Research Article
The Periodic Boundary Value Problem for a Quasilinear Evolution Equation in Besov Spaces

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Received 9 March 2015; Accepted 7 May 2015

Academic Editor: Valentin Keyantuo

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This paper is concerned with the periodic boundary value problem for a quasilinear evolution equation of the following type:

\[ \partial_t u + f(u)\partial_x u + F(u) = 0, \quad x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \quad t > 0, \]

(1)

\[ u(0, x) = u_0(x), \quad x \in \mathbb{T}. \]

(2)

Clearly, (1) can be viewed as a generalized Burgers-type equation:

\[ \partial_t u + f(u)\partial_x u = 0, \]

(3)

with a nonlocal perturbation \( F(u) \). Note that (1) contains the following important equations.

When \( f(u) = u \) and \( F(u) = \partial_x(1 - \partial_x^2)^{-1}(u^2 + (1/2)u_x^2) \), (1) changes into the CH equation:

\[ u_t - u_{xxt} + 3u u_x = 2u_x u_{x} + uu_{xxx}. \]

(4)

When \( f(u) = u \) and \( F(u) = \partial_x(1 - \partial_x^2)^{-1}(3/2)u^2 \), (1) reduces to the DP equation:

\[ u_t - u_{xxt} + 4u u_x = 3u_x u_{x} + uu_{xxx}. \]

(5)

When \( f(u) = u \) and \( F(u) = \partial_x(1 - \partial_x^2)^{-1}((b/2)u^2 + ((3 - b)/2)u_x^2) \), (1) becomes the b-equation:

\[ u_t - u_{xxt} + (b + 1)u u_x = bu_x u_{x} + uu_{xxx}. \]

(6)

When \( f(u) = u^2 \) and \( F(u) = (1 - \partial_x^2)^{-1}\partial_x(3/2)u u_x^2 + (1 - \partial_x^2)^{-1}(1/2)u_x^2) \), (1) turns out to be the Novikov [1] equation:

\[ u_t - u_{xxt} + 4u^2 u_x = 3uu_x u_{x} + u u_{xxx}. \]

(7)

The CH equation was derived independently by two groups of researchers, Fuchssteiner and Fokas [2] and Camassa and Holm [3]. Fuchssteiner and Fokas derived (4) in studying completely integrable generalizations of the KdV equation with bi-Hamiltonian structures, while Camassa and Holm proposed (4) to describe the unidirectional propagation of shallow water waves over a flat bottom.

Many physicists and mathematicians have paid much attention to the CH equation and a series of achievements had been made. For example, consider the following authors. Constantin [4] and Misiolak [5], investigated the Cauchy problem. Constantin, Escher, and McKean, and so forth [6–8] studied the wave-breaking and so on. Xin and Zhang [9] proved that there are global weak solutions for any data in \( H^1(\mathbb{R}) \) without any sign conditions on the initial value. Bressan and Constantin [10, 11] and Holden and Raynaud [12, 13]...
proved the existence of the global conservative and dissipative solutions. The CH equation arises also as an equation of the geodesic flow for the $H^3$ metrics on the Bott-Virasoro group [14, 15]. Ivanov [16] extended the CH hierarchy and obtained their conserved quantities. Constantin and Ivanov [17] studied an integrable two-component CH shallow water system.

The DP equation can be regarded as a model for nonlinear shallow water dynamics derived by Degasperis and Procesi [18] in 1999. In 2003, the DP equation was derived by Dullin et al. [19] as a shallow water approximation to the Euler equation. The Cauchy problem for the DP equation has been studied extensively. The local and global well-posedness for the strong solutions, the global existence of weak solution, the blow-up phenomena, and nonuniformly continuous dependence on the initial data can be seen in [20, 21] and the references therein.

In 2003, Holm and Staley [22] studied the exchange of stability in the dynamics of solitary wave solutions for $b$-equation. The well-posedness, blow-up phenomena, and global solutions for the $b$-equation can be found in [23, 24] and the additional references therein.

It is shown that Novikov equation possesses a bi-Hamiltonian structure and an infinite sequence of conserved quantities and admits exact peaked solutions $u(t, x) = \pm \sqrt{c}e^{-|x-ct|}$ with $c > 0$ (see [25]), as well as the explicit formulas for multipeakon solutions (see [25, 26]). By using the Littlewood-Paley decomposition and Kato's theory, the formulas for multipeakon solutions (see [25, 26]). By using the Littlewood-Paley decomposition and Kato's theory, the Novikov equation's well-posedness was studied in Besov spaces $B^s_{p,r}(\mathbb{R})$ and in the Sobolev space $H^s(\mathbb{R})$ (see [27, 28]). Wu and Yin [29] established some results on the existence and uniqueness of global weak solutions to the Novikov equation. Jiang and Ni [30] obtained some results about blow-up phenomena of the strong solution to the Cauchy problem for the Novikov equation. For the periodic case, Tığlay [31] investigated the Cauchy problem for the periodic Novikov equation and proved that for $s > 5/2$ the periodic Novikov equation is local well-posed in $H^s(\mathbb{T})$. The range of local well-posedness for the periodic Novikov equation was extended to the $s > 3/2$ in [32]. When $s < 3/2$, Grayshan [33] proved that the properties of the solution map for the Novikov equation are not uniformly continuous in Sobolev spaces $H^s$.

The issue of continuity of the solution map has been the subject of many papers. For the Burgers equation, Kato [34] showed that the solution map $u_0 \mapsto u$ is not Hölder continuous regardless of the Hölder exponent. However, for certain general quasilinear hyperbolic systems, Kato obtained uniform continuity of the data to solution map for initial data in Sobolev spaces with integer index (measured in a weaker Sobolev norm). Tao [35] obtained Lipschitz continuity of the solution map for the Benjamin-Ono equation for $H^1(\mathbb{R})$ initial data measured in $L^2$. Herr et al. and so forth [36] have also obtained Lipschitz continuity in a weaker topology for the Benjamin-Ono with generalized dispersion. Davidson [37] studied the continuity of the solution map for the generalized reduced Ostrovsky equation. Karapetyan [38] proved the Hölder continuity of the solution map for the hyperelastic rod equation. For the continuity of solution map for some CH type equation and incompressible Euler equations in Besov spaces, we refer to [39–41]. These works lead to a natural question, whether a result similar to these holds for (1) when $f, F$ satisfy some assumptions.

Motivated by the results mentioned above, this paper deals with the problem (1)-(2). The aim of this paper is to prove local well-posedness of strong solutions in Besov spaces, Hölder continuity of the solution map in $B^s_{p,r}$ equipped with a weak topology.

We formulate the periodic boundary value problem (1)-(2) in Besov space as

$$\partial_t u + f(u) \partial_x u + F(u) = 0, \quad x \in \mathbb{T}, \ t \in \mathbb{R}^+, \quad u(0, x) = u_0(x) \in B^s_{p,r}(\mathbb{T}).$$

(8)

Now we introduce some notations and make some assumptions to $f(u)$ and $F(u)$.

**Notations.** In this paper we adopt the following notations.

The notation $\leq$ denotes the estimates that hold up to some “harmless” constant which may change from line to line but whose meaning is clear from the context. $\mathcal{D}^{\alpha}((\mathbb{T}))$ is the space of all infinitely differentiable functions on $\mathbb{T}$ and $\mathcal{D}^{\alpha}((\mathbb{T})^c)$ is its dual space (the details on the periodic distributions can be found in, e.g., [42]). Assuming that all function spaces are over $\mathbb{T}$, hence we drop $\mathbb{T}$ in our notations of function spaces if there is no ambiguity.

For $T > 0, s \in \mathbb{R}$, and $1 \leq p \leq \infty$, we define

$$E^s_{p,r}(T) = C([0,T]; B^s_{p,r}) \cap C^1([0,T]; B^{s-1}_{p,r}),$$

$$F^s_{p,r}(T) = C([0,T]; B^s_{p,r})$$

if $r < \infty$,

$$F^{s}_{p,\infty}(T) = L^{\infty}_{r\to\infty}(0,T; B^{s}_{p,\infty}) \cap \text{Lip}([0,T]; B^{s-1}_{p,\infty}),$$

$$F^{s}_{p,\infty}(T) = L^{\infty}_{r\to\infty}(0,T; B^{s}_{p,\infty})$$

if $r = \infty$,

$$E^s_{p,r} = \bigcap_{T>0} E^s_{p,r}(T).$$

**Assumptions.** Assume that, for $1 \leq p, r \leq \infty$, there exists a real number $s_0 \geq 1 + 1/p$ and an $n \in \mathbb{N}$ such that $f(\cdot), F(\cdot)$ satisfy the following conditions.

(A1) $f \in C^1(\mathbb{R})$ and $\|f(u)\|_{B^s_{p,r}} \leq \|u^n\|_{B^s_{p,r}}$ for $s > 1/p$.

Besides, for $s > 1/p$ and any $u, v \in B^s_{p,r}, f$ satisfies

$$\|f(u) - f(v)\|_{B^s_{p,r}}$$

$$\leq \left(\|u\|_{B^s_{p,r}} + \|v\|_{B^s_{p,r}}\right)^{n-1} \|\partial_x u - \partial_x v\|_{B^{s-1}_{p,r}}$$

$$\leq \left(\|u\|_{B^s_{p,r}} + \|v\|_{B^s_{p,r}}\right)^{n-1} \|u - v\|_{B^s_{p,r}}.$$
For $s > s_0$, $F : B^{s}_{p,r} \mapsto B^{s}_{p,r}$ is continuous and
\[ \|F(u)\|_{B^{s}_{p,r}} \leq \|u\|^{n+1}_{B^{s}_{p,r}}. \]
Moreover, for $s > s_0$ and any $u, v$ in $B^{s}_{p,r}$, $F$ satisfies
\[ \|F(u) - F(v)\|_{B^{s}_{p,r}} \leq \left( \|u\|_{B^{s}_{p,r}} + \|v\|_{B^{s}_{p,r}} \right) \|\partial_{x}u - \partial_{x}v\|_{B^{s}_{p,r}}. \]

For $s_0 - 1 < s < s_0$ and any $u, v$ in $B^{s+1}_{p,r}$, it holds that
\[ \|F(u) - F(v)\|_{B^{s}_{p,r}} \leq \left( \|u\|_{B^{s+1}_{p,r}} + \|v\|_{B^{s+1}_{p,r}} \right) \|\partial_{x}u - \partial_{x}v\|_{B^{s}_{p,r}}. \]

Our main results are as follows.

**Theorem 1.** For $1 \leq p, r \leq \infty$, if there exist $s_0 \geq 1 + 1/p$ and $n$ such that $f(u)$ and $F(u)$ satisfy the assumptions $(A1)$–$(A3)$, then we have the following results.

\[ \alpha = \begin{cases} 1, & \text{if } s \neq 2 + \frac{1}{p}, \ s_0 - 1 < q_1 = q_2 \leq s - 1, \ q_1 = q_2 \neq 1 + \frac{1}{p}, \\ s - q_2, & \text{if } s \neq 2 + \frac{1}{p}, \ s - 1 = q_1 < q_2 < s. \end{cases} \]

Remark 2. As mentioned before, we can view (1) as a generalized Burgers-type equation:
\[ \partial_{t}u + f(u)\partial_{x}u = 0, \]
with a nonlocal perturbation $F(u)$. To obtain the well-posedness, we need to assume some smooth properties on $f(u)$ and $F(u)$. Condition $(A1)$ implies that, for $1 \leq p, r \leq \infty$, $s > 1/p$, $f : B^{s}_{p,r} \mapsto B^{s}_{p,r}$ is continuous and $(A2)$ shows that when $1 \leq p, r \leq \infty$, $s > s_0 \geq 1 + 1/p$, $F : B^{s}_{p,r} \mapsto B^{s}_{p,r}$ is continuous. Therefore, it is reasonable to expect the well-posedness of (1) in $B^{s}_{p,r}$ with $1 \leq p, r \leq \infty, s > s_0$. Basically, condition $(A3)$ is used to estimate $\|u - v\|_{B^{s}_{p,r}}$ for two solutions $u, v$ and $s' < s - 1$.

We note that the issue of nonuniform dependence on initial data has been the subject of many papers (see, e.g., [32, 43]). In this paper, it is to be regretted that we can not find a feasible method to study the uniform continuity of the solution map $u_0 \mapsto u$ defined by problem (1)-(2) for general $f(u)$ and $F(u)$. But when $f(u) = u$ and $F(u) = u^2$, the solution map $u_0 \mapsto u$ defined by problem (1)-(2) is not uniformly continuous from any bounded subset of $B^{s}_{p,r}$ into $E^{s}_{2,r}(T)$ for any $T > 0$ with $s > 3/2, 1 \leq r \leq \infty$. As a special case, we have the following proposition.

**Proposition 3.** When $f(u) = u$ and $F(u) = u^2$, problem (1)-(2) satisfies $(A1)$–$(A3)$ automatically. In this case, Theorem 1 holds for $s_0 = 3/2$. However, the solution map $u_0 \mapsto u$ defined by problem (1)-(2) is not uniformly continuous from any bounded subset of $B^{s}_{p,r}$ into $E^{s}_{2,r}(T)$ for any $T > 0$ with $s > 3/2$ and $1 \leq r \leq \infty$. More precisely, there exist two sequences of solutions $u_k$ and $v_k$ in $E^{s}_{2,r}(T)$ such that
\[ \|u_k(t)\|_{E^{s}_{2,r}} + \|v_k(t)\|_{E^{s}_{2,r}} \leq 1, \]
\[ \lim_{k \to \infty} \|u_k(0) - v_k(0)\|_{E^{s}_{2,r}} = 0, \]
\[ \lim_{k \to \infty} \|u_k(t) - v_k(t)\|_{E^{s}_{2,r}} \geq \left| \sin \left( \frac{1}{2} \right) \right|, \ 0 \leq t \leq T. \]

We now conclude this introduction by outlining the rest of the paper. In Section 2, we give some preliminaries. In Section 3, we prove Theorem 1. In Section 4, we demonstrate Proposition 3.
2. Preliminaries

In this section, we shall recall some basic facts on the Littlewood-Paley theory, Besov spaces, and the transport equations theory that will be used in this paper. We refer to [44–46] for the elementary properties of them.

Let \( \chi, \phi \) be two functions satisfying \( \chi, \phi \in C_0^\infty(\mathbb{R}), 0 \leq \chi, \phi \leq 1, \chi(\xi) = 1 \) for \( |\xi| \leq 3/4, \) supp \( \chi = \{ \xi \in \mathbb{R}, |\xi| \leq 4/3 \} \) and \( \phi(\xi) = \chi(2^{-1}\xi) - \chi(\xi). \) Then

\[
\chi(\xi) + \sum_{j \in \mathbb{N}} \phi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R},
\]

\[
supp \phi(2^{-j} \cdot) \cap \sup \phi(2^{-j} \cdot) = \emptyset \quad \text{if} \quad |j - j'| \geq 2,
\]

\[
supp \chi(\cdot) \cap \sup \phi(2^{-j} \cdot) = \emptyset \quad \text{if} \quad j \geq 1.
\]

We decompose \( u \in \mathcal{D}'(\mathbb{T}) \) on the circle \( \mathbb{T} \) into Fourier series \( u = (1/2\pi) \sum_{\xi \in \mathbb{Z}} \mathcal{F}_x u(\xi) e^{i\xi x}, \) where \( \mathcal{F}_x u(\xi) = \int_{\mathbb{T}} e^{-i\xi x} u(x) dx \) is the Fourier transform on the circle. The inverse relation is given by \( (\mathcal{F}_x^{-1}u)(x) = (1/2\pi) \sum_{\xi \in \mathbb{Z}} u(\xi) e^{i\xi x}. \) Define the periodic dyadic blocks as

\[
\Delta_{-j} u = \mathcal{F}_x^{-1} \chi(\xi) \mathcal{F}_x u = \frac{1}{2\pi} \sum_{\xi \in \mathbb{Z}} \chi(\xi) \mathcal{F}_x u(\xi) e^{i\xi x},
\]

\[
\Delta_j u = \mathcal{F}_x^{-1} \phi(2^{-j} \xi) \mathcal{F}_x u = \frac{1}{2\pi} \sum_{\xi \in \mathbb{Z}} \phi(2^{-j} \xi) \mathcal{F}_x u(\xi) e^{i\xi x}, \quad \text{if} \quad j \geq 0.
\]

Then, we define the the low frequency cut-off \( S_j \) as \( S_j u = \sum_{|\xi| \leq 1} \Delta_j u. \) Direct computation implies that, for any \( 1 \leq p \leq \infty, \) we have the quasi-orthogonality properties with our choice of \( \phi \) and \( \forall u, v \in \mathcal{D}'(\mathbb{T}): \)

\[
\Delta_j S_{j-1} u \Delta_j v = 0, \quad \text{if} \quad |j - j| \geq 2,
\]

\[
\Delta_j (S_{j-1} u \Delta_j v) = 0, \quad \text{if} \quad |j - j| \geq 5.
\]

Furthermore, for all \( u \in L^p, \| \Delta_j u \|_{L^p} \leq C \| u \|_{L^p}, \) and \( \| S_j u \|_{L^p} \leq C \| u \|_{L^p}. \)

**Definition 4** (Besov spaces). Let \( s \in \mathbb{R}, 1 \leq p, r \leq +\infty. \) The nonhomogeneous Besov space \( B^s_{p,r}(\mathbb{T}) \) is defined by

\[
B^s_{p,r} = \left\{ f \in \mathcal{D}'(\mathbb{T}) : \| f \|_{B^s_{p,r}(\mathbb{T})} < \infty \right\},
\]

where \( \| f \|_{B^s_{p,r}(\mathbb{T})} = \| 2^{|s|} \Delta_j f \|_{L^p(\mathbb{T})} = \| (2^{|s|} \Delta_j f) \|_{L^p(\mathbb{T})}. \) In particular, \( B^s_{p,\infty} = \bigcap_{r \leq 1} B^s_{p,r}. \)

**Remark 5.** When \( p = 2, 1 \leq r \leq +\infty, \) the above definition is equivalent to the following:

\[
\| u \|_{B^s_{p,r}(\mathbb{T})} = \left( \left( \sum_{\xi \in \mathbb{Z}} (1 + \xi^2)^{|s|} |\mathcal{F}_x u(\xi)|^2 \right)^{r/2} + \sum_{j \in \mathbb{N}} \left( \sum_{|\xi| \leq 2^j} (1 + \xi^2)^{|s|} |\mathcal{F}_x u(\xi)|^2 \right)^{r/2} \right)^{1/r}, \quad r < \infty,
\]

\[
\| u \|_{B^s_{p,1}(\mathbb{T})} = \left( \left( \sum_{\xi \in \mathbb{Z}} (1 + \xi^2)^{|s|} |\mathcal{F}_x u(\xi)|^2 \right)^{1/2} \right)^{1/r}, \quad r = \infty.
\]

When \( p = r = 2, \) then \( B^s_{2,2} \) turn out to be the Sobolev spaces \( H^s. \)

The following lemma summarizes some useful properties of \( B^s_{p,r}. \)

**Lemma 6** (see [44–46]). Let \( s \in \mathbb{R}, 1 \leq p, r, p_j, r_j \leq \infty, j = 1, 2; \) then consider the following.

1. \( B^s_{p_r}(\mathbb{T}) \) is a Banach space. When \( 1 < p, r \leq \infty, \) then the set of all trigonometric polynomial is dense in \( B^s_{p_r}(\mathbb{T}). \)

2. \( B^s_{p,r_1} \hookrightarrow B^s_{p,r_2}, \) if \( p_1 \leq p_2; r_1 \leq r_2 \) and \( s_1 = s_2 = s_1 - (1/p_1 - 1/p_2). \)

\( B^s_{p,r_1} \hookrightarrow B^s_{p,r_2} \) is locally compact if \( s_2 < s_1; r_1 \leq r_2. \)

(3) \( \forall s > 0, B^s_{p,r} \cap L^\infty \) is a Banach algebra. \( B^s_{p,r} \) is a Banach algebra \( \Leftrightarrow B^s_{p_r} \hookrightarrow L^\infty \Leftrightarrow s > 1/p \) (or \( s \geq 1/p \) and \( r = 1). \)

(4) For all \( \theta \in [0, 1], \) \( s = \theta s_1 + (1 - \theta) s_2, \)

\[
\| f \|_{B^s_{p,r}} \leq C \| f \|_{B^{\theta s_1}_{p_1,r_1}} \| f \|_{B^{(1 - \theta) s_2}_{p_2,r_2}}, \quad \forall f \in B^s_{p_r} \cap B^{s_1}_{p_1,r_1}
\]

(5) For all \( \theta \in (0, 1), s_1 > s_2, s = \theta s_1 + (1 - \theta) s_2, \)

\[
\| u \|_{B^s_{p,r}} \leq \frac{C \theta}{s_1 - s_2} \| u \|_{B^{\theta s_1}_{p_1,r_1}} \| u \|_{B^{(1 - \theta) s_2}_{p_2,r_2}}, \quad \forall u \in B^s_{p_r} \cap B^{s_1}_{p_1,r_1}
\]

(6) If \( \{ u_k \}_{k \in \mathbb{N}} \) is bounded in \( B^s_{p,r} \) and \( u_k \) converges to \( u \) in \( \mathcal{D}'(\mathbb{T}), \) then \( u \in B^s_{p,r} \) and

\[
\| u \|_{B^s_{p,r}} \leq \liminf_{k \to \infty} \| u_k \|_{B^s_{p,r}}.
\]
Now we state some results in Besov spaces $B^s_{p,r}$ of the transport equation. We refer to [45–48] for the details.

**Lemma 7** (a priori estimates). Let $1 \leq p, r \leq \infty$ and $s > -\min\{1/p, 1-1/p\}$. Assume that $f_0 \in B^s_{p,r}$, $F \in L^1(0,T; B^{s-1}_{p,\infty})$ and $\partial_x v \in L^1(0,T; B^{-1}_{p,\infty})$ if $s > 1 + 1/p$ or $\partial_x v \in L^1(0,T; B^{1/p,\infty} \cap L^\infty)$ otherwise. If $f \in L^\infty(0,T; B^{s,1}_{p,\infty}) \cap C([0,T]; \mathbb{R}^n)$, then $f_t + v f_x = F$, $\quad x \in \mathbb{T}, \ t > 0,$

$$ f(0,x) = f_0, \quad x \in \mathbb{T}, $$

then there exists a constant $C$ depending only on $s, p, r$ as follows.

1. For all $t \in [0,T]$

$$ \|f\|_{B^s_{p,r}} \leq \|f_0\|_{B^s_{p,r}} + \int_0^t \|F(r)\|_{B^{s-1}_{p,\infty}} \, dr $$

$$ + C \int_0^t \|v(r)\|_{B^s_{p,r}} \|f(r)\|_{B^s_{p,r}} \, dr, $$

and hence,

$$ \|f\|_{B^s_{p,r}} \leq e^{C(t)} \left(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{C(t)} \|F(r)\|_{B^{s-1}_{p,\infty}} \, dr, \right) $$

where $V(t)$

$$ V(t) = \begin{cases} \int_0^t \|v_x(r)\|_{B^{s-1}_{p,\infty}} \, dr, & \text{if } s < 1 + \frac{1}{p}, \\ \int_0^t \|v_x(r)\|_{B^{s-1}_{p,\infty}} \, dr, & \text{if } s > 1 + \frac{1}{p} \text{ or } \{s = 1 + \frac{1}{p}, r = 1\}. \end{cases} $$

Moreover, if $r < \infty$, then $f \in C([0,T]; B^s_{p,r})$. If $r = \infty$, then $f \in C([0,T]; B^s_{p,\infty})$ for all $s' < s$. (2) If $v = f$, then (1) holds true for all $s > 0$ with $V(t) = \int_0^t \|\partial_x v^2(r)\|_{L^\infty} \, dr$.

**Lemma 8** (existence and uniqueness). Let $p, r, s$, $f_0$ and $F$ be as in the statement of Lemma 7. Assume that $v \in L^p(0,T; B^{-M}_{p,\infty})$ for some $p > 1, M > 0$ and $v_x \in L^1(0,T; B^{s-1}_{p,\infty})$ if $s > 1 + 1/p$ or $s = 1 + 1/p$ and $r = 1$ and $v_x \in L^1(0,T; B^{1/p,\infty} \cap L^\infty)$ if $s > 1 + 1/p$. Then, problem (28) has a unique solution $f \in L^\infty(0,T; B^s_{p,r}) \cap C([0,T]; B^s_{p,r})$ for any $s' < s$ and the inequalities of Lemma 7 hold true. Moreover, if $r < \infty$, then $f \in C([0,T]; B^s_{p,r})$.

**Lemma 9** (see [49]). Letting $1 \leq p, r \leq \infty$, then we have the following estimates.

1. For $s > 0$,

$$ \|fg\|_{B^s_{p,r}} \leq C \left(\|f\|_{B^s_{p,r}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{B^s_{p,r}} \right) $$

$$ \forall f, g \in B^s_{p,r} \cap L^\infty. $$
Hence, we obtain $W(t) \leq 1$. Since $s - 1 > s_0 - 1$, by (A2) (if $s' > s_0$) or (A3) (if $s_0 - 1 < s' < s_0$) and (40), we have

$$
\|F(u) - F(v)\|_{B^s_{p,r}} \leq \left(\|u\|_{B^s_{p,r}} + \|v\|_{B^s_{p,r}}\right) \|u - v\|_{B^s_{p,r}}
$$

\[ (41) \]

\[ \leq \|u - v\|_{B^s_{p,r}} . \]

Similarly, when $2 + 1/p > s > s_0$, it follows that $s_0 \geq 1 + 1/p > s' > s_0 - 1$. Then (A3) and (40) also yield

$$
\|F(u) - F(v)\|_{B^s_{p,r}} \leq \left(\|u\|_{B^s_{p,r}} + \|v\|_{B^s_{p,r}}\right) \|u - v\|_{B^s_{p,r}}
$$

\[ (42) \]

\[ \leq \|u - v\|_{B^s_{p,r}} . \]

As $s' > s_0 - 1 \geq 1/p$ and $B^s_{p,r}$ is an algebra, by (A1) and (40), we obtain

$$
\|f(u) - f(v)\|_{B^s_{p,r}} \leq \|f(u)\|_{B^s_{p,r}} \|u - v\|_{B^s_{p,r}}.
$$

\[ (43) \]

Putting all of these results together we arrive at $\|F\|_{B^s_{p,r}} \leq \|u - v\|_{B^s_{p,r}}$ and therefore

$$
\|w(t)\|_{B^s_{p,r}} \leq \|w(0)\|_{B^s_{p,r}} + \int_0^t \|u - v\|_{B^s_{p,r}} \, dt.
$$

\[ (44) \]

By Gronwall’s inequality we obtain (34). Finally, if $s - 1 < s' < s$, by interpolation and (40) again, we have

$$
\|w\|_{B^s_{p,r}} \leq \|w\|_{B^{s'}_{p,r}}^{\theta} \|w\|_{B^{s}_{p,r}}^{1-\theta} \leq \|w\|_{B^{s'}_{p,r}}^{\theta} \left(\|w\|_{B^{s'}_{p,r}} + \|w\|_{B^{s}_{p,r}}\right)^{1-\theta} \|w\|_{B^{s'}_{p,r}}^{\theta},
$$

\[ (45) \]

where $\theta = s - s'$. Using (34), we complete the proof for this lemma.

\[ \square \]

3. Proof of Theorem 1

The proof of Theorem 1 includes the following several steps.

3.1. Approximate Solution. Starting from $u_t = 0$ and by induction, we define a sequence of smooth functions $\{u_k\}$, $k \in \mathbb{N}$ by solving the following transport equation iteratively:

$$
\partial_t u_{k+1} + f(u_k) \partial_x u_{k+1} = -F(u_k),
$$

\[ (46) \]

$$
u_{k+1}(0, x) = S_{k+1}u_0.
$$

\[ (47) \]

Since $f(u), F(u)$ satisfy the continuous assumption and all the data belong to $B^{s_0}_{p,r}$, from Lemma 8 and by induction, we see that, for all $k \geq 1$, the above equation has a global solution $u_{k+1}$ belonging to $C(R^+, B^{s_0}_{p,r})$.

3.2. Uniform Bounds and Lifespan of the Approximate Solutions. For $k \in \mathbb{N}$, set $U_k(t) = \int_0^t \|u_k\|_{B^s_{p,r}}^p \, dt$ with $1 \leq p, r \leq \infty$ and $s > s_0 \geq 1 + 1/p$; then we have

$$
\|u_{k+1}(t)\|_{B^s_{p,r}} \leq e^{\mathcal{L}_{k+1}(t)} \left(\|u_0\|_{B^s_{p,r}}^p + C \int_0^t e^{-\mathcal{L}_{k+1}(\tau)} \|u_{k+1}(\tau)\|_{B^s_{p,r}}^{p-1} d\tau\right).
$$

\[ (48) \]

In fact, since $\|u_{k+1}(0, x)\|_{B^s_{p,r}} \leq \|u_0\|_{B^s_{p,r}}$, from the estimate (30) of Lemma 7, the assumptions (A1), (A2), and the algebra property of $B^s_{p,r}$, we have

$$
\|u_{k+1}(t)\|_{B^s_{p,r}} \leq \|u_0\|_{B^s_{p,r}}^p 
$$

\[ \cdot \exp \left\{ C \int_0^t \|\partial_x f(u_k(t))\|_{B^s_{p,r}} \, dt\right\}
$$

\[ + \int_0^t \exp \left\{ C \int_\tau^t \|\partial_x f(u_k(t'))\|_{B^s_{p,r}} \, dt'\right\} \|u_{k+1}\|_{B^s_{p,r}} \, d\tau \leq \|u_0\|_{B^s_{p,r}}^p 
$$

\[ \cdot \exp \left\{ C \int_0^t \|u_k(t')\|_{B^s_{p,r}}^n \, dt'\right\} \int_0^t \exp \left\{ C \int_\tau^t \|u_k(t')\|_{B^s_{p,r}}^n \, dt'\right\} \|u_{k+1}\|_{B^s_{p,r}}^n \, d\tau, \]

which gives (48). Let $T > 0$ such that $2n \|u_0\|_{B^s_{p,r}}^n < 1$; we claim that, for any $k \in \mathbb{N}$,

$$
\|u_k\|_{B^s_{p,r}} \leq \frac{\|u_0\|_{B^s_{p,r}}^p}{(1 - 2nC \|u_0\|_{B^s_{p,r}}^n)^{1/n}}, \quad \forall t \in [0, T].
$$

\[ (50) \]

Assume that (50) is true for $k$. We now prove that it also holds true for $k + 1$. Since $U_k(t) = \int_0^t \|u_k\|_{B^s_{p,r}}^p \, dt$, by using (50), we have

$$
e^{\mathcal{L}_{k+1}(t) - \mathcal{L}_{k+1}(t)} \leq \exp \left\{ C \int_0^t \frac{\|u_0\|_{B^s_{p,r}}^p}{(1 - 2nC \|u_0\|_{B^s_{p,r}}^n)^{1/n}} \, dt'\right\}
$$

\[ \leq \exp \left\{ \frac{1}{2n} \int_0^t \frac{d}{dt'} \left(1 - 2nC \|u_0\|_{B^s_{p,r}}^n \right)^{1/2n} \right\}
$$

\[ \leq \exp \left\{ \frac{1}{2n} \int_0^t \frac{d}{dt'} \left(1 - 2nC \|u_0\|_{B^s_{p,r}}^n \right)^{-1/2n} \right\}
$$

\[ = \exp \left\{ \frac{1}{2n} \int_0^t \frac{d}{dt'} \left(1 - 2nC \|u_0\|_{B^s_{p,r}}^n \right)^{-1/2n} \right\}. \]

From the above equation, we see clearly that when $\tau = 0$, $U_k(0) = 0$. We obtain

$$
e^{\mathcal{L}_{k+1}(t)} \leq \left(1 - \frac{1}{2nC \|u_0\|_{B^s_{p,r}}^n} \right)^{1/2n}.
$$

\[ (52) \]
Using (48), (50), (51), and (52) gives rise to
\[
\|u_{k+1}(t)\|_{B^s_{p,r}} \leq e^{C \mu_1(t)} \|u_0\|_{B^s_{p,r}} + C \int_0^t e^{C \mu_1(t)} \|u_k(t)\|_{B^{s+1}_{p,r}} \, dt
\]
\[
\leq \left( \frac{1}{1 - 2nC \|u_0\|_{B^s_{p,r}} t} \right)^{1/2n} \left[ 2 \|u_0\|_{B^s_{p,r}} \right] + \int_0^t \frac{C \|u_0\|_{B^s_{p,r}}^{s+1}}{(1 - 2nC \|u_0\|_{B^s_{p,r}} t)^{1+1/2n}} \, dt
\]
\[
= \left( \frac{1}{1 - 2nC \|u_0\|_{B^s_{p,r}} t} \right)^{1/2n} \left[ 2 \|u_0\|_{B^s_{p,r}} \right]
\]
\[
\leq \left( \frac{1}{1 - 2nC \|u_0\|_{B^s_{p,r}} t} \right)^{1/2n} \left[ 2 \|u_0\|_{B^s_{p,r}} \right].
\]
Hence, (50) is true and \( \{u_k\} \) is uniformly bounded in \( C([0,T]; B^s_{p,r}) \) for all \( t \in [0,T] \). Setting \( T > 0 \) such that \( 2nC \|u_0\|_{B^s_{p,r}} T < 1 \), for \( n \in \mathbb{N} \), we can conclude that the solution \( u_k \) exists for \( 0 \leq t \leq T \) and satisfies the bound
\[
\|u_k(t)\|_{B^s_{p,r}} \leq 2 \|u_0\|_{B^s_{p,r}}, \quad 0 \leq t \leq T.
\]
Furthermore, for \( s > 1 + 1/p \) and \( t \in [0,T] \), using (46) yields
\[
\|\partial_t u_k\|_{B^{s-1}_{p,r}} \leq \|f(u_k)\|_{B^{s-1}_{p,r}} + \|F(u_k)\|_{B^{s-1}_{p,r}}
\]
\[
\leq \|u_k\|_{B^s_{p,r}}^{s-1} \|\partial_t u_{k+1}\|_{B^{s-1}_{p,r}} + \|F(u_k)\|_{B^{s-1}_{p,r}}
\]
\[
\leq \|u_0\|_{B^s_{p,r}}^{s-1} \|\partial_t u_{k+1}\|_{B^{s-1}_{p,r}} \leq \|u_k\|_{B^s_{p,r}}^{s-1} \|\partial_t u_{k+1}\|_{B^{s-1}_{p,r}} + \|F(u_k)\|_{B^{s-1}_{p,r}}.
\]
Hence, we conclude that \( \{u_k\} \subset E^s_{p,r}(T) \) is uniformly bounded.

3.3. Convergence in \( C([0,T], B^{s-1}_{p,r}) \). We are going to show that \( \{u_k\} \) is a Cauchy sequence in \( C([0,T], B^{s-1}_{p,r}) \). It is sufficient to estimate the difference of the iteration. For \( k, l \in \mathbb{N} \), from (46) we have
\[
\partial_t (u_{k+l+1} - u_{k+1}) + f(u_{k+l}) \partial_t (u_{k+l+1} - u_{k+1})
\]
\[
= \left[ f(u_k) - f(u_{k+l}) \right] \partial_t u_{k+l+1} + [F(u_k) - F(u_{k+l})].
\]
\[
3.3.1. \text{Case of } s > 2 + 1/p > s_0 \text{ or } 2 + 1/p > s > s_0 \text{ or } s > s_0 \geq 2 + 1/p.
\]
Using estimates analogous to those in Lemma 10, we obtain that
\[
\|u_{k+l+1} - u_{k+1}\|_{B^{s-1}_{p,r}} \leq C_T \left( \|u_{k+l+1} - u_{k+1}\|_{B^{s-1}_{p,r}} \right)
\]
\[
+ \int_0^t \|u_{k+l} - u_k\|_{B^{s-1}_{p,r}} \, dt,
\]
where \( C_T \) is a big constant depending on \( \|u_0\|_{B^s_{p,r}} \) and \( T \). Note that (see [50] for detailed computations)
\[
\|u_{k+l+1} - u_{k+1}\|_{B^{s-1}_{p,r}} = \sum_{i=k+1}^{k+l} \Delta_i \|u_0\|_{B^{s-1}_{p,r}} \leq C 2^{-k} \|u_0\|_{B^{s-1}_{p,r}}.
\]
It follows that
\[
\|u_{k+l+1} - u_{k+1}\|_{B^{s-1}_{p,r}} \leq C_T \left( 2^{-k} + \int_0^t \|u_{k+l} - u_k\|_{B^{s-1}_{p,r}} \, dt \right).
\]
For simplicity, we denote \( I = [0,T] \). Arguing by induction with respect to the index \( k \), we obtain
\[
\|u_{k+l+1} - u_{k+1}\|_{C(I; B^{s-1}_{p,r})} \leq (TC_T)^{k+1} (k+1)! \|u_0\|_{C(I; B^{s-1}_{p,r})} + 2^{-k} C_T \sum_{j=0}^k \frac{(TC_T)^{j}}{j!}.
\]
Since \( \|u_0\|_{B^{s-1}_{p,r}} \) is uniformly bounded by \( \|u_0\|_{B^s_{p,r}} \), hence (60) implies that \( \{u_k\} \) is a Cauchy sequence in \( C(I; B^{s-1}_{p,r}) \) and therefore \( \{u_k\} \) converges to some function \( u \in C(I; B^{s-1}_{p,r}) \).

3.3.2. Case of \( s = 2 + 1/p > s_0 \geq 1 + 1/p \). A simple interpolation argument like the one in Lemma 10 leads to
\[
\|u_{k+l+1} - u_{k+1}\|_{B^{s-1}_{p,r}} \leq \|u_{k+l+1} - u_{k+1}\|_{B^s_{p,r}} \left( \|u_0\|_{B^s_{p,r}} \right)^{-\theta},
\]
where we choose \( s_1, s_2 \) such that \( s_0 - 1 < s_1 < s_2 < 1 + 1/p \), \( s_0 < s_2 < 2 + 1/p = s \) and \( s - 1 = 1 + 1/p = \theta s_1 + (1 - \theta) s_2 \) with \( 0 < \theta < 1 \). Therefore, \( \{u_k\} \) is also a Cauchy sequence in \( C(I; B^{s-1}_{p,r}) \).

Gathering the above results, we know that \( \{u_k\} \subset C(I; B^{s-1}_{p,r}) \) is a Cauchy sequence for \( s > s_0 \) and converges to some function \( u \in C(I; B^{s-1}_{p,r}) \).

3.4. Existence of the Solution. Firstly, as \( \|u_0\|_{B^s_{p,r}} \) is uniformly bounded by \( \|u_0\|_{B^s_{p,r}} \), Fatou property guarantees \( \|u_k\|_{B^s_{p,r}} \leq 2 \|u_0\|_{B^s_{p,r}} \), which gives (13). Repeat the process in Lemma 10; for any \( s' < s - 1 \), we obtain
\[
\|u_k - u\|_{B^{s'}_{p,r}} \leq \|u_0 - u\|_{B^{s-1}_{p,r}}.
\]
If \( s - 1 < s' < s \), by using interpolation again, we have
\[
\|u_k - u\|_{B_{p,r}^{s'}} \leq \|u_k - u\|_{B_{p,r}^{s}}^{\theta} \|u_k - u\|_{B_{p,r}^{1-s}}^{1-\theta},
\]
(63)
where \( \theta = s - s' \). From (62) and (63), we see that \( \{u_k\} \) converges to \( u \) in \( C(I; B_{p,r}^{s-1}) \) for all \( s' < s \); this enables us to deduce that \( u \) indeed solves (1) in the sense of distributions.

### 3.5. Regularity and Uniqueness of the Solution

When \( r < \infty \), from Lemma 8, we know that \( u \in C([0,T]; B_{p,r}^{s-1}) \); thus by the equation itself in (8), we know that \( u_t \in C([0,T]; B_{p,r}^{s-1}) \).

When \( r = \infty \), directly from the equation in (8), we know \( u_t \in L^\infty(0,T; B_{p,r}^{s-1}) \). Therefore we have \( u \in E_{p,\infty}^s(T) \). The uniqueness of the solution is a corollary of Lemma 10.

### 3.6. Continuity of the Solution Map

The continuity with respect to the initial data in
\[
C([0,T]; B_{p,r}^{s-1}) \cap C([0,T]; B_{p,r}^{s-1}) \subset C([0,T]; B_{p,r}^{s-1}), \quad \forall s' < s
\]
(64)
or in
\[
C([0,T]; B_{p,r}^{s-1}) \subset C([0,T]; B_{p,r}^{s-1}), \quad s = 2 + \frac{1}{p}
\]
(65)
can be obtained directly by Lemma 10. We now prove that the continuity holds true up to index \( s \geq 1 + 1/p \). We state the following lemma first.

**Lemma 11.** Denote \( \mathbb{N} = \mathbb{N} \cup \{\infty\} \). Let \( n, f \) satisfy the assumption (A1). Suppose that \((p, r) \in [1, +\infty]^2 \), \( r < \infty \), and \( s > 1 + 1/p \), \( s \neq 2 + 1/p \). Let \( \{a_j\}_{j \in \mathbb{N}} \) be a sequence of continuous bounded functions on \([0,T) \times \mathbb{R} \) with \( \sup_{t \in [0,T)} a_j(t) \|B_{p,r}^{s-1}\| \leq \alpha(t) \) for some \( \alpha(t) \in L^\infty(0,T) \) if \( n > 1 \) or \( \alpha(t) \in L^1(0,T) \) if \( n = 1 \).

Assume that \( \{v_j\}_{j \in \mathbb{N}} \) is the solution to
\[
\partial_t v_j + f(a_j) \partial_x v_j = g, \quad v_j(0) = v_0,
\]
(66)
with \( v_0 \in B_{p,r}^{s-1}, g \in L^1(0,T; B_{p,r}^{s-1}) \). If \( a_k \to a_\infty \) in \( L^1(0,T; B_{p,r}^{s-1}) \), then \( v_k \to v_\infty \) in \( C([0,T]; B_{p,r}^{s-1}) \).

**Proof.** Let \( u_k = v_k - v_\infty \). It is obvious that \( u_k \) solves the transport equation:
\[
w_t + f(a_k) \partial_x w = -(f(a_k) - f(a_\infty)) \partial_x v_\infty.
\]
(67)
We first consider \( v_0 \in B_{p,r}^{s-1} \) and \( g \in L^1(0,T; B_{p,r}^{s-1}) \). Note that \( r < \infty \); by Lemma 7, we obtain that, for \( k \in \mathbb{N} \), \( v_k \in C([0,T]; B_{p,r}^{s-1}) \) and
\[
\|v_k(t)\|_{B_{p,r}^{s-1}} \leq \|v_0\|_{B_{p,r}^{s-1}} \exp \left\{ C \int_0^t \alpha(r) \, dr \right\} + \int_0^t \exp \left\{ C \int_r^t \alpha(r') \, dr' \right\} \|g\|_{B_{p,r}^{s-1}} \, dr
\]
(68)
Since \( s \neq 2 + 1/p \), for \( k \in \mathbb{N} \), Lemma 7 tells us that
\[
\|w_k(t)\|_{B_{p,r}^{s-1}} \leq \int_0^t \exp \left\{ C \int_r^t \alpha(r') \, dr' \right\} \times \left\| (f(a_k) - f(a_\infty)) \partial_x v_\infty \right\|_{B_{p,r}^{s-1}} \, dt
\]
(69)
\[
\leq \int_0^t \left\| (f(a_k) - f(a_\infty)) \partial_x v_\infty \right\|_{B_{p,r}^{s-1}} \, dt.
\]
Since \( s > 1 + 1/p \), using (A1) yields that
\[
\left\| (f(a_k) - f(a_\infty)) \partial_x v_\infty \right\|_{B_{p,r}^{s-1}} \leq \left( \|a_k\|_{B_{p,r}^{s-1}} + \|a_\infty\|_{B_{p,r}^{s-1}} \right)^{s-1} \|a_k - a_\infty\|_{B_{p,r}^{s-1}} \|v_\infty\|_{B_{p,r}^{s-1}} = (1 + \|\alpha(t)\|_{L^\infty}) \|a_k - a_\infty\|_{B_{p,r}^{s-1}} \|v_\infty\|_{B_{p,r}^{s-1}}.
\]
(70)
Therefore, we obtain the following estimate by gathering the above inequalities:
\[
\|w_k(t)\|_{B_{p,r}^{s-1}} \leq \|w_0\|_{B_{p,r}^{s-1}} + \|v_0' - v_\infty'\|_{B_{p,r}^{s-1}} + \|v_\infty' - v_\infty\|_{B_{p,r}^{s-1}}
\]
(72)
where \( v_\infty' \) satisfies the “cut-off” equation:
\[
\partial_t v_\infty' + f(a_\infty) \partial_x v_\infty' = S_j g = \sum_{i=1}^{j-1} \Delta_i g,
\]
(73)
\[
v_\infty'(0) = S_j v_0 = \sum_{i=1}^{j-1} \Delta_i v_0.
\]
For \( v_k - v_\infty' \), we see that \( v_k - v_\infty' \) solves the equation:
\[
\partial_t u + f(a_k) \partial_x u = g - S_j g, \quad u(0) = v_0 - S_j v_0.
\]
(74)
By Lemma 7 and (A1) once again, it follows that
\[
\|
\nu_k - \nu_i
\|_{B^s_{p,r}} \leq \| \nu_0 - S_j \|_{B^s_{p,r}} + \int_0^t \|C_i \|_{\alpha(r)} \|g - S_j \|_{B^s_{p,r}} \, dr
\]
\[
\leq \| \nu_0 - S_j \|_{B^s_{p,r}} + \int_0^t \|g - S_j \|_{B^s_{p,r}} \, dr.
\] (75)

Note that \( \nu_i(0) \in B^s_{p,r} \) and \( S_j g \in L^1(0, T; B^s_{p,r}) \). Thanks to (71), we have
\[
\|
\nu_i - \nu_i(0)
\|_{B^s_{p,r}} \leq \|
\nu_0 - S_j \nu_0
\|_{B^s_{p,r}} + \int_0^t \|S_j g \|_{B^s_{p,r}} \, dr.
\] (76)

Similarly, we can obtain
\[
\|
\nu_i - \nu_i(0)
\|_{B^s_{p,r}} \leq \|
\nu_0 - S_j \nu_0
\|_{B^s_{p,r}} + \int_0^t \|S_j g \|_{B^s_{p,r}} \, dr.
\] (77)

Inserting (75), (76), and (77) into (72), we have
\[
\|
\nu_k
\|_{B^s_{p,r}} \leq \|
\nu_0 - S_j \nu_0
\|_{B^s_{p,r}} + \int_0^t \|S_j g \|_{B^s_{p,r}} \, dr.
\] (81)

By the definition of \( S_j \), we have \( \|g(r) - S_j g(r)\|_{B^s_{p,r}} \leq \|g(r)\|_{B^s_{p,r}} \leq \|
\nu_0 - S_j \nu_0
\|_{B^s_{p,r}} \) \( \leq L^1(0, T) \). Using the Lebesgue dominated convergence theorem yields that the first two terms of the right-hand side of the above estimate may be arbitrary small for \( j \) large enough. Take \( j \) large enough; then fix \( j \). Let \( k \) tend to infinity so that the last term of right-hand side tends to zero. We complete the proof of Lemma 11.

Set \( s > s_0 \geq 1 + 1/p, s \neq 2 + 1/p \). Fix \( u_{0,\infty} \in B^s_{p,r} \) and let \( u_{0,k} \in B^s_{p,r} \) be a sequence with \( u_{0,k} \to u_{0,\infty} \). For simplicity, we denote \( I = [0, T] \) and we let \( u_{\infty} \) (resp., \( u_{0,k} \)) be the solution to (1) with initial data \( u_{0,\infty} \) (resp., \( u_{0,k} \)); we need to demonstrate that
\[
\lim_{k \to \infty} u_k = u_{\infty} \text{ in } C(I; B^s_{p,r}).
\] (79)

By Lemma 10, we have \( u_k \to u_{\infty} \) in \( C(I; B^{s-1}_{p,r}) \). To prove (79), it suffices to prove that
\[
\lim_{k \to \infty} \partial_x u_k = \partial_x u_{\infty} \text{ in } C(I; B^{s-1}_{p,r}).
\] (80)

Set \( v_k = \partial_x u_k, k \in \mathbb{N} \); then \( v_k \) satisfies
\[
\partial_t v_k + f(u_k) \partial_x v_k = -f'(u_k) (\partial_x u_k)^2 - \partial_x F(u_k)
\]
\[
= g_k,
\]
\[
v_k(0, x) = \partial_x u_{0,k}.
\] (81)

Motivated by the work in [34], for \( k \in \mathbb{N} \), we decompose \( v_k = z_k + h_k \) such that
\[
\partial_t z_k + f(u_k) \partial_x z_k = g_k - g_{\infty},
\]
\[
z_k(0, x) = \partial_x u_{0,k} - \partial_x u_{0,\infty},
\]
\[
\partial_t h_k + f(u_k) \partial_x h_k = g_{\infty},
\]
\[
h_k(0, x) = \partial_x u_{0,\infty}.
\] (82)

Since \( u_{0,k} \to u_{0,\infty} \) in \( B^{s-1}_{p,r} \), we have \( \| u_{0,k} \|_{B^s_{p,r}} \leq \| u_{0,\infty} \|_{B^s_{p,r}} \leq 1 \). Then (13) shows that
\[
\| u_k \|_{B^s_{p,r}} + \| u_{\infty} \|_{B^s_{p,r}} \leq 2 \| u_{0,k} \|_{B^s_{p,r}} + 2 \| u_{0,\infty} \|_{B^s_{p,r}} \leq 1.
\] (83)

By the algebra property of \( B^{s-1}_{p,r} \), the assumptions (A1), (A2), and (83), we have
\[
\| g_k - g_{\infty} \|_{B^s_{p,r}} = \| f'(u_{\infty}) (\partial_x u_{\infty})^2 - f'(u_k) (\partial_x u_k)^2 + \partial_x F(u_{\infty}) - \partial_x F(u_k) \|_{B^s_{p,r}} \leq \| \partial_x u_{\infty} - \partial_x u_k \|_{B^{s-1}_{p,r}}.
\] (84)

Since \( s > 1 + 1/p, s \neq 2 + 1/p \), by Lemma 7, we arrive at
\[
\| z_k(t) \|_{B^s_{p,r}} \leq \| z_0 \|_{B^s_{p,r}} + \int_0^t \| C_i \|_{\alpha(r)} \| g_k - g_{\infty} \|_{B^s_{p,r}} \, dr
\]
\[
\leq \| z_0 \|_{B^s_{p,r}} + \int_0^t \| C_i \|_{\alpha(r)} \| g_k \|_{B^s_{p,r}} \, dr.
\] (85)

By (83) and the convergence \( u_k \to u_{\infty} \) in \( C(I; B^{s-1}_{p,r}) \), we can get \( h_k \to h_{\infty} = v_{\infty} = \partial_x u_{\infty} \) in \( C(I; B^{s-1}_{p,r}) \) by applying Lemma 11. That is to say, there is a \( K > 0 \) such that
\[
k > K \implies \| h_k - h_{\infty} \|_{C(I; B^{s-1}_{p,r})} < \epsilon.
\] (86)

Therefore, when \( k > K \), we have
\[
\| \partial_x u_k - \partial_x u_{\infty} \|_{B^{s-1}_{p,r}} = \| v_k - v_{\infty} \|_{B^{s-1}_{p,r}} \leq \| z_k(t) \|_{B^{s-1}_{p,r}} + \| h_k(t) \|_{B^{s-1}_{p,r}} \leq \| \partial_x u_{0,k} - \partial_x u_{0,\infty} \|_{B^{s-1}_{p,r}} \]
\[
+ \int_0^t \| \partial_x u_k - \partial_x u_{\infty} \|_{B^{s-1}_{p,r}} \, dt + \epsilon.
\] (87)
By using the Gronwall’s inequality, we have
\[
\|\partial_x u_{\infty} - \partial_x u_k\|_{C([0,T];B^s_r)} \leq \varepsilon C(T, u_{0,\infty}) \quad \text{as } k > K, \quad (88)
\]
where \(C(T, u_{0,\infty})\) is a enough big constant depending on \(T\) and \(u_{0,\infty}\). Hence we obtain (80).

3.7. Hölder Continuity of the Solution Map with Weak Topology. Basically, direct from Lemma 10, (14) is proved. Hereto, we complete the proof of Theorem 1.

4. Proof of Proposition 3
In this section, we will show that when \(f(u) = u, F(u) = u^2\), the solution map defined by the problem (1)-(2) is not uniformly continuous from any bounded subset of \(B^s_{2r}\) into \(E^{s}_{2r}(T)\) for any \(T > 0\) with \(s > 3/2, 1 \leq r < \infty\). Actually, in this case, \(s_0 = 3/2\) and \(f(\cdot), F(\cdot)\) satisfy the assumptions (A1)–(A3) with \(n = 1\).

To prove Proposition 3, we need to show that, for any given \(T > 0\), there are two sequences of solutions \(u_k\) and \(v_k\) which satisfy (17)–(19) when \(t \in [0, T]\).

We will use the following estimates for trigonometric functions cosine and sine.

Lemma 12 (see [43, 51]). Let \(\sigma, \alpha \in \mathbb{R}\). If \(\lambda \in \mathbb{Z}^+\) and \(\lambda \gg 1\), then
\[
\|\sin(\lambda x - \alpha)\|_{B^s_{2r}(T)} = \|\cos(\lambda x - \alpha)\|_{B^s_{2r}(T)} = \begin{cases} 2^{1/s} \pi \left(1 + \lambda^2\right)^{s/2} \approx \lambda^s, & r < \infty, \\ \pi \left(1 + \lambda^2\right)^{s/2} \approx \lambda^s, & r = \infty. \end{cases} \quad (89)
\]

From now on, we assume that \(1/2 < \sigma < \min\{s - 1, 3/2\}, f(u) = u,\) and \(F(u) = u^2\).

4.1. Approximate Solutions and Actual Solutions. Following the approach in [32], we choose approximate solutions as
\[
u^{\omega, k}(t,x) = \omega^{-1} k^{-s} \cos \theta, \quad \theta = kx - \omega t, \quad k \in \mathbb{Z}^+, \quad \omega = 0, 1. \quad (90)
\]
Substituting these functions into (8) gives rise to the error which is defined to be
\[
E = \partial_t \left(\nu^{\omega, k}\right) + \nu^{\omega, k} \partial_x \left(\nu^{\omega, k}\right) + \left(\nu^{\omega, k}\right)^2. \quad (91)
\]
We begin by estimating the \(B^s_{2r}\) norm of \(E\).

Lemma 13. For \(k \gg 1, 1 \leq r < \infty,\) and \(1/2 < \sigma < \min\{s - 1, 3/2\}\) and \(s + 2 > s\), then
\[
\|E\|_{B^s_{2r}} \leq k^r, \quad (92)
\]
where \(r_s < 0\) and
\[
r_s = \begin{cases} -2s + 1 + \sigma < 0, & \text{if } \frac{3}{2} < s \leq \frac{3 + \sigma}{2}, \\ -2, & \text{if } s > \frac{3 + \sigma}{2}. \end{cases} \quad (93)
\]
Proof. Firstly, we have
\[
\|E\|_{B^s_{2r}} \leq \|\partial_t \left(\nu^{\omega, k}\right) + \left(\nu^{\omega, k}\right)^2\|_{B^s_{2r}}, \quad (94)
\]
As the \(\nu^{\omega, k}\) are explicitly given, direct computation shows that
\[
\|\partial_t \left(\nu^{\omega, k}\right) + \left(\nu^{\omega, k}\right)^2\|_{B^s_{2r}} \leq k^{-2s + 1}. \quad (95)
\]
Use Lemma 12 to get
\[
\|\left(\nu^{\omega, k}\right)^2\|_{B^s_{2r}} \leq k^{-2}. \quad (96)
\]
Putting our estimates for both terms of (94) together, it follows that
\[
\|E\|_{B^s_{2r}} \leq \max\left\{k^{-2s + 1}, k^{-2}\right\} \leq k^r, \quad (97)
\]
which completes the estimate.

When \(1 \leq r < \infty\), Lemma 12 shows that for \(k \gg 1\) and any \(t \geq 0\)
\[
\|\nu^{\omega, k}(t, x)\|_{B^s_{2r}} = \|\nu^{\omega, k} + k^{-s} \cos(kx - \omega t)\|_{B^s_{2r}} \leq 1. \quad (98)
\]
Therefore, according to Theorem 1, the following periodic boundary value problem
\[
\partial_t u_{\omega, k} + u_{\omega, k} \partial_x (u_{\omega, k}) = -F(u_{\omega, k}), \quad x \in \bar{T}, \ t > 0, \quad (99)
\]
has a unique solution \(u_{\omega, k}(t, x)\) in \(E^{s}_{2r}(\bar{T})\), where \(\bar{T} > 0\) can be chosen independent of \(k \gg 1\) by using (98) and
\[
\bar{T} = \frac{1}{2C_{s,r}} \left\|\nu^{\omega, k}(0)\right\|_{B^s_{2r}} \geq 1. \quad (100)
\]
Let \(T_1 = \min\{\bar{T}, T\}\). Then, \(u_{\omega, k} \in E^{s}_{2r}(T_1)\) and (13) turn out to be
\[
\|u_{\omega, k}(t, x)\|_{B^s_{2r}} \leq \|u^{\omega, k}(0, x)\|_{B^s_{2r}} \leq 1, \quad 0 \leq t \leq T_1. \quad (101)
\]
Let \(v = u^{\omega, k} - u_{\omega, k}\). Then, \(v\) satisfies the initial value problem:
\[
\partial_t v + u^{\omega, k} \partial_x v = E - \nu \partial_x u_{\omega, k} - \left[F(u^{\omega, k}) - F(u_{\omega, k})\right], \quad x \in \bar{T}, \ t \in \mathbb{R}^+. \quad (102)
\]
\[v(0, x) = 0, \quad x \in \bar{T}.\]
4.2. Estimating the Difference between Approximate and True Solutions. We now calculate the $B_{s,r}^{2}\sigma$ norm and $B_{s,r}^{2}\sigma$ norm of $v$ for $1/2 < \sigma < \min\{s - 1, 3/2\}$. 

**Lemma 14.** Let $k \gg 1, 1/2 < \sigma < \min\{s - 1, 3/2\}, \sigma > 2 > s$, and $r_s$ be given in Lemma 13; then

\[
\|v(t)\|_{B_{s,r}^{2}\sigma} \leq k^{s}, \\
\|v(t)\|_{B_{s,r}^{2}\sigma} \leq k^{s}, \\
\] 
for $0 < t \leq T_1$.

**Proof.** Rewrite the equation in (102) as

\[
\partial_t v + u_{w,k}(t) \partial_x v = E - \partial_t u_{w,k} - \left[ F(u_{w,k}) - F(u_{w,k}) \right]. \tag{104}
\]

By Lemma 7 and $v(0, x) = 0$, we have

\[
\|v(t)\|_{B_{s,r}^{2}\sigma} \leq C \int_0^t \left\{ \int \|\partial_t u_{w,k}(\tau')\|_{B_{s,r}^{2}\sigma} d\tau' \right\} \times \|F\|_{B_{s,r}^{2}\sigma} d\tau,
\] 
where $F' = E - \partial_t u_{w,k} - \left[ F(u_{w,k}) - F(u_{w,k}) \right]$. Note that $B_{s,r}^{2}\sigma$ is a Banach algebra; from (98), (101), we obtain

\[
\|v(t)\|_{B_{s,r}^{2}\sigma} \leq \int_0^t \|v\|_{B_{s,r}^{2}\sigma} d\tau + \int_0^t \|E\|_{B_{s,r}^{2}\sigma} d\tau. \tag{105}
\]

Applying the Gronwall’s inequality and Lemma 13 yields

\[
\|v(t)\|_{B_{s,r}^{2}\sigma} \leq e^{C_{s,r}T} \int_0^t k^{s} d\tau \leq k^{s}. \tag{106}
\]

To estimate the $B_{s,r}^{2}\sigma$ norm of $v$, we first estimate the $B_{s,r}^{2}\sigma$ norm of $u_{w,k}(t)$. Applying the property (2) in Lemma 7, the embedding $B_{s,r}^{2}\sigma^{-1} \rightarrow L^\infty$, and (101), it follows that

\[
\|u_{w,k}\|_{B_{s,r}^{2}\sigma}^{-1} \leq \|u_{w,k}(0)\|_{B_{s,r}^{2}\sigma}^{-1} \leq C \int_0^t \|F(u_{w,k})\|_{B_{s,r}^{2}\sigma} d\tau \\
+ \|\partial_t (u_{w,k})\|_{L^\infty} \|u_{w,k}\|_{B_{s,r}^{2}\sigma}^{-1} \leq C \int_0^t \|u_{w,k}\|_{L^\infty} \|u_{w,k}\|_{B_{s,r}^{2}\sigma}^{-1} d\tau \\
+ \|u_{w,k}\|_{B_{s,r}^{2}\sigma} \|u_{w,k}\|_{B_{s,r}^{2}\sigma}^{-1} \leq C \int_0^t \|u_{w,k}\|_{B_{s,r}^{2}\sigma}^{-1} d\tau.
\] 

From the above inequality and Lemma 12, we have

\[
\|u_{w,k}(t)\|_{B_{s,r}^{2}\sigma} \leq \|u_{w,k}(0)\|_{B_{s,r}^{2}\sigma} C_{s,r} T e^{C_{s,r}T} \leq k^{s}. \tag{107}
\]

Thanks to (109), we have

\[
\|v(t)\|_{B_{s,r}^{2}\sigma} \leq \|u_{w,k}(t)\|_{B_{s,r}^{2}\sigma} + \|u_{w,k}(t)\|_{B_{s,r}^{2}\sigma} \leq \|u_{w,k}(t)\|_{B_{s,r}^{2}\sigma} \leq k^{s}. \tag{110}
\]

Combining these results gives rise to the desired estimates. □

4.3. Proof for Proposition 3. It suffices to show that $u_{0,k}$ and $u_{1,k}$ are two sequences of solutions satisfying the three conditions (17), (18), and (19) for $t \in [0, T_1]$.

By the construction of $u_{w,k}$, we see that (101) implies (17), and

\[
\|u_{0,k}(0) - u_{1,k}(0)\|_{B_{s,r}^{2}\sigma} = \|u_{0,k}(0) - u_{1,k}(0)\|_{B_{s,r}^{2}\sigma} = k^{-1} \rightarrow 0, \tag{111}
\]

when $k \rightarrow \infty$ gives (18). For (19), first we have that, for $t \in [0, T_1]$,

\[
\liminf_{k \rightarrow \infty} \|u_{0,k}(t) - u_{1,k}(t)\|_{B_{s,r}^{2}\sigma} \geq \liminf_{k \rightarrow \infty} \|u_{0,k}(t) - u_{1,k}(t)\|_{B_{s,r}^{2}\sigma} \\
- \lim_{k \rightarrow \infty} \|u_{0,k}(t) - u_{k}(t)\|_{B_{s,r}^{2}\sigma} - \lim_{k \rightarrow \infty} \|u_{k}(t) - u_{1,k}(t)\|_{B_{s,r}^{2}\sigma}. \tag{112}
\]

By Lemma 14 and interpolation inequality, we obtain

\[
\|v\|_{B_{s,r}^{2}\sigma} \leq \|v\|_{B_{s,r}^{2}\sigma}^{1/2} \|v\|_{B_{s,r}^{2}\sigma}^{1/2} \leq k^{s} \sqrt{k^{s}} = k^{s}. \tag{113}
\]

where $r'_s$ can be simplified as

\[
r'_s = r_s \cdot \frac{1}{2} + (s - \sigma), \tag{114}
\]

As $r'_s$, we can deduce that

\[
\lim_{k \rightarrow \infty} \|u_{0,k}(t) - u_{k}(t)\|_{B_{s,r}^{2}\sigma} = 0. \tag{115}
\]

Therefore, using Lemma 12, for $t \in [0, T_1]$, we have

\[
\liminf_{k \rightarrow \infty} \|u_{0,k}(t) - u_{k}(t)\|_{B_{s,r}^{2}\sigma} \geq \liminf_{k \rightarrow \infty} \|u_{0,k}(t) - u_{k}(t)\|_{B_{s,r}^{2}\sigma} \\
- \limsup_{k \rightarrow \infty} \|u_{k}(t) - u_{k}(t)\|_{B_{s,r}^{2}\sigma} \geq \liminf_{k \rightarrow \infty} \left(2k^{-\sigma} \sin \left(\frac{\sin T}{2}\right) \sin \left(\frac{t}{2}\right)\right)_{B_{s,r}^{2}\sigma} \\
- \limsup_{k \rightarrow \infty} \left(2k^{-\sigma} \sin \left(\frac{\sin T}{2}\right) \sin \left(\frac{t}{2}\right)\right)_{B_{s,r}^{2}\sigma} = \left|\sin \left(\frac{t}{2}\right)\right|.
\] 

We complete the proof for Proposition 3.
Conflict of Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments
The authors would like to express their great gratitude to the referees for their valuable suggestions, which have led to a meaningful improvement of the paper. This work is supported by the National Natural Science Foundation of China (no. 11401125).

References


