Research Article

The Boundedness of Some Integral Operators on Weighted Hardy Spaces Associated with Schrödinger Operators

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Let $L = -\Delta + V$ be a Schrödinger operator acting on $L^2(\mathbb{R}^n)$, $n \geq 1$, where $V \neq 0$ is a nonnegative locally integrable function on $\mathbb{R}^n$. In this paper, we will first define molecules for weighted Hardy spaces $H_L^p(w)$ ($0 < p \leq 1$) associated with L and establish their molecular characterizations. Then, by using the atomic decomposition and molecular characterization of $H_L^p(w)$, we will show that the imaginary power L^{iy} is bounded on $H_L^p(w)$ for $n/(n+1) < p \leq 1$, and the fractional integral operator $L^{-\alpha/2}$ is bounded from $H_L^p(w)$ to $H_L^q(w^{q/p})$, where $0 < \alpha < \min\{n/2, 1\}$, $n/(n+1) < p \leq n/(n+\alpha)$, and $1/q = 1/p - \alpha/n$.

1. Introduction

Let $n \geq 1$ and V be a nonnegative locally integrable function defined on $\mathbb{R}^n$, not identically zero. We define the form $\mathcal{Q}$ by

$$\mathcal{Q}(u, v) := \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^n} Vuv \, dx \quad (1)$$

with domain $\mathcal{D}(\mathcal{Q}) = \mathcal{V} \times \mathcal{V}$, where

$$\begin{aligned} \mathcal{V} &:= \left\{ u \in L^2(\mathbb{R}^n) : \frac{\partial u}{\partial x_k} \in L^2(\mathbb{R}^n) \text{ for } k \right. \\ &\quad \left. = 1, 2, \dots, n, \sqrt{V}u \in L^2(\mathbb{R}^n) \right\}. \end{aligned} \quad (2)$$

It is well known that this symmetric form is closed. Note also that it was shown by Simon [1] that this form coincides with the minimal closure of the form given by the same expression but defined on $C_0^\infty(\mathbb{R}^n)$ (the space of C^∞ functions with compact supports). In other words, $C_0^\infty(\mathbb{R}^n)$ is a core of the form $\mathcal{Q}$.

Let us denote by L the self-adjoint operator associated with $\mathcal{Q}$. The domain of L is given by

$$\mathcal{D}(L) := \left\{ u \in \mathcal{V} : \exists v \in L^2(\mathbb{R}^n) \text{ such that } \mathcal{Q}(u, \varphi) = \int_{\mathbb{R}^n} v \cdot \varphi \, dx, \forall \varphi \in \mathcal{V} \right\}. \quad (3)$$

Formally, we write $L = -\Delta + V$ as a Schrödinger operator with potential V . Let $\{e^{-tL}\}_{t>0}$ be the semigroup of linear operators generated by $-L$ and let $p_t(x, y)$ be their kernels. Since V is a locally integrable nonnegative function on $\mathbb{R}^n$, then the Feynman-Kac formula implies that the semigroup kernels $p_t(x, y)$ associated with e^{-tL} satisfy the estimates:

$$0 \leq p_t(x, y) \leq \frac{1}{(4\pi t)^{n/2}} \exp \left\{ -\frac{|x-y|^2}{4t} \right\} \quad (4)$$

for all $t > 0$ and all $x, y \in \mathbb{R}^n$.

Since the Schrödinger operator L is a self-adjoint and positive definite operator acting on $L^2(\mathbb{R}^n)$, then L admits the following spectral decomposition:

$$L = \int_0^\infty \lambda \, dE_L(\lambda), \quad (5)$$

where $E_L(\lambda)$ denotes its spectral resolution. For any $\gamma \in \mathbb{R}$, we can define the imaginary power $L^{i\gamma}$ associated with L by the formula

$$L^{i\gamma} := \int_0^\infty \lambda^{i\gamma} dE_L(\lambda). \quad (6)$$

By the functional calculus for L , we can also define the operator $L^{i\gamma}$ as follows:

$$L^{i\gamma}(f)(x) := \frac{1}{\Gamma(-i\gamma)} \int_0^\infty t^{-i\gamma-1} e^{-tL}(f)(x) dt. \quad (7)$$

By the spectral theory, we know that $\|L^{i\gamma}\|_{L^2 \rightarrow L^2} = 1$ for all $\gamma \in \mathbb{R}$. Moreover, it was proved by Shen [2] that $L^{i\gamma}$ is a Calderón-Zygmund operator provided that $V \in \text{RH}_{n/2}$ (reverse Hölder class). We refer the reader to [2–4] for related results concerning the imaginary powers of self-adjoint operators.

For any $0 < \alpha < n$, the fractional integrals $L^{-\alpha/2}$ associated with L are defined by

$$L^{-\alpha/2}(f)(x) := \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{-tL}(f)(x) dt. \quad (8)$$

Since the kernel $p_t(x, y)$ of $\{e^{-tL}\}_{t>0}$ satisfies the Gaussian upper bounds (4), then it is easy to check that $|L^{-\alpha/2}(f)(x)| \leq C \cdot I_\alpha(|f|)(x)$ for all $x \in \mathbb{R}^n$, where I_α denotes the classical fractional integral operator (see [5]):

$$I_\alpha(f)(x) := \frac{\Gamma((n-\alpha)/2)}{2^\alpha \pi^{n/2} \Gamma(\alpha/2)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy. \quad (9)$$

Hence, by using the L^p - L^q boundedness of I_α (see [5]), we have

$$\|L^{-\alpha/2}(f)\|_{L^q} \leq C \|I_\alpha(f)\|_{L^q} \leq C \|f\|_{L^p}, \quad (10)$$

where $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. For more information about the fractional integrals $L^{-\alpha/2}$ associated with some classes of operators, one can see [6–9].

In [10], Song and Yan introduced the weighted Hardy space $H_L^1(w)$ associated with L in terms of the square function and established its atomic decomposition theory. Furthermore, they also showed that the Riesz transform $\nabla L^{-1/2}$ associated with L is bounded on $L^p(w)$ for $1 < p < 2$ and bounded from $H_L^1(w)$ to the classical weighted Hardy space $H^1(w)$ (see [11, 12] for $H^1(w)$).

Recently, in [13], we defined the weighted Hardy spaces $H_L^p(w)$ associated with L for $0 < p < 1$ and gave their atomic decompositions. We also obtained that $\nabla L^{-1/2}$ is bounded from $H_L^p(w)$ to the classical weighted Hardy space $H^p(w)$ (see also [11, 12] for $H^p(w)$) when $n/(n+1) < p < 1$. In this paper, we first define molecules for the weighted Hardy spaces $H_L^p(w)$ associated with L and then establish their molecular characterizations. As applications of the molecular characterization combining with the atomic decomposition of $H_L^p(w)$, we will obtain some estimates of $L^{i\gamma}$ and $L^{-\alpha/2}$ on $H_L^p(w)$ for $n/(n+1) < p \leq 1$. Our main results are stated as follows.

Theorem 1. *Let $L = -\Delta + V$, $n/(n+1) < p \leq 1$, and $w \in A_1 \cap \text{RH}_{(2/p)'}'$. Then, for any $\gamma \in \mathbb{R}$, the imaginary power $L^{i\gamma}$ is bounded from $H_L^p(w)$ to the weighted Lebesgue space $L^p(w)$.*

Theorem 2. *Let $L = -\Delta + V$, $n/(n+1) < p \leq 1$, and $w \in A_1 \cap \text{RH}_{(2/p)'}'$. Then, for any $\gamma \in \mathbb{R}$, the imaginary power $L^{i\gamma}$ is bounded on $H_L^p(w)$.*

Theorem 3. *Suppose that $L = -\Delta + V$. Let $0 < \alpha < n/2$, $n/(n+1) < p \leq 1$, $1/q = 1/p - \alpha/n$, and $w \in A_1 \cap \text{RH}_{(2/p)'}'$. Then, the fractional integral operator $L^{-\alpha/2}$ is bounded from $H_L^p(w)$ to $L^q(w^{q/p})$.*

Theorem 4. *Suppose that $L = -\Delta + V$. Let $0 < \alpha < \min\{n/2, 1\}$, $n/(n+1) < p \leq n/(n+\alpha)$, $1/q = 1/p - \alpha/n$, and $w \in A_1 \cap \text{RH}_{(2/p)'}'$. Then, the fractional integral operator $L^{-\alpha/2}$ is bounded from $H_L^p(w)$ to $H_L^q(w^{q/p})$.*

2. Notations and Preliminaries

Let us first recall some standard definitions and notations. The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^p boundedness of Hardy-Littlewood maximal functions in [14]. A weight w is a nonnegative, locally integrable function defined on $\mathbb{R}^n$, and $B = B(x_0, r_B)$ denotes the ball with the center x_0 and radius r_B . We say that $w \in A_1$, if

$$\frac{1}{|B|} \int_B w(x) dx \leq C \cdot \text{ess inf}_{x \in B} w(x), \quad (11)$$

for every ball $B \subseteq \mathbb{R}^n$,

where C is a positive constant which is independent of B . A weight function w is said to belong to the reverse Hölder class RH_s , if there exist two constants $s > 1$ and $C > 0$ such that the following reverse Hölder inequality holds:

$$\left(\frac{1}{|B|} \int_B w(x)^s dx \right)^{1/s} \leq C \left(\frac{1}{|B|} \int_B w(x) dx \right), \quad (12)$$

for every ball $B \subseteq \mathbb{R}^n$.

Given a ball B and $\lambda > 0$, λB denotes the ball with the same center as B whose radius is λ times that of B . For a given weight function w and a measurable set E , we denote the Lebesgue measure of E by $|E|$ and the weighted measure of E by $w(E)$, where $w(E) = \int_E w(x) dx$.

We give the following results which will be often used in the sequel.

Lemma 5 (see [15]). *Let $w \in A_1$. Then, for any ball B , there exists an absolute constant $C > 0$ such that*

$$w(2B) \leq C w(B). \quad (13)$$

In general, for any $\lambda > 1$, we have

$$w(\lambda B) \leq C \cdot \lambda^n w(B), \quad (14)$$

where C does not depend on B nor on λ .

Lemma 6 (see [15]). *Let $w \in A_1$. Then, there exists a constant $C > 0$ such that*

$$C \cdot \frac{|E|}{|B|} \leq \frac{w(E)}{w(B)} \quad (15)$$

for any measurable subset E of a ball B .

Given a weight function w on $\mathbb{R}^n$, for $0 < p < \infty$, we denote by $L^p(w)$ the weighted space of all functions f satisfying

$$\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty. \quad (16)$$

In particular, when w equals a constant function, we will denote $L^p(w)$ simply by $L^p(\mathbb{R}^n)$ and define

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} < \infty. \quad (17)$$

Throughout this paper, we will use C to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $1/C \leq A/B \leq C$. Moreover, we denote the conjugate exponent of $s > 1$ by $s' = s/(s-1)$.

3. Atomic Decomposition and Molecular Characterization of Weighted Hardy Spaces

Let $L = -\Delta + V$. For any $t > 0$, we define $P_t = e^{-tL}$ and

$$Q_{t,k} := (-t)^k \frac{d^k P_s}{ds^k} \Big|_{s=t} = (tL)^k e^{-tL}, \quad k = 1, 2, \dots \quad (18)$$

We denote $Q_{t,k}$ simply by Q_t when $k = 1$. First note that Gaussian upper bounds carry over from heat kernels to their time derivatives.

Lemma 7 (see [16, 17]). *For every $k = 1, 2, \dots$, there exist two positive constants C_k and c_k such that the kernel $p_{t,k}(x, y)$ of the operator $Q_{t,k}$ satisfies*

$$|p_{t,k}(x, y)| \leq \frac{C_k}{(4\pi t)^{n/2}} \exp \left\{ -\frac{|x-y|^2}{c_k t} \right\} \quad (19)$$

for all $t > 0$ and almost all $x, y \in \mathbb{R}^n$.

Let $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty)$. For any $x \in \mathbb{R}^n$, we set

$$\Gamma(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x-y| < t\}. \quad (20)$$

For a given function $f \in L^2(\mathbb{R}^n)$, we consider the square function associated with Schrödinger operator L , which is defined by (see [18, 19])

$$S_L(f)(x) := \left(\iint_{\Gamma(x)} |Q_{t^2}(f)(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}, \quad (21)$$

$x \in \mathbb{R}^n$.

Set

$$H_L^2(\mathbb{R}^n) = \overline{\mathcal{R}(L)} = \overline{\{Lu \in L^2(\mathbb{R}^n) : u \in L^2(\mathbb{R}^n)\}}, \quad (22)$$

where $\mathcal{R}(L)$ stands for the range of L in $L^2(\mathbb{R}^n)$. Given a weight function w on $\mathbb{R}^n$, in [10, 13], the authors defined the weighted Hardy spaces $H_L^p(w)$ for $0 < p \leq 1$ as the completion of $H_L^2(\mathbb{R}^n)$ in the norm given by the $L^p(w)$ -quasinorm of square functions; that is,

$$\|f\|_{H_L^p(w)} := \|S_L(f)\|_{L^p(w)}. \quad (23)$$

For the Schrödinger operator $L = -\Delta + V$, it can be shown that $H_L^2(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ (see, e.g., [19]). In [10], Song and Yan characterized weighted Hardy spaces $H_L^1(w)$ in terms of atoms in the following way and obtained their atomic characterizations.

Definition 8 (see [10]). Let $M \in \mathbb{N}$ and $p = 1$. A function $a(x) \in L^2(\mathbb{R}^n)$ is called a $(1, 2, M)$ -atom with respect to w (or a w -(1, 2, M)-atom) if there exist a ball $B = B(x_0, r_B)$ and a function $b \in \mathcal{D}(L^M)$ such that

- (a) $a = L^M b$;
- (b) $\text{supp } L^k b \subseteq B, k = 0, 1, \dots, M$;
- (c) $\|(r_B^2 L)^k b\|_{L^2(B)} \leq r_B^{2M} |B|^{1/2} [w(B)]^{-1}, k = 0, 1, \dots, M$.

Theorem 9 (see [10]). *Let $M \in \mathbb{N}$ and $w \in A_1 \cap RH_2$. If $f \in H_L^1(w) \cap L^2(\mathbb{R}^n)$, then there exist a family of w -(1, 2, M)-atoms $\{a_j\}$ and a sequence of real numbers $\{\lambda_j\}$ with $\sum_j |\lambda_j| \leq C \|f\|_{H_L^1(w)}$ such that f can be represented in the form $f(x) = \sum_j \lambda_j a_j(x)$, and the sum converges in the sense of both $L^2(\mathbb{R}^n)$ -norm and $H_L^1(w)$ -norm.*

Similarly, in [13], we introduced the notion of weighted atoms for $H_L^p(w)$ ($0 < p < 1$) and proved their atomic characterizations.

Definition 10 (see [13]). Let $M \in \mathbb{N}$ and $0 < p < 1$. A function $a(x) \in L^2(\mathbb{R}^n)$ is called a $(p, M; L)$ -atom with respect to w (or a w -($p, M; L$)-atom) if there exist a ball $B = B(x_0, r_B)$ and a function $b \in \mathcal{D}(L^M)$ such that

- (a') $a = L^M b$;
- (b') $\text{supp } L^k b \subseteq B, k = 0, 1, \dots, M$;
- (c') $\|(r_B^2 L)^k b\|_{L^2(B)} \leq r_B^{2M} |B|^{1/2} [w(B)]^{-1/p}, k = 0, 1, \dots, M$.

Theorem 11 (see [13]). *Let $M \in \mathbb{N}$ and $0 < p < 1$.*

- (i) *If $w \in A_1$ and $f \in H_L^p(w) \cap L^2(\mathbb{R}^n)$, then there exist a family of w -($p, M; L$)-atoms $\{a_j\}$ and a sequence of real numbers $\{\lambda_j\}$ with $\sum_j |\lambda_j|^p \leq C \|f\|_{H_L^p(w)}^p$ such that f can be represented in the form $f(x) = \sum_j \lambda_j a_j(x)$, and the sum converges in the sense of both $L^2(\mathbb{R}^n)$ -norm and $H_L^p(w)$ -norm.*

(ii) If $w \in A_1 \cap RH_{(2/p)'}'$ and $n/(n+1) < p < 1$, then every w -($p, M; L$)-atom a is in $H_L^p(w)$ and its $H_L^p(w)$ -norm is uniformly bounded; that is, there exists a constant $C > 0$ independent of a such that $\|a\|_{H_L^p(w)} \leq C$.

For every bounded Borel function $F : [0, \infty) \rightarrow \mathbb{C}$, we define the operator $F(L) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ by the following formula:

$$F(L) := \int_0^\infty F(\lambda) dE_L(\lambda), \quad (24)$$

where $E_L(\lambda)$ is the spectral decomposition of L . Therefore, the operator $\cos(t\sqrt{L})$ is well defined on $L^2(\mathbb{R}^n)$. Moreover, it follows from [20] that there exists a constant c_0 such that the Schwartz kernel $K_{\cos(t\sqrt{L})}(x, y)$ of $\cos(t\sqrt{L})$ has support contained in $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq c_0 t\}$. By the functional calculus for L and Fourier inversion formula, whenever F is an even bounded Borel function with $\hat{F} \in L^1(\mathbb{R})$, we can write $F(\sqrt{L})$ in terms of $\cos(t\sqrt{L})$. More precisely

$$F(\sqrt{L}) = (2\pi)^{-1} \int_{-\infty}^\infty \hat{F}(t) \cos(t\sqrt{L}) dt, \quad (25)$$

which gives

$$\begin{aligned} K_{F(\sqrt{L})}(x, y) &= (2\pi)^{-1} \int_{|t| \geq c_0^{-1}|x-y|} \hat{F}(t) K_{\cos(t\sqrt{L})}(x, y) dt. \end{aligned} \quad (26)$$

Lemma 12 (see [10, 19]). Let $\varphi \in C_0^\infty(\mathbb{R})$ be even and $\text{supp } \varphi \subseteq [-c_0^{-1}, c_0^{-1}]$. Let Φ denote the Fourier transform of φ . Then, for each $j = 0, 1, \dots$ and for all $t > 0$, the Schwartz kernel $K_{(t^2 L)^j \Phi(t\sqrt{L})}(x, y)$ of $(t^2 L)^j \Phi(t\sqrt{L})$ satisfies

$$\text{supp } K_{(t^2 L)^j \Phi(t\sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq t\}. \quad (27)$$

For a given real number $s > 0$, we define

$$\begin{aligned} \mathcal{F}(s) := \left\{ \psi : \mathbb{C} \longrightarrow \mathbb{C} \text{ measurable, } |\psi(z)| \leq C \right. \\ \left. \cdot \frac{|z|^s}{1 + |z|^{2s}} \right\}. \end{aligned} \quad (28)$$

Then, for any nonzero function $\psi \in \mathcal{F}(s)$, we can prove the following estimate (see [10, 19]):

$$\left(\int_0^\infty \|\psi(t\sqrt{L})f\|_{L^2}^2 \frac{dt}{t} \right)^{1/2} = C \|f\|_{L^2}, \quad (29)$$

where $C = (\int_0^\infty |\psi(t)|^2 (dt/t))^{1/2}$. Inspired by [19, 21, 22], we are now going to define the weighted molecules corresponding to the weighted atoms mentioned above.

Definition 13. Let $\varepsilon > 0$, $M \in \mathbb{N}$, and $0 < p \leq 1$. A function $m(x) \in L^2(\mathbb{R}^n)$ is called a w -($p, M, \varepsilon; L$)-molecule associated with L , if there exist a ball $B = B(x_0, r_B)$ and a function $b \in \mathcal{D}(L^M)$ such that

- (A) $m = L^M b$;
- (B) $\|(r_B^2 L)^k b\|_{L^2(2B)} \leq r_B^{2M} |B|^{1/2} [w(B)]^{-1/p}$, $k = 0, 1, \dots, M$;
- (C) $\|(r_B^2 L)^k b\|_{L^2(2^{j+1}B \setminus 2^j B)} \leq 2^{-j\varepsilon} r_B^{2M} |2^j B|^{1/2} [w(2^j B)]^{-1/p}$, $k = 0, 1, \dots, M$, $j = 1, 2, \dots$.

Clearly, for every w -($p, M; L$)-atom a , it is also a w -($p, M, \varepsilon; L$)-molecule for all $\varepsilon > 0$. Then, we are able to establish the following molecular characterizations for the weighted Hardy spaces $H_L^p(w)$ ($0 < p \leq 1$) associated with L .

Theorem 14. Let $\varepsilon > 0$, $M \in \mathbb{N}$, and $0 < p \leq 1$.

- (i) If $f \in H_L^p(w) \cap L^2(\mathbb{R}^n)$ and $w \in A_1$, then there exist a family of w -($p, M, \varepsilon; L$)-molecules $\{m_j\}$ and a sequence of real numbers $\{\lambda_j\}$ with $\sum_j |\lambda_j|^p \leq C \|f\|_{H_L^p(w)}^p$ such that $f(x) = \sum_j \lambda_j m_j(x)$, and the sum converges in the sense of both $L^2(\mathbb{R}^n)$ -norm and $H_L^p(w)$ -norm.
- (ii) Assume that $M > (n/2)(1/p - 1/2)$ and $w \in A_1 \cap RH_{(2/p)'}'$. Then, every w -($p, M, \varepsilon; L$)-molecule m is in $H_L^p(w)$. Moreover, there exists a constant $C > 0$ independent of m such that $\|m\|_{H_L^p(w)} \leq C$.

Proof. (i) This is a straightforward consequence of Theorems 9 and 11. (ii) We will use some ideas from [19, 22]. Suppose that m is a w -($p, M, \varepsilon; L$)-molecule associated with a ball $B = B(x_0, r_B)$. Let $\varphi \in C_0^\infty(\mathbb{R})$ be even with $\text{supp } \varphi \subseteq [-(2c_0)^{-1}, (2c_0)^{-1}]$ and let Φ denote the Fourier transform of φ . We set $\Psi(x) = x^2 \Phi(x)$, $x \in \mathbb{R}$. By the L^2 -functional calculus of L , for every $m \in L^2(\mathbb{R}^n)$, we can establish the following version of the Calderón reproducing formula:

$$m(x) = c_\Psi \int_0^\infty (t^2 L)^M \Psi^2(t\sqrt{L})(m)(x) \frac{dt}{t}, \quad (30)$$

where the above equality holds in the sense of $L^2(\mathbb{R}^n)$ -norm. Set

$$\begin{aligned} U_0(B) &= 2B, \\ U_j(B) &= 2^{j+1}B \setminus 2^j B, \quad j = 1, 2, \dots \end{aligned} \quad (31)$$

Then, we can decompose

$$\begin{aligned} \mathbb{R}^n \times (0, \infty) &= \left(\bigcup_{j=0}^\infty U_j(B) \times (0, 2^j r_B] \right) \\ &\cup \left(\bigcup_{j=1}^\infty 2^j B \times (2^{j-1} r_B, 2^j r_B] \right). \end{aligned} \quad (32)$$

For any measurable set E in $\mathbb{R}^n$, we denote χ_E the characteristic function of the set E . Hence, by formula (30), we are able to write

$$\begin{aligned} m(x) &= c_\Psi \sum_{j=0}^{\infty} \int_0^{2^j r_B} (t^2 L)^M \Psi^2(t\sqrt{L}) (m\chi_{U_j(B)})(x) \frac{dt}{t} \\ &\quad + c_\Psi \sum_{j=1}^{\infty} \int_{2^{j-1} r_B}^{2^j r_B} (t^2 L)^M \Psi^2(t\sqrt{L}) (m\chi_{2^j B})(x) \frac{dt}{t} \quad (33) \\ &:= \sum_{j=0}^{\infty} m_j^{(1)}(x) + \sum_{j=1}^{\infty} m_j^{(2)}(x). \end{aligned}$$

Let us first consider the terms $\{m_j^{(1)}\}_{j=0}^{\infty}$. We will show that each $m_j^{(1)}$ is a multiple of a w -($p, M; L$)-atom with a sequence of coefficients in ℓ^p . Indeed, for every $j = 0, 1, 2, \dots$, one can write

$$m_j^{(1)}(x) = L^M b_j(x), \quad (34)$$

where

$$b_j(x) = c_\Psi \int_0^{2^j r_B} t^{2M} \Psi^2(t\sqrt{L}) (m\chi_{U_j(B)})(x) \frac{dt}{t}. \quad (35)$$

By using Lemma 12, we can easily conclude that, for every $k = 0, 1, \dots, M$, $\text{supp}(L^k b_j) \subseteq 2^{j+1} B$. In addition, by the duality argument, we get

$$\begin{aligned} &\left\| \left[(2^{j+1} r_B)^2 L \right]^k b_j \right\|_{L^2(2^{j+1} B)} \\ &= \sup_{\|h\|_{L^2(2^{j+1} B)} \leq 1} \left| \int_{2^{j+1} B} \left[(2^{j+1} r_B)^2 L \right]^k b_j(x) h(x) dx \right|. \quad (36) \end{aligned}$$

Then, it follows from Hölder's inequality and estimate (29) that

$$\begin{aligned} &\left| \int_{2^{j+1} B} \left[(2^{j+1} r_B)^2 L \right]^k b_j(x) h(x) dx \right| = c_\Psi (2^{j+1} r_B)^{2k} \\ &\quad \cdot \left| \int_0^{2^j r_B} \int_{2^{j+1} B} t^{2M} L^k \Psi(t\sqrt{L}) (m\chi_{U_j(B)})(y) \right. \\ &\quad \cdot \Psi(t\sqrt{L})(h)(y) \frac{dy dt}{t} \left. \right| \leq c_\Psi (2^{j+1} r_B)^{2k} \\ &\quad \cdot (2^j r_B)^{2M-2k} \left| \int_0^{2^j r_B} \int_{2^{j+1} B} (t^2 L)^k \Psi(t\sqrt{L}) \right. \\ &\quad \cdot (m\chi_{U_j(B)})(y) \cdot \Psi(t\sqrt{L})(h)(y) \frac{dy dt}{t} \left. \right| \end{aligned}$$

$$\begin{aligned} &\leq c_\Psi (2^{j+1} r_B)^{2M} \left(\int_0^\infty \left\| (t^2 L)^k \Psi(t\sqrt{L}) \right. \right. \\ &\quad \cdot (m\chi_{U_j(B)}) \left. \left. \right\|_{L^2}^2 \frac{dt}{t} \right)^{1/2} \left(\int_0^\infty \left\| \Psi(t\sqrt{L}) \right. \right. \\ &\quad \cdot (h\chi_{2^{j+1} B}) \left. \left. \right\|_{L^2}^2 \frac{dt}{t} \right)^{1/2} \leq c_\Psi (2^{j+1} r_B)^{2M} \\ &\quad \cdot \left\| m\chi_{U_j(B)} \right\|_{L^2} \cdot \|h\chi_{2^{j+1} B}\|_{L^2} \leq C \cdot 2^{-j\varepsilon} (2^{j+1} r_B)^{2M} \\ &\quad \cdot |2^{j+1} B|^{1/2} [w(2^{j+1} B)]^{-1/p}. \quad (37) \end{aligned}$$

Hence,

$$\begin{aligned} &\left\| \left[(2^{j+1} r_B)^2 L \right]^k b_j \right\|_{L^2(2^{j+1} B)} \\ &\leq C \cdot 2^{-j\varepsilon} (2^{j+1} r_B)^{2M} |2^{j+1} B|^{1/2} [w(2^{j+1} B)]^{-1/p}, \quad (38) \end{aligned}$$

which implies our desired result. Next we consider the terms $\{m_j^{(2)}\}_{j=1}^{\infty}$. For every $j = 1, 2, \dots$, we write

$$\begin{aligned} m_j^{(2)}(x) &= c_\Psi \int_{2^{j-1} r_B}^{2^j r_B} (t^2 L)^M \Psi^2(t\sqrt{L}) (m)(x) \frac{dt}{t} \\ &\quad - c_\Psi \int_{2^{j-1} r_B}^{2^j r_B} (t^2 L)^M \Psi^2(t\sqrt{L}) (m\chi_{(2^j B)^c})(x) \frac{dt}{t} \\ &:= m_j^{(21)}(x) - m_j^{(22)}(x). \quad (39) \end{aligned}$$

To deal with the term $m_j^{(21)}$, we recall that $m = L^M b$ for some $b \in \mathcal{D}(L^M)$, and then

$$\begin{aligned} m_j^{(21)}(x) &= c_\Psi \int_{2^{j-1} r_B}^{2^j r_B} (t^2 L)^M \Psi^2(t\sqrt{L}) (L^M b)(x) \frac{dt}{t} \\ &= L^M b_j^{(21)}(x), \quad (40) \end{aligned}$$

where

$$b_j^{(21)}(x) = c_\Psi \int_{2^{j-1} r_B}^{2^j r_B} (t^2 L)^M \Psi^2(t\sqrt{L}) (b)(x) \frac{dt}{t}. \quad (41)$$

Since $b(x) = b(x)\chi_{2^j B}(x) + \sum_{l=j}^{\infty} b(x)\chi_{U_l(B)}(x)$, then we can further write

$$b_j^{(21)}(x) = b_{1,j}^{(21)}(x) + \sum_{l=j}^{\infty} b_{l,j}^{(21)}(x), \quad (42)$$

where

$$\begin{aligned} b_{1,j}^{(21)}(x) &= c_\Psi \int_{2^{j-1}r_B}^{2^j r_B} (t^2 L)^M \Psi^2(t\sqrt{L}) (b\chi_{2^j B})(x) \frac{dt}{t}, \\ b_{ij}^{(21)}(x) &= c_\Psi \int_{2^{j-1}r_B}^{2^j r_B} (t^2 L)^M \Psi^2(t\sqrt{L}) (b\chi_{U_l(B)})(x) \frac{dt}{t}. \end{aligned} \quad (43)$$

By using Lemma 12 again, we have $\text{supp}(L^k b_{1,j}^{(21)}) \subseteq 2^j B$ and $\text{supp}(L^k b_{ij}^{(21)}) \subseteq 2^{l+1} B$ for every $k = 0, 1, \dots, M$. Moreover, it follows from Minkowski's integral inequality that

$$\begin{aligned} &\left\| \left[(2^j r_B)^2 L \right]^k b_{1,j}^{(21)} \right\|_{L^2(2^j B)} = c_\Psi (2^j r_B)^{2k} \\ &\cdot \left\| \int_{2^{j-1}r_B}^{2^j r_B} t^{2M} L^{M+k} \Psi^2(t\sqrt{L}) (b\chi_{2^j B}) \frac{dt}{t} \right\|_{L^2(2^j B)} \\ &\leq c_\Psi (2^j r_B)^{2k} \\ &\cdot \int_{2^{j-1}r_B}^{2^j r_B} \left\| (t^2 L)^{M+k} \Psi^2(t\sqrt{L}) (b\chi_{2^j B}) \right\|_{L^2(2^j B)} \frac{dt}{t^{2k+1}} \\ &\leq C \|b\chi_{2^j B}\|_{L^2(2^j B)} \leq C \sum_{l=0}^{j-1} \|b\chi_{U_l(B)}\|_{L^2(2^j B)} \\ &\leq C \sum_{l=0}^{j-1} 2^{-l\epsilon} r_B^{2M} |2^l B|^{1/2} [w(2^l B)]^{-1/p}. \end{aligned} \quad (44)$$

When $0 \leq l \leq j-1$, then $2^l B \subseteq 2^j B$. Since $w \in A_1$, then, by using Lemma 6, we can get

$$\frac{w(2^l B)}{w(2^j B)} \geq C \cdot \frac{|2^l B|}{|2^j B|}. \quad (45)$$

Consequently,

$$\begin{aligned} &\left\| \left[(2^j r_B)^2 L \right]^k b_{1,j}^{(21)} \right\|_{L^2(2^j B)} \leq C \cdot 2^{-j[2M-n(1/p-1/2)]} \\ &\cdot (2^j r_B)^{2M} |2^j B|^{1/2} [w(2^j B)]^{-1/p} \sum_{l=0}^{\infty} \frac{1}{2^{l\epsilon}} \cdot \frac{1}{2^{l(n/p-n/2)}} \\ &\leq C \cdot 2^{-j[2M-n(1/p-1/2)]} \cdot (2^j r_B)^{2M} |2^j B|^{1/2} \\ &\cdot [w(2^j B)]^{-1/p}. \end{aligned} \quad (46)$$

On the other hand,

$$\begin{aligned} &\left\| \left[(2^{l+1} r_B)^2 L \right]^k b_{ij}^{(21)} \right\|_{L^2(2^{l+1} B)} = c_\Psi (2^{l+1} r_B)^{2k} \\ &\cdot \left\| \int_{2^{j-1}r_B}^{2^j r_B} t^{2M} L^{M+k} \Psi^2(t\sqrt{L}) (b\chi_{U_l(B)}) \frac{dt}{t} \right\|_{L^2(2^{l+1} B)} \\ &\leq c_\Psi (2^{l+1} r_B)^{2k} \int_{2^{j-1}r_B}^{2^j r_B} \left\| (t^2 L)^{M+k} \Psi^2(t\sqrt{L}) \right. \\ &\cdot (b\chi_{U_l(B)}) \left. \right\|_{L^2(2^{l+1} B)} \frac{dt}{t^{2k+1}} \leq C (2^{l+1} r_B)^{2k} \\ &\cdot \|b\chi_{U_l(B)}\|_{L^2(2^{l+1} B)} \cdot \frac{1}{(2^j r_B)^{2k}} \leq C \cdot 2^{-l\epsilon} (2^{l+1} r_B)^{2M} \\ &\cdot |2^{l+1} B|^{1/2} [w(2^{l+1} B)]^{-1/p}. \end{aligned} \quad (47)$$

Observe that $2M > n(1/p - 1/2)$. Thus, from the above discussions, we have already proved that each $m_j^{(21)}$ is a multiple of a w -($p, M; L$)-atom with a sequence of coefficients in ℓ^p . Finally, we estimate the terms $\{m_j^{(22)}\}_{j=1}^{\infty}$. For every $j = 1, 2, \dots$, we decompose $m_j^{(22)}$ as follows:

$$\begin{aligned} m_j^{(22)}(x) &= c_\Psi \sum_{l=j}^{\infty} \int_{2^{j-1}r_B}^{2^j r_B} (t^2 L)^M \Psi^2(t\sqrt{L}) (m\chi_{U_l(B)})(x) \frac{dt}{t} \\ &= \sum_{l=j}^{\infty} L^M b_{lj}^{(22)}(x), \end{aligned} \quad (48)$$

where

$$b_{lj}^{(22)}(x) = c_\Psi \int_{2^{j-1}r_B}^{2^j r_B} t^{2M} \Psi^2(t\sqrt{L}) (m\chi_{U_l(B)})(x) \frac{dt}{t}. \quad (49)$$

It follows immediately from Lemma 12 that $\text{supp}(L^k b_{lj}^{(22)}) \subseteq 2^{l+1} B$ for every $k = 1, 2, \dots, M$ and $l \geq j$. Moreover,

$$\begin{aligned} &\left\| \left[(2^{l+1} r_B)^2 L \right]^k b_{lj}^{(22)} \right\|_{L^2(2^{l+1} B)} = c_\Psi (2^{l+1} r_B)^{2k} \\ &\cdot \left\| \int_{2^{j-1}r_B}^{2^j r_B} t^{2M} L^k \Psi^2(t\sqrt{L}) (m\chi_{U_l(B)}) \frac{dt}{t} \right\|_{L^2(2^{l+1} B)} \\ &\leq c_\Psi (2^{l+1} r_B)^{2k} (2^j r_B)^{2M-2k} \\ &\cdot \int_{2^{j-1}r_B}^{2^j r_B} \left\| (t^2 L)^k \Psi^2(t\sqrt{L}) (m\chi_{U_l(B)}) \right\|_{L^2(2^{l+1} B)} \frac{dt}{t} \\ &\leq c_\Psi (2^{l+1} r_B)^{2M} \|m\chi_{U_l(B)}\|_{L^2(2^{l+1} B)} \leq C \\ &\cdot 2^{-l\epsilon} (2^{l+1} r_B)^{2M} |2^{l+1} B|^{1/2} [w(2^{l+1} B)]^{-1/p}. \end{aligned} \quad (50)$$

Therefore, we have showed that each $m_j^{(22)}$ is also a multiple of a w -($p, M; L$)-atom with a sequence of coefficients in ℓ^p . This completes the proof of Theorem 14. $\square$

4. Proof of Theorems 1 and 2

Proof of Theorem 1. By the known result, we have that, for any $\gamma \in \mathbb{R}$, the operator $L^{i\gamma}$ is linear and bounded on $L^2(\mathbb{R}^n)$ (see, e.g., [2, 23]). Since $H_L^p(w) \cap L^2(\mathbb{R}^n)$ is dense in $H_L^p(w)$, then, by Theorems 9 and 11 and a standard density argument, it is enough for us to show that, for any w -($p, M; L$)-atom a , $M \in \mathbb{N}$, there exists a constant $C > 0$ independent of a such that $\|L^{i\gamma}(a)\|_{L^p(w)} \leq C$. Let a be a w -($p, M; L$)-atom associated with a ball $B = B(x_0, r_B)$, $\|a\|_{L^2(B)} \leq |B|^{1/2}[w(B)]^{-1/p}$. We write

$$\begin{aligned} \|L^{i\gamma}(a)\|_{L^p(w)}^p &= \int_{2B} |L^{i\gamma}(a)(x)|^p w(x) dx \\ &\quad + \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |L^{i\gamma}(a)(x)|^p w(x) dx \quad (51) \\ &:= I_1 + I_2. \end{aligned}$$

We set $s = 2/p > 1$. Note that $w \in \text{RH}_{s'}$, and then it follows from Hölder's inequality, the L^2 boundedness of $L^{i\gamma}$, and Lemma 5 that

$$\begin{aligned} I_1 &\leq \left(\int_{2B} |L^{i\gamma}(a)(x)|^2 dx \right)^{p/2} \left(\int_{2B} w(x)^{s'} dx \right)^{1/s'} \\ &\leq C \|a\|_{L^2(B)}^p \cdot \frac{w(2B)}{|2B|^{1/s}} \leq C. \end{aligned} \quad (52)$$

On the other hand, for any $x \in 2^{k+1}B \setminus 2^k B$, $k = 1, 2, \dots$, by expression (7), we can write

$$\begin{aligned} |L^{i\gamma}(a)(x)| &\leq C \int_0^{\infty} |e^{-tL}(a)(x)| \frac{dt}{t} \\ &\leq C \int_0^{r_B^2} |e^{-tL}(a)(x)| \frac{dt}{t} \quad (53) \\ &\quad + C \int_{r_B^2}^{\infty} |e^{-tL}(a)(x)| \frac{dt}{t} := \text{I} + \text{II}. \end{aligned}$$

For the term I, we observe that when $x \in 2^{k+1}B \setminus 2^k B$, $y \in B$, then $|x - y| \geq 2^{k-1}r_B$. Hence, by using Hölder's inequality and estimate (4), we deduce

$$\begin{aligned} |e^{-tL}(a)(x)| &\leq C \cdot \frac{\sqrt{t}}{(2^{k-1}r_B)^{n+1}} \int_B |a(y)| dy \\ &\leq C \cdot \frac{\sqrt{t}}{(2^k r_B)^{n+1}} \cdot \|a\|_{L^2(B)} |B|^{1/2} \quad (54) \\ &\leq C \cdot [w(B)]^{-1/p} \cdot \frac{\sqrt{t}}{2^{k(n+1)} \cdot r_B}. \end{aligned}$$

So we have

$$\begin{aligned} \text{I} &\leq C \cdot \frac{1}{2^{k(n+1)} [w(B)]^{1/p}} \cdot \frac{1}{r_B} \int_0^{r_B^2} \frac{dt}{\sqrt{t}} \\ &\leq C \cdot \frac{1}{2^{k(n+1)} [w(B)]^{1/p}}. \end{aligned} \quad (55)$$

We now turn to estimate the other term II. In this case, since there exists a function $b \in \mathcal{D}(L^M)$ such that $a = L^M b$ and $\|b\|_{L^2(B)} \leq r_B^{2M} |B|^{1/2} [w(B)]^{-1/p}$, then it follows from Hölder's inequality and Lemma 7 that

$$\begin{aligned} |e^{-tL}(a)(x)| &= |(tL)^M e^{-tL}(b)(x)| \cdot \frac{1}{t^M} \\ &\leq C \cdot \frac{1}{(2^{k-1}r_B)^{n+1}} \int_B |b(y)| dy \cdot \frac{1}{t^{M-1/2}} \\ &\leq C \cdot \frac{1}{(2^k r_B)^{n+1}} \|b\|_{L^2(B)} |B|^{1/2} \cdot \frac{1}{t^{M-1/2}} \quad (56) \\ &\leq C \cdot \frac{r_B^{2M-1}}{2^{k(n+1)} [w(B)]^{1/p}} \cdot \frac{1}{t^{M-1/2}}. \end{aligned}$$

Consequently,

$$\begin{aligned} \text{II} &\leq C \cdot \frac{1}{2^{k(n+1)} [w(B)]^{1/p}} \cdot r_B^{2M-1} \int_{r_B^2}^{\infty} \frac{dt}{t^{M+1/2}} \\ &\leq C \cdot \frac{1}{2^{k(n+1)} [w(B)]^{1/p}}, \end{aligned} \quad (57)$$

where in the last inequality we have used the fact that $M \geq 1$. Therefore, by combining the above estimates for I and II, we obtain the following pointwise inequality:

$$|L^{i\gamma}(a)(x)| \leq C \cdot \frac{1}{2^{k(n+1)} [w(B)]^{1/p}}, \quad (58)$$

when $x \in 2^{k+1}B \setminus 2^k B$.

Notice that $w \in A_1$. Substituting inequality (58) into the term I_2 and using Lemma 5, then we have

$$\begin{aligned} I_2 &\leq C \sum_{k=1}^{\infty} \frac{1}{2^{kp(n+1)} w(B)} \cdot w(2^{k+1}B) \\ &\leq C \sum_{k=1}^{\infty} \frac{1}{2^{k[p(n+1)-n]}} \leq C, \end{aligned} \quad (59)$$

where the last series is convergent since $p > n/(n+1)$. Summarizing estimates (52) and (59) derived above, we complete the proof of Theorem 1. $\square$

Proof of Theorem 2. Since $H_L^p(w) \cap L^2(\mathbb{R}^n)$ is dense in $H_L^p(w)$ and the operator $L^{i\gamma}$ is linear and bounded on $L^2(\mathbb{R}^n)$, then, in view of Theorems 9, 11, and 14, it suffices to verify that, for every w -($p, 2M; L$)-atom a , the function $m = L^{i\gamma}(a)$ is a multiple of a w -($p, M, \varepsilon; L$)-molecule for some $\varepsilon > 0$, and

the multiplicative constant is independent of a . Let a be a w -($p, 2M; L$)-atom with $\text{supp } a \subseteq B = B(x_0, r_B)$. By the definition, there exists a function $b \in \mathcal{D}(L^{2M})$ such that

$$a = L^{2M}(b)$$

$$\left\| (r_B^2 L)^k b \right\|_{L^2(B)} \leq r_B^{4M} |B|^{1/2} [w(B)]^{-1/p}, \quad (60)$$

$$k = 0, 1, \dots, 2M.$$

We set $\tilde{b} = L^{iy}(L^M b)$, and then $m = L^M(\tilde{b})$. Obviously, we have $m(x) \in L^2(\mathbb{R}^n)$. Moreover, for $k = 0, 1, \dots, M$, we can deduce

$$\begin{aligned} \left\| (r_B^2 L)^k \tilde{b} \right\|_{L^2(2B)} &= \frac{1}{r_B^{2M}} \left\| L^{iy} \left[(r_B^2 L)^{M+k} b \right] \right\|_{L^2(2B)} \\ &\leq C \cdot \frac{1}{r_B^{2M}} \left\| (r_B^2 L)^{M+k} b \right\|_{L^2(B)} \\ &\leq C \cdot r_B^{2M} |B|^{1/2} [w(B)]^{-1/p}. \end{aligned} \quad (61)$$

It remains to estimate $\|(r_B^2 L)^k \tilde{b}\|_{L^2(2^{j+1}B \setminus 2^j B)}$ for $k = 0, 1, \dots, M, j = 1, 2, \dots$. We write

$$\begin{aligned} \left| (r_B^2 L)^k \tilde{b}(x) \right| &= \left| L^{iy} \left[(r_B^2 L)^k L^M b \right](x) \right| \\ &\leq C \int_0^{r_B^2} \left| e^{-tL} \left[r_B^{2k} L^{M+k} b \right](x) \right| \frac{dt}{t} \\ &\quad + C \int_{r_B^2}^\infty \left| e^{-tL} \left[r_B^{2k} L^{M+k} b \right](x) \right| \frac{dt}{t} \\ &:= I' + II'. \end{aligned} \quad (62)$$

As mentioned in the proof of Theorem 1, we know that when $x \in 2^{j+1}B \setminus 2^j B, y \in B$, then $|x - y| \geq 2^{j-1}r_B, j = 1, 2, \dots$. It follows from Hölder's inequality and estimate (4) that

$$\begin{aligned} I' &\leq C \int_0^{r_B^2} \frac{\sqrt{t}}{(2^{j-1}r_B)^{n+1}} \left\| r_B^{2k} L^{M+k} b \right\|_{L^2} |B|^{1/2} \frac{dt}{t} \leq C \\ &\quad \cdot \frac{1}{(2^j r_B)^{n+1}} \left(\frac{1}{r_B^{2M}} \cdot r_B^{4M} |B|^{1/2} [w(B)]^{-1/p} \right) |B|^{1/2} \\ &\quad \cdot \int_0^{r_B^2} \frac{dt}{\sqrt{t}} \leq C \cdot \frac{1}{2^{j(n+1)}} \cdot r_B^{2M} [w(B)]^{-1/p}. \end{aligned} \quad (63)$$

Since $w \in A_1$ and $B \subseteq 2^j B, j = 1, 2, \dots$, then, by using Lemma 6, we can get

$$\frac{w(B)}{w(2^j B)} \geq C \cdot \frac{|B|}{|2^j B|}. \quad (64)$$

Hence,

$$\begin{aligned} I' &\leq C \cdot \frac{1}{2^{j[(n+1)-n/p]}} \cdot r_B^{2M} [w(2^j B)]^{-1/p}, \\ &\quad \text{when } x \in 2^{j+1}B \setminus 2^j B. \end{aligned} \quad (65)$$

Applying Hölder's inequality and Lemma 7, we obtain

$$\begin{aligned} II' &\leq C \cdot r_B^{2k} \int_{r_B^2}^\infty \left| (tL)^{M+k} e^{-tL}(b)(x) \right| \frac{dt}{t^{M+k+1/2}} \\ &\leq C \cdot r_B^{2k} \int_{r_B^2}^\infty \frac{1}{(2^{j-1}r_B)^{n+1}} \|b\|_{L^2(B)} |B|^{1/2} \frac{dt}{t^{M+k+1/2}} \\ &\leq C \\ &\quad \cdot \frac{1}{(2^j r_B)^{n+1}} (r_B^{4M+2k} |B| [w(B)]^{-1/p}) \int_{r_B^2}^\infty \frac{dt}{t^{M+k+1/2}} \\ &\leq C \cdot \frac{1}{2^{j(n+1)}} \cdot r_B^{2M} [w(B)]^{-1/p}. \end{aligned} \quad (66)$$

It follows immediately from inequality (64) that

$$\begin{aligned} II' &\leq C \cdot \frac{1}{2^{j[(n+1)-n/p]}} \cdot r_B^{2M} [w(2^j B)]^{-1/p}, \\ &\quad \text{when } x \in 2^{j+1}B \setminus 2^j B. \end{aligned} \quad (67)$$

Combining the above estimates for I' and II' , we thus obtain

$$\begin{aligned} \left\| (r_B^2 L)^k \tilde{b} \right\|_{L^2(2^{j+1}B \setminus 2^j B)} \\ \leq C \cdot \frac{1}{2^{j[(n+1)-n/p]}} \cdot r_B^{2M} |2^j B|^{1/2} [w(2^j B)]^{-1/p}. \end{aligned} \quad (68)$$

Observe that $p > n/(n+1)$. If we set $\varepsilon = (n+1) - n/p$, then we have $\varepsilon > 0$. Therefore, we have proved that the function $m = L^{iy}(a)$ is a multiple of a w -($p, M, \varepsilon; L$)-molecule. This completes the proof of Theorem 2. $\square$

5. Proof of Theorems 3 and 4

Proof of Theorem 3. As in the proof of Theorem 1, assuming first that $f \in H_L^p(w) \cap L^2(\mathbb{R}^n)$, we are going to prove that, for every w -($p, M; L$)-atom a associated with a ball $B = B(x_0, r_B)$, $M > (3n)/4$, there exists a constant $C > 0$ independent of a such that $\|L^{-\alpha/2}(a)\|_{L^q(w^{q/p})} \leq C$. We write

$$\begin{aligned} &\|L^{-\alpha/2}(a)\|_{L^q(w^{q/p})}^q \\ &= \int_{2B} \left| L^{-\alpha/2}(a)(x) \right|^q w(x)^{q/p} dx \\ &\quad + \sum_{k=1}^\infty \int_{2^{k+1}B \setminus 2^k B} \left| L^{-\alpha/2}(a)(x) \right|^q w(x)^{q/p} dx \\ &:= J_1 + J_2. \end{aligned} \quad (69)$$

By our assumption $0 < \alpha < n/2$, $1/q = 1/p - \alpha/n$, then we are able to choose a number $\mu > q$ such that $1/\mu = 1/2 - \alpha/n$. Set $s = 2/p$. By a simple calculation, we can see that

$$\begin{aligned} \left(\frac{q}{p}\right) \cdot \left(\frac{\mu}{q}\right)' &= \left(\frac{1}{p}\right) \cdot \left(\frac{q\mu}{(\mu - q)}\right) \\ &= \left(\frac{1}{p}\right) \cdot \left(\frac{1}{p} - \frac{1}{2}\right)^{-1} = s', \\ 1 - \frac{q}{\mu} &= q \left(\frac{1}{q} - \frac{1}{\mu}\right) = q \left(\frac{1}{p} - \frac{1}{2}\right) = \frac{q}{(ps')}. \end{aligned} \quad (70)$$

Applying Hölder's inequality with exponent $\nu = \mu/q > 1$, the L^2 - L^μ boundedness of $L^{-\alpha/2}$ (see (10)), Lemma 5, and $w \in \text{RH}_{s'}$, we get

$$\begin{aligned} J_1 &\leq \left(\int_{2B} |L^{-\alpha/2}(a)(x)|^{q(\mu/q)} dx \right)^{q/\mu} \\ &\quad \cdot \left(\int_{2B} w(x)^{(q/p) \cdot (\mu/q)'} dx \right)^{1-q/\mu} \\ &= \left(\int_{2B} |L^{-\alpha/2}(a)(x)|^\mu dx \right)^{q/\mu} \\ &\quad \cdot \left(\int_{2B} w(x)^{s'} dx \right)^{q/(ps')} \leq C \|a\|_{L^2(B)}^q \left(\frac{w(2B)}{|2B|^{1/s}} \right)^{q/p} \\ &\leq C. \end{aligned} \quad (71)$$

We now turn to deal with J_2 . For any $x \in 2^{k+1}B \setminus 2^k B$, $k = 1, 2, \dots$, by expression (8), we can write

$$\begin{aligned} |L^{-\alpha/2}(a)(x)| &\leq C \int_0^\infty |e^{-tL}(a)(x)| \frac{dt}{t^{1-\alpha/2}} \\ &\leq C \int_0^{r_B^2} |e^{-tL}(a)(x)| \frac{dt}{t^{1-\alpha/2}} \\ &\quad + C \int_{r_B^2}^\infty |e^{-tL}(a)(x)| \frac{dt}{t^{1-\alpha/2}} \\ &:= \text{III} + \text{IV}. \end{aligned} \quad (72)$$

For the term III, it follows immediately from (54) that

$$\begin{aligned} \text{III} &\leq C \cdot \frac{1}{2^{k(n+1)} [w(B)]^{1/p}} \cdot \frac{1}{r_B} \int_0^{r_B^2} \frac{dt}{t^{1/2-\alpha/2}} \\ &\leq C \cdot \frac{r_B^\alpha}{2^{k(n+1)} [w(B)]^{1/p}}. \end{aligned} \quad (73)$$

For the other term IV, by the previous estimate (56), we thus have

$$\begin{aligned} \text{IV} &\leq C \cdot \frac{1}{2^{k(n+1)} [w(B)]^{1/p}} \cdot r_B^{2M-1} \int_{r_B^2}^\infty \frac{dt}{t^{M+1/2-\alpha/2}} \\ &\leq C \cdot \frac{r_B^\alpha}{2^{k(n+1)} [w(B)]^{1/p}}, \end{aligned} \quad (74)$$

where the last inequality holds since $M > (3n)/4 > 1/2 + \alpha/2$. Combining the above estimates for III and IV, we obtain the following pointwise inequality:

$$|L^{-\alpha/2}(a)(x)| \leq C \cdot \frac{r_B^\alpha}{2^{k(n+1)} [w(B)]^{1/p}}, \quad \text{when } x \in 2^{k+1}B \setminus 2^k B. \quad (75)$$

Note that $w \in A_1$. Substituting inequality (75) into the term J_2 , we get

$$J_2 \leq C \sum_{k=1}^\infty \left(\frac{r_B^\alpha}{2^{k(n+1)} [w(B)]^{1/p}} \right)^q \int_{2^{k+1}B} w(x)^{q/p} dx. \quad (76)$$

Using Hölder's inequality with exponent $\nu = \mu/q > 1$, Lemma 5, and the condition $w \in A_1 \cap \text{RH}_{s'}$, we can deduce that

$$\begin{aligned} J_2 &\leq C \sum_{k=1}^\infty \left(\frac{r_B^\alpha}{2^{k(n+1)} [w(B)]^{1/p}} \right)^q \cdot |2^{k+1}B|^{q/\mu} \\ &\quad \cdot \left(\int_{2^{k+1}B} w(x)^{(q/p) \cdot (\mu/q)'} dx \right)^{1-q/\mu} \\ &\leq C \sum_{k=1}^\infty \left(\frac{r_B^\alpha}{2^{k(n+1)} [w(B)]^{1/p}} \right)^q \cdot |2^{k+1}B|^{q/\mu} \\ &\quad \cdot \left(\frac{w(2^{k+1}B)}{|2^{k+1}B|^{1/s}} \right)^{q/p} \leq C \sum_{k=1}^\infty \frac{1}{2^{k[q(n+1)-n]}} \leq C, \end{aligned} \quad (77)$$

where in the last inequality we have used the fact that $q > p > n/(n+1)$. By combining inequality (77) with (71), we obtain the desired result. Therefore, by a standard density argument, we can show that the same conclusion holds for all $f \in H_L^p(w)$. This concludes the proof of Theorem 3. $\square$

Proof of Theorem 4. As in the proof of Theorem 2, it is enough to show that, for every w -($p, 2M; L$)-atom a , function $m = L^{-\alpha/2}(a)$ is a multiple of a w -($p, M, \varepsilon; L$)-molecule for some $\varepsilon > 0$ and the multiplicative constant is independent of a . Let a be a w -($p, 2M; L$)-atom with $\text{supp } a \subseteq B = B(x_0, r_B)$ and $a = L^{2M}(b)$, where $M > (3n)/4 > \max\{(n/2)(1/p-1/2), 1/2 + \alpha/2\}$, $b \in \mathcal{D}(L^{2M})$. Setting $\tilde{b} = L^{-\alpha/2}(L^M b)$, then we have $m = L^M(\tilde{b})$. It is easy to check that $m(x) \in L^2(\mathbb{R}^n)$. As before, since $0 < \alpha < n/2$, then we may choose a number $\mu > 2$ such that $1/\mu = 1/2 - \alpha/n$. For $k = 0, 1, \dots, M$, by using Hölder's inequality and the L^2 - L^μ boundedness of $L^{-\alpha/2}$, we obtain

$$\begin{aligned} &\left\| (r_B^2 L)^k \tilde{b} \right\|_{L^2(2B)} \\ &\leq \frac{1}{r_B^{2M}} \left\| L^{-\alpha/2} \left[(r_B^2 L)^{M+k} b \right] \right\|_{L^\mu(2B)} |2B|^{1/2-1/\mu} \\ &\leq C \cdot \frac{1}{r_B^{2M}} \left\| (r_B^2 L)^{M+k} b \right\|_{L^2(B)} |B|^{1/2-1/\mu} \\ &\leq C \cdot r_B^{2M} |B|^{1/2+\alpha/n} [w(B)]^{-1/p}. \end{aligned} \quad (78)$$

Noting that $1/q = 1/p - \alpha/n$, then a straightforward computation yields that $q/p < (2/p)'$ whenever $0 < \alpha < n/2$. By our assumption $w \in \text{RH}_{(2/p)'}$, then we have $w \in \text{RH}_{q/p}$. Consequently,

$$\left[w^{q/p}(B) \right]^{p/q} \leq C \cdot \frac{w(B)}{|B|^{1-p/q}}, \quad (79)$$

which implies

$$[w(B)]^{-1/p} \leq C \cdot |B|^{1/q-1/p} [w^{q/p}(B)]^{-1/q}. \quad (80)$$

Substituting inequality (80) into (78), we can get

$$\left\| (r_B^2 L)^k \tilde{b} \right\|_{L^2(2B)} \leq C \cdot r_B^{2M} |B|^{1/2} [w^{q/p}(B)]^{-1/q}. \quad (81)$$

It remains to estimate $\|(r_B^2 L)^k \tilde{b}\|_{L^2(2^{j+1}B \setminus 2^j B)}$ for $k = 0, 1, \dots, M, j = 1, 2, \dots$. We write

$$\begin{aligned} \left| (r_B^2 L)^k \tilde{b}(x) \right| &= \left| L^{-\alpha/2} \left[(r_B^2 L)^k L^M b \right](x) \right| \\ &\leq C \int_0^{r_B^2} \left| e^{-tL} [r_B^{2k} L^{M+k} b](x) \right| \frac{dt}{t^{1-\alpha/2}} \\ &\quad + C \int_{r_B^2}^{\infty} \left| e^{-tL} [r_B^{2k} L^{M+k} b](x) \right| \frac{dt}{t^{1-\alpha/2}} \\ &:= \text{III}' + \text{IV}'. \end{aligned} \quad (82)$$

For the term III' , by using the same arguments as in the proof of (63), we have

$$\begin{aligned} \text{III}' &\leq C \cdot \frac{1}{2^{j(n+1)}} \cdot r_B^{2M} [w(B)]^{-1/p} \frac{1}{r_B} \int_0^{r_B^2} \frac{dt}{t^{1/2-\alpha/2}} \\ &\leq C \cdot \frac{1}{2^{j(n+1)}} \cdot r_B^{2M+\alpha} [w(B)]^{-1/p}. \end{aligned} \quad (83)$$

For the term IV' , we proceed as that of (66) and then obtain

$$\begin{aligned} \text{IV}' &\leq C \cdot \frac{1}{2^{j(n+1)}} \\ &\quad \cdot [w(B)]^{-1/p} r_B^{4M+2k-1} \int_{r_B^2}^{\infty} \frac{dt}{t^{M+k+1/2-\alpha/2}} \\ &\leq C \cdot \frac{1}{2^{j(n+1)}} \cdot r_B^{2M+\alpha} [w(B)]^{-1/p}. \end{aligned} \quad (84)$$

Combining the above estimates for III' and IV' , we can get

$$\begin{aligned} \left| (r_B^2 L)^k \tilde{b}(x) \right| &\leq C \cdot \frac{1}{2^{j(n+1)}} \cdot r_B^{2M+\alpha} [w(B)]^{-1/p}, \\ &\quad \text{when } x \in 2^{j+1}B \setminus 2^j B. \end{aligned} \quad (85)$$

Since $w \in A_1$, then it follows from inequality (64) that

$$\begin{aligned} \left| (r_B^2 L)^k \tilde{b}(x) \right| &\leq C \cdot \frac{1}{2^{j[(n+1)-n/p]}} \\ &\quad \cdot r_B^{2M+\alpha} [w(2^j B)]^{-1/p}, \\ &\quad \text{when } x \in 2^{j+1}B \setminus 2^j B. \end{aligned} \quad (86)$$

Similar to the proof of (80), we can also show that

$$[w(2^j B)]^{-1/p} \leq C \cdot |2^j B|^{1/q-1/p} [w^{q/p}(2^j B)]^{-1/q}. \quad (87)$$

Hence, we finally obtain

$$\begin{aligned} \left| (r_B^2 L)^k \tilde{b}(x) \right| &\leq C \cdot \frac{1}{2^{j[(n+1)-n/q]}} \\ &\quad \cdot r_B^{2M} [w^{q/p}(2^j B)]^{-1/q}, \\ &\quad \text{when } x \in 2^{j+1}B \setminus 2^j B. \end{aligned} \quad (88)$$

Therefore,

$$\begin{aligned} \left\| (r_B^2 L)^k \tilde{b} \right\|_{L^2(2^{j+1}B \setminus 2^j B)} &\leq C \cdot \frac{1}{2^{j[(n+1)-n/q]}} \cdot r_B^{2M} |2^j B|^{1/2} [w^{q/p}(2^j B)]^{-1/q}. \end{aligned} \quad (89)$$

Observe that $1 \geq q > p > n/(n+1)$. If we set $\varepsilon = (n+1) - n/q$, then $\varepsilon > 0$. Summarizing estimates (81) and (89) derived above, we conclude the proof of Theorem 4. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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