} \|e_k\|_{\infty} \\ &\geq \|e_n\|_{\infty} - \frac{1}{2} \sup_{k \leq N} \|e_k\|_{\infty}, \end{aligned} \quad (25)$$

from which we obtain $\sup_k \|e_k\|_{\infty} \leq 4 \left\| \sum_{k=1}^N e_k \right\|_{\infty}$. Moreover, using (24) we see that $\sup_n \left\| \sum_{k=1}^n e_k \right\|_{\infty} \leq 2 \left\| \sum_{k=1}^N e_k \right\|_{\infty}$ which proves the first inequality of the proposition. The second inequality is trivial. $\square$

Now consider, for fixed p and $N > 0$, the maps

$$\begin{aligned} (T_N f)(t) &= N^{1/p} t^{(N-1)/p} f(t^N), \\ (S_N f)(t) &= N^{1/p} t^{\lceil (N-1)/p \rceil} f(t^N), \end{aligned} \quad (26)$$

$t \in [0, 1]$. (Here $\lceil r \rceil$ is the smallest integer $\geq r$.) We include the case $p = \infty$ by putting $1/\infty = 0$. We clearly have $|T_N f| \geq |S_N f|$ and we obtain $\|T_N f\|_p = \|f\|_p$ for $f \in L_p(0, 1)$. If $p = \infty$, then $S_N = T_N$.

Lemma 9. *One has*

$$\begin{aligned} (a) \quad \alpha(T_N E, p, \epsilon) &= \alpha^{1/N}(E, p, \epsilon), \\ \beta(T_N E, p, \epsilon) &= \beta^{1/N}(E, p, \epsilon), \\ \gamma(T_N E, \epsilon) &= \gamma^{1/N}(E, \epsilon), \end{aligned} \quad (27)$$

$$\begin{aligned} (b) \quad \alpha\left(S_N E, p, \frac{\epsilon}{(1-2\epsilon)}\right) &\geq \alpha^{1/N}(E, p, \epsilon), \\ \beta\left(S_N E, p, \frac{\epsilon}{(1-2\epsilon)\alpha^{1/N}(E, p, \epsilon)}\right) &\leq \beta^{1/N}(E, p, \epsilon), \end{aligned} \quad (28)$$

(c)

$$\alpha^{1/N}(E, p, \epsilon) (1-2\epsilon) \|e\|_p \leq \|S_N e\|_p \leq \|e\|_p, \quad e \in E, \quad (29)$$

for $0 < \epsilon < 1/2$.

Proof. (a) follows from simple substitution. (Recall that γ is only defined for $p = \infty$.)

(b): Let $\alpha_0 = \alpha^{1/N}(E, p, \epsilon) = \alpha(T_N E, p, \epsilon)$ and $\beta_0 = \beta^{1/N}(E, p, \epsilon) = \beta(T_N E, p, \epsilon)$. At first, let $p < \infty$. Consider $e \in E$. Put $\theta = \lceil (N-1)/p \rceil - (N-1)/p$. Since $0 \leq \theta \leq 1$ and $(S_N e)(t) = t^\theta (T_N e)(t)$ for $t \in [0, 1]$, we obtain, with Lemma 6,

$$\begin{aligned} \|(S_N e) \cdot 1_{[0, \alpha_0]}\|_p &\leq \alpha_0^\theta \|(T_N e) \cdot 1_{[0, \alpha_0]}\|_p \leq \alpha_0^\theta \epsilon \|T_N e\|_p \\ &\leq \alpha_0^\theta \frac{\epsilon}{(1-2\epsilon)} \|T_N e \cdot 1_{[\alpha_0, \beta_0]}\|_p \\ &\leq \frac{\epsilon}{(1-2\epsilon)} \|S_N e\|_p. \end{aligned} \quad (30)$$

This yields the first inequality if $p < \infty$.

Similarly, we obtain

$$\begin{aligned} \|(S_N e) \cdot 1_{[\beta_0, 1]}\|_p &\leq \|(T_N e) \cdot 1_{[\beta_0, 1]}\|_p \leq \epsilon \|T_N e\|_p \\ &\leq \frac{\epsilon}{(1-2\epsilon)} \|T_N e \cdot 1_{[\alpha_0, \beta_0]}\|_p \\ &\leq \alpha_0^{-\theta} \frac{\epsilon}{(1-2\epsilon)} \|S_N e \cdot 1_{[\alpha_0, \beta_0]}\|_p \\ &\leq \frac{\epsilon}{(1-2\epsilon)\alpha_0} \|S_N e\|_p. \end{aligned} \quad (31)$$

This yields the second inequality of (c) if $p < \infty$.

If $p = \infty$, then the two inequalities follow from (a) and the fact that α is increasing and β is decreasing with respect to ϵ . (Recall that here is $S_N = T_N$.)

(c): If $p = \infty$, then (c) is trivial. Now let $p < \infty$. By definition, $\|S_N e\|_p \leq \|T_N e\|_p = \|e\|_p$. On the other hand, with α_0, β_0 of (b), and Lemma 6,

$$\begin{aligned} \|S_N e\|_p &\geq \|(S_N e) \cdot 1_{[\alpha_0, \beta_0]}\|_p \\ &\geq \alpha_0^{\lceil (N-1)/p \rceil - (N-1)/p} \|T_N e \cdot 1_{[\alpha_0, \beta_0]}\|_p \\ &\geq \alpha_0 (1-2\epsilon) \|T_N e\|_p = \alpha_0 (1-2\epsilon) \|e\|_p \end{aligned} \quad (32)$$

since $0 \leq \lceil (N-1)/p \rceil - (N-1)/p \leq 1$. $\square$

3. Some Facts about Müntz Spaces

If Λ satisfies (1), then any $f \in M_p(\Lambda)$ can be represented as $f(t) = \sum_{k=1}^{\infty} \alpha_k t^{\lambda_k}$ where the series converges pointwise on $[0, 1]$. There are universal constants $c_k > 0$ with $|\alpha_k| \leq c_k \|f\|_p$ for all k [8]. Moreover, it is easily seen that for any $\lambda > 0$ we have

$$\lim_{\mu \rightarrow \lambda} \|t^\lambda - t^\mu\|_{\infty} = \lim_{\mu \rightarrow \lambda} \left(\frac{\lambda}{\mu} \right)^{\lambda/(\mu-\lambda)} \left(1 - \frac{\lambda}{\mu} \right) = 0. \quad (33)$$

Lemma 10. Let $\epsilon_k > 0$ such that $\sum_{k=1}^{\infty} \epsilon_k c_k < 1/2$. Moreover, let μ_k satisfy $\|t^{\lambda_k} - t^{\mu_k}\|_{\infty} \leq \epsilon_k$ for all k . Then, T with $T(\sum_{n=1}^{\infty} \alpha_k t^{\lambda_k}) = \sum_{n=1}^{\infty} \alpha_k t^{\mu_k}$ is an isomorphism between $M_p(\Lambda)$ and $M_p(\{\mu_k : k = 1, 2, \dots\})$.

Proof. For $f(t) = \sum_{k=1}^{\infty} \alpha_k t^{\lambda_k}$ we obtain

$$\|Tf - f\|_p \leq \left(\sum_{k=1}^{\infty} c_k \epsilon_k \right) \|f\|_p \leq \frac{1}{2} \|f\|_p. \quad (34)$$

This proves $2^{-1} \|f\|_p \leq \|Tf\|_p \leq 3 \cdot 2^{-1} \|f\|_p$. $\square$

Using Lemma 10, we can always assume in the following that $\Lambda \subset \mathbb{Q}$.

Put

$$\begin{aligned} M_{\infty}^0(\Lambda) &= \{f \in M_{\infty}(\Lambda) : f(1) = 0\} \\ &= \overline{\text{span}} \{t^{\lambda_k} - t^{\lambda_{k+1}} : k = 1, 2, \dots\}. \end{aligned} \quad (35)$$

Since $M_{\infty}^0(\Lambda)$ is 1-codimensional in $M_{\infty}(\Lambda)$ and $M_{\infty}(\Lambda)$ contains a complemented subspace isomorphic to c_0 , we obtain $M_{\infty}^0(\Lambda) \sim M_{\infty}(\Lambda)$.

Proposition 11. Fix $X = M_p(\Lambda)$, $1 \leq p < \infty$, or $X = M_{\infty}^0(\Lambda)$, $p = \infty$. Then, there are linear bounded finite rank operators $R_n : X \rightarrow X$ with $\lim_{n \rightarrow \infty} R_n f = f$, $f \in X$, $R_n R_m = R_{\min(n,m)}$ if $n \neq m$ and

$$\begin{aligned} c_1 \left(\sum_{k=1}^{\infty} \|(R_k - R_{k-1}) f\|_p^p \right)^{1/p} \\ \leq \|f\|_p \leq c_2 \left(\sum_{k=1}^{\infty} \|(R_k - R_{k-1}) f\|_p^p \right)^{1/p}, \end{aligned} \quad (36)$$

if $1 \leq p < \infty$, or

$$\begin{aligned} c_1 \sup_k \|(R_k - R_{k-1}) f\|_{\infty} \\ \leq \|f\|_{\infty} \leq c_2 \sup_k \|(R_k - R_{k-1}) f\|_{\infty}, \end{aligned} \quad (37)$$

if $p = \infty$. Moreover, there are indices $m_k < n_k$, $k = 1, 2, \dots$, with

$$(R_k - R_{k-1})X = \text{span} \{t^{\lambda_j} - t^{\lambda_{j+1}} : m_k \leq j \leq n_k\}, \quad (38)$$

if $p = \infty$, or

$$(R_k - R_{k-1})X = \text{span} \{t^{\lambda_j} : m_k \leq j \leq n_k\}, \quad (39)$$

if $1 \leq p < \infty$.

For a proof see [5, Proposition 9.1.4].

The condition $\lim_{n \rightarrow \infty} R_n f = f$ implies the following.

Corollary 12. For the operators R_k of Proposition 11 one has $f = \sum_{k=1}^{\infty} (R_k - R_{k-1})f$ (with $R_0 = 0$), $f \in X$, where the series converges in norm.

However, $R_k - R_{k-1}$ are no projections and the “summands overlap”; that is, $(R_k - R_{k-1})^2 \neq R_k - R_{k-1}$ and $(R_k - R_{k-1})(R_{k+1} - R_k) \neq 0$ in general.

4. Final Proofs

Let Λ satisfy (1). Fix $1 \leq p \leq \infty$. Take R_k , m_k , and n_k as in Proposition 11 and put $E_k = (R_k - R_{k-1})M_p(\Lambda)$, if $1 \leq p < \infty$, or $E_k = (R_k - R_{k-1})M_{\infty}^0(\Lambda)$, if $p = \infty$. Fix $0 < \epsilon_k < 1/4$ and let $0 < N_1 < N_2 < \dots$ be such that

$$\alpha^{1/N_k}(E_k, p, \epsilon_k) \geq \frac{1}{2} \quad \forall k. \quad (40)$$

Let F_j be finite dimensional Banach spaces and put $X = (\sum_{j=1}^{\infty} \oplus F_j)_p$, if $1 \leq p < \infty$, or $X = (\sum_{j=1}^{\infty} \oplus F_j)_0$, if $p = \infty$.

Lemma 13. Assume that $F_{j_k} = S_{N_k} E_k$ for some j_k , $k = 1, 2, \dots$. Then, $M_p(\Lambda)$ is isomorphic to a complemented subspace of X .

Proof. For $e \in M_p(\Lambda)$, $1 \leq p < \infty$, or $e \in M_{\infty}^0(\Lambda)$, $p = \infty$, put $Ue = (f_j)$ where $f_{j_k} = S_{N_k}(R_k - R_{k-1})e$ and $f_j = 0$ if $j \neq j_k$, $k = 1, 2, \dots$. Using Proposition 11, Corollary 12, and Lemma 9(c), we see that Ue is a well-defined element in X and U is a bounded linear operator $M_p(\Lambda) \rightarrow X$ if $p < \infty$ or $M_{\infty}^0(\Lambda) \rightarrow X$. By Lemma 9(c) and condition (40), U is even an into-isomorphism. Hence, there are universal constants $c_1 > 0$ and $c_2 > 0$ such that $c_1 \|e\|_p \leq \|Ue\|_p \leq c_2 \|e\|_p$ for all e .

It remains to prove that $UM_p(\Lambda)$ and $UM_{\infty}^0(\Lambda)$, respectively, are complemented in X . To this end, let $f \in X$. At first assume that, for some $N \in \mathbb{N}$, $f = (f_1, \dots, f_N, 0, \dots)$ with $f_j \in F_j$ and $f_{j_k} = S_{N_k} e_k$ for some $e_k \in E_k$ if $j_k \leq N$. Put $Vf = \sum_{j_k \leq N} e_k$. The fact that $R_m R_n = R_{\min(m,n)}$ if $m \neq n$ implies $(R_n - R_{n-1})(R_k - R_{k-1}) = 0$ if $|n - k| \geq 2$. Hence,

$$(R_n - R_{n-1})Vf = (R_n - R_{n-1})(e_{n-1} + e_n + e_{n+1}). \quad (41)$$

We obtain a universal constant $c_3 > 0$ with

$$\begin{aligned} \|(R_n - R_{n-1})Vf\|_p \\ \leq c_3 (\|e_{n-1}\|_p^p + \|e_n\|_p^p + \|e_{n+1}\|_p^p)^{1/p} \\ \leq 4c_3 (\|S_{N_{n-1}} e_{n-1}\|_p^p + \|S_{N_n} e_n\|_p^p + \|S_{N_{n+1}} e_{n+1}\|_p^p)^{1/p} \end{aligned} \quad (42)$$

if $1 \leq p < \infty$. Here again, (40) and Lemma 9(c) were used. In view of Proposition 11, this implies that V is a bounded linear operator on the dense subspace of X consisting of finite sequences. Hence, V can be uniquely extended to a linear bounded operator $V : X \rightarrow M_p(\Lambda)$, called V again. The same argument shows that V is also bounded if $p = \infty$.

We obtain by definition $VUe = e$ if $e \in M_p(\Lambda)$, $1 \leq p < \infty$, or $e \in M_{\infty}^0(\Lambda)$, $p = \infty$. This follows from Corollary 12. Indeed, we have $Ue = (f_j)$ with $f_{j_k} = S_{N_k} e_k$ and $f_l = 0$ for $l \notin \{j_k : k = 1, 2, \dots\}$. Hence,

$$VUe = V(f_j) = \sum_{k=1}^{\infty} (R_k - R_{k-1})e = e. \quad (43)$$

(Here we used $R_0 = 0$.) This shows that UV is a bounded projection from X onto $UM_p(\Lambda)$, if $1 \leq p < \infty$, or onto $UM_{\infty}^0(\Lambda)$, if $p = \infty$. Since $M_{\infty}^0(\Lambda) \sim M_{\infty}(\Lambda)$, this completes the proof. $\square$

Proposition 14. Let $1 \leq p < \infty$. Assume that $\Omega_k, k = 1, 2, \dots$, are finite subsets of positive numbers and put

$$F_k = \text{span} \{t^\lambda : \lambda \in \Omega_k\} \subset L_p(0, 1). \quad (44)$$

Then, there is a Müntz space $M_p(\Lambda)$ which is isomorphic to $(\sum_{k=1}^{\infty} \oplus F_k)_p$.

Here $\Lambda = \cup_{k=1}^{\infty} ([\lceil (N_k - 1)/p \rceil + N_k \Omega_k)$ for suitable integers $N_k > 0$. If all Ω_k consist of rational numbers, then N_k can be taken such that in addition $\Lambda \subset \mathbb{N}$.

Proof. Fix $\epsilon_k > 0$ such that

$$\sum_{k=1}^{\infty} \frac{\epsilon_k}{1 - 2\epsilon_k} < \frac{1}{16}. \quad (45)$$

Find integers $1 < N_1 < N_2 < \dots$ such that $\max N_k \Omega_k < \min N_{k+1} \Omega_{k+1}$ for all k and

$$\begin{aligned} \frac{1}{2} &< \alpha^{1/N_k}(F_k, p, \epsilon_k) \leq \beta^{1/N_k}(F_k, p, \epsilon_k) \\ &< \alpha^{1/N_{k+1}}(F_{k+1}, p, \epsilon_{k+1}) \end{aligned} \quad (46)$$

for all k . Put $\alpha_k = \alpha^{1/N_k}(F_k, p, \epsilon_k)$. Then, we obtain

$$\sum_{k=1}^{\infty} \frac{\epsilon_k}{(1 - 2\epsilon_k) \alpha_k} < \frac{1}{8}. \quad (47)$$

Define $\Lambda = \cup_{k=1}^{\infty} ([\lceil (N_k - 1)/p \rceil + N_k \Omega_k)$. (Of course we can take N_k so large that Λ satisfies (1) and consists of integers if all Ω_k consist of rational numbers.)

We obtain, by Lemma 5, Lemma 9 ((a) and (b)), and (46),

$$\begin{aligned} &\alpha \left(S_{N_{k+1}} F_{k+1}, p, \frac{\epsilon_{k+1}}{(1 - 2\epsilon_{k+1}) \alpha_{k+1}} \right) \\ &\geq \alpha \left(S_{N_{k+1}} F_{k+1}, p, \frac{\epsilon_{k+1}}{(1 - 2\epsilon_{k+1})} \right) \\ &\geq \alpha^{1/N_{k+1}}(F_{k+1}, p, \epsilon_{k+1}) \geq \beta^{1/N_k}(F_k, p, \epsilon_k) \\ &\geq \beta \left(S_{N_k} F_k, p, \frac{\epsilon_k}{(1 - 2\epsilon_k) \alpha_k} \right). \end{aligned} \quad (48)$$

Here we used that $\alpha(\cdot, \cdot, \epsilon)$ is increasing with respect to ϵ . Now Proposition 7 yields

$$M_p(\Lambda) = \overline{\text{span}} \left(\bigcup_{k=1}^{\infty} S_{N_k} F_k \right) \sim \left(\sum_{k=1}^{\infty} \oplus S_{N_k} F_k \right)_p. \quad (49)$$

Finally, with Lemma 9(c) we obtain

$$\left(\sum_{k=1}^{\infty} \oplus S_{N_k} F_k \right)_p \sim \left(\sum_{k=1}^{\infty} \oplus F_k \right)_p. \quad (50)$$

□

Now we turn to the case $p = \infty$.

Proposition 15. Let $\Omega_k, k = 1, 2, \dots$, be finite subsets of positive numbers and put

$$F_k = \text{span} \{t^\lambda : \lambda \in \Omega_k\} \subset C_0(0, 1), \quad (51)$$

$$G_k = \text{span} \{t^\lambda - t^\mu : \lambda, \mu \in \Omega_k\} \subset C_{0,0}(0, 1).$$

Then, there is a Müntz space $M_\infty(\Lambda)$, where $\Lambda = \cup_{k=1}^{\infty} N_k \Omega_k$ for suitable integers N_k , such that

$$\left(\sum_{k=1}^{\infty} \oplus F_k \right)_0 \sim M_\infty(\Lambda) \sim \left(\sum_{k=1}^{\infty} \oplus G_k \right)_0. \quad (52)$$

If all Ω_k consist of rational numbers, then N_k can be taken such that in addition $\Lambda \subset \mathbb{N}$.

Proof. Recall that for $p = \infty$ we have $S_N = T_N$ and

$$\|S_N f\|_\infty = \|T_N f\|_\infty = \|f\|_\infty. \quad (53)$$

Fix $\bar{\lambda}_k \in \Omega_k$ and put $H_k = \text{span}\{t^{\bar{\lambda}_k}\}$. For any $f \in F_k$, we have $f = (f - f(1)t^{\bar{\lambda}_k}) + f(1)t^{\bar{\lambda}_k}$ which implies

$$F_k = G_k \oplus H_k. \quad (54)$$

Fix $\epsilon_k > 0$ with $\sum_{k=1}^{\infty} \epsilon_k < 1/8$. Then, we find integers $1 < N_1 < N_2 < \dots$ with $N_k \bar{\lambda}_k < N_{k+1} \bar{\lambda}_{k+1}$,

$$\frac{1}{2} < \beta^{1/N_k}(G_k, \infty, \epsilon_k) < \alpha^{1/N_{k+1}}(G_{k+1}, \infty, \epsilon_{k+1}), \quad (55)$$

$$\frac{1}{2} < \gamma^{1/N_k}(F_k, \epsilon_k) < \alpha^{1/N_{k+1}}(F_{k+1}, \infty, \epsilon_{k+1}).$$

Put

$$\begin{aligned} Y &= \overline{\text{span}} \left(\bigcup_{k=1}^{\infty} T_{N_k} G_k \right), \\ Z &= \overline{\text{span}} \left(\bigcup_{k=1}^{\infty} T_{N_k} H_k \right), \\ \Lambda &= \bigcup_{k=1}^{\infty} N_k \Omega_k. \end{aligned} \quad (56)$$

(The integers N_k should be taken so large that Λ satisfies (1) and that $\Lambda \subset \mathbb{N}$ if all Ω_k consist of rational numbers.) Hence $Y \subset M_\infty^0(\Lambda)$ and $Z \subset M_\infty(\Lambda)$.

Let $g_k \in G_k$ and fix $M \in \mathbb{N}$. Lemma 9(a) and Proposition 7 imply

$$\frac{7}{8} \sup_{k \leq M} \|T_{N_k} g_k\|_\infty \leq \left\| \sum_{k=1}^M T_{N_k} g_k \right\|_\infty \leq \frac{9}{8} \sup_{k \leq M} \|T_{N_k} g_k\|_\infty. \quad (57)$$

Moreover, let $f_k \in F_k$. Then, Lemma 9(a) and Proposition 8 show

$$\begin{aligned} \frac{1}{2} \sup_{n \leq M} \left\| \sum_{k=1}^n T_{N_k} f_k \right\|_\infty &\leq \left\| \sum_{k=1}^M T_{N_k} f_k \right\|_\infty \\ &\leq \sup_{n \leq M} \left\| \sum_{k=1}^n T_{N_k} f_k \right\|_\infty. \end{aligned} \quad (58)$$

Let

$$P \left(\sum_{k=1}^M T_{N_k} f_k \right) = \sum_{j=1}^M f_j(1) t^{N_j \bar{\lambda}_j}. \quad (59)$$

Then, P is a projection. (Recall that $f_j(1) = T_{N_j} f_j(1)$).

We have $\|P\| \leq 2$. Indeed, we obtain, in view of (58),

$$\begin{aligned} \left\| P \left(\sum_{k=1}^M T_{N_k} f_k \right) \right\|_{\infty} &= \left\| \sum_{j=1}^M f_j(1) t^{N_j \bar{\lambda}_j} \right\|_{\infty} \\ &= \left\| \sum_{j=1}^{M-1} \left(\sum_{k=1}^j f_k(1) \right) \left(t^{N_j \bar{\lambda}_j} - t^{N_{j+1} \bar{\lambda}_{j+1}} \right) \right. \\ &\quad \left. + \left(\sum_{k=1}^M f_k(1) \right) t^{N_M \bar{\lambda}_M} \right\|_{\infty} \leq \sup_{j \leq M} \left| \sum_{k=1}^j f_k(1) \right| \\ &= \sup_{j \leq M} \left| \sum_{k=1}^j (T_{N_k} f_k)(1) \right| \leq 2 \left\| \sum_{k=1}^M T_{N_k} f_k \right\|_{\infty}. \end{aligned} \quad (60)$$

Here we used that $t^{N_j \bar{\lambda}_j} - t^{N_{j+1} \bar{\lambda}_{j+1}} \geq 0$ and

$$\sum_{j=1}^{M-1} \left(t^{N_j \bar{\lambda}_j} - t^{N_{j+1} \bar{\lambda}_{j+1}} \right) + t^{N_M \bar{\lambda}_M} \leq 1. \quad (61)$$

Hence, P can be extended to a bounded projection from $M_{\infty}(\Lambda) = \overline{\text{span}}(\cup_{k=1}^{\infty} T_{N_k} F_k)$ onto Z whose kernel is Y . Equation (58) implies that $H \sim c \sim c_0$ since $\dim T_{N_k} H_k = 1$ for all k . Thus, we have $M_{\infty}(\Lambda) \sim Y \oplus c_0$. Since Y contains a complemented isomorphic copy of c_0 , we obtain $M_{\infty}(\Lambda) \sim Y$ and (53) and (57) show

$$Y \sim \left(\sum_{k=1}^{\infty} \oplus T_{N_k} G_k \right)_0 \sim \left(\sum_{k=1}^{\infty} \oplus G_k \right)_0. \quad (62)$$

Finally, put $X = (\sum_{k=1}^{\infty} \oplus F_k)_0$. As a consequence of (54), we have $X \sim (\sum_{k=1}^{\infty} \oplus G_k)_0 \oplus c_0$ which implies by the preceding $M_{\infty}(\Lambda) \sim X$. $\square$

Propositions 14 and 15 prove Theorem 4. Indeed, in view of (33) and the fact that $\|\cdot\|_p \leq \|\cdot\|_{\infty}$ for all p , we may assume that all Ω_k consist of rational numbers.

We conclude with the following.

Proof of Theorem 1. Let $\Omega_k \subset \mathbb{N}$, $k = 1, 2, \dots$, be a family of finite subsets such that each finite subset of $\mathbb{N}$ occurs infinitely many times among the Ω_k . Put $F_k = \text{span}\{t^{\lambda} : \lambda \in \Omega_k\}$, $\mathcal{F}_p = (\sum_{k=1}^{\infty} \oplus F_k)_p$, if $1 \leq p < \infty$, and $\mathcal{F}_{\infty} = (\sum_{k=1}^{\infty} \oplus F_k)_0$. According to Propositions 14 and 15, these spaces are isomorphic to Müntz spaces $M_p(\widehat{\Lambda})$ with $\widehat{\Lambda} \subset \mathbb{N}$. This proves Theorem 1(a).

To prove (b), let Λ be a given set of positive numbers satisfying (1). According to Lemma 10, we may assume that Λ consists of rational numbers. As in the beginning of this section, consider $E_k = \text{span}\{t^{\lambda_j} : m_k \leq j \leq n_k\}$, if $1 \leq p < \infty$, and $E_k = \text{span}\{t^{\lambda_j} - t^{\lambda_{j+1}} : m_k \leq j \leq n_k\}$, if $p = \infty$.

Moreover, let N_k be integers fulfilling inequality (40). Put $\Lambda_k = \{(N_k - 1)/p + N_k \lambda_j : m_k \leq j \leq n_k\}$ if $1 \leq p < \infty$ and $\Lambda_k = \{N_k \lambda_j : m_k \leq j \leq n_k + 1\}$ if $p = \infty$. At the same time we can take N_k so large that all Λ_k consist of positive integers. Then, Lemma 13 shows that $M_p(\Lambda)$ is isomorphic to a complemented subspace of $(\sum_{k=1}^{\infty} \oplus S_{N_k} E_k)_p$ if $1 \leq p < \infty$ which itself is complemented in $\mathcal{F}_p$ since the sets Λ_k appear among the sets Ω_n introduced in the proof of part (a). For $p = \infty$, Lemma 13 implies that $M_{\infty}(\Lambda)$ is isomorphic to a complemented subspace of $(\sum_{k=1}^{\infty} \oplus S_{N_k} E_k)_0$ which, in view of Proposition 15, is isomorphic to $(\sum_{k=1}^{\infty} \oplus B_k)_0$ where $B_k = \text{span}\{t^{N_k \lambda_j} : m_k \leq j \leq n_k\}$. $(\sum_{k=1}^{\infty} \oplus B_k)_0$, of course, is complemented in $\mathcal{F}_{\infty}$. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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