Research Article

A Reverse Theorem on the $\|\cdot\|$ - w^* Continuity of the Dual Map

Mienie de Kock¹ and Francisco Javier García-Pacheco²

¹Department of Mathematics and Physics, Texas A&M University Central Texas, Killeen, TX 76548, USA

²Department of Mathematics, University of Cadiz, 11519 Puerto Real, Spain

Correspondence should be addressed to Francisco Javier García-Pacheco; garcia.pacheco@uca.es

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Given a Banach space X , $x \in S_X$, and $J_X(x) = \{x^* \in S_{X^*} : x^*(x) = 1\}$, we define the set $J_X^*(x)$ of all $x^* \in S_{X^*}$ for which there exist two sequences $(x_n)_{n \in \mathbb{N}} \subseteq S_X \setminus \{x\}$ and $(x_n^*)_{n \in \mathbb{N}} \subseteq S_{X^*}$ such that $(x_n)_{n \in \mathbb{N}}$ converges to x , $(x_n^*)_{n \in \mathbb{N}}$ has a subnet w^* -convergent to x^* , and $x_n^*(x_n) = 1$ for all $n \in \mathbb{N}$. We prove that if X is separable and reflexive and X^* enjoys the Radon-Riesz property, then $J_X^*(x)$ is contained in the boundary of $J_X(x)$ relative to S_{X^*} . We also show that if X is infinite dimensional and separable, then there exists an equivalent norm on X such that the interior of $J_X(x)$ relative to S_{X^*} is contained in $J_X^*(x)$.

1. Preliminaries and Background

Recall that a point x in the unit sphere S_X of a real or complex normed space X is said to be a smooth point of B_X , provided that there is only one functional in S_{X^*} attaining its norm at x . This unique functional is usually denoted by $J_X(x)$. The set of smooth points of the (closed) unit ball B_X of X is usually denoted as $\text{smo}(B_X)$. X is said to be smooth provided that $S_X = \text{smo}(B_X)$. If X is smooth, then the dual map of X is defined as the map $J_X : X \rightarrow X^*$ such that $\|J_X(x)\| = \|x\|$ and $J_X(x)(x) = \|x\|^2$ for all $x \in X$. It is well known that the dual map is $\|\cdot\|$ - w^* continuous and that $J_X(\lambda x) = \bar{\lambda} J_X(x)$ for all $\lambda \in \mathbb{C}$ and all $x \in X$. We refer the reader to [1, 2] for a better perspective on these concepts.

On the other hand, recall that a normed space is said to be rotund (or strictly convex) provided that its unit sphere is free of nontrivial segments. It is well known among Banach Space Geometers that smoothness and rotundity are dual concepts in the following sense: if a dual space is rotund (smooth), then the predual is smooth (rotund). The converse does not hold though. Next, we will gather some of the most relevant results in terms of rotund and smooth renormings into the following (see [3, Theorem 1 (VII.4)] and [1, Corollary 4.3]).

Theorem 1 (see [1, 3]). *Let X be a real or complex normed space. Then one has the following.*

- (i) *If X is separable, then X admits an equivalent norm so that both X and X^* are rotund.*
- (ii) *If X is reflexive, then X admits an equivalent norm so that X is rotund and smooth.*

In case $x \in S_X$ is not a smooth point then $J_X(x)$ is defined as $x^{-1}(1) \cap B_{X^*}$, that is, the set $\{x^* \in B_{X^*} : x^*(x) = 1\}$. We will now continue with a brief introduction on faces and the impact of surjective linear isometries on them. The following definition is very well known amid Banach Space Geometers.

Definition 2. Let X be a real or complex normed space and consider a nonempty convex subset C of B_X . Then one has the following.

- (i) C is said to be a face of B_X provided that C verifies the extremal condition with respect to B_X ; that is, if $x, y \in B_X$ and $t \in (0, 1)$ with $tx + (1 - t)y \in C$, then $x, y \in C$.
- (ii) C is said to be an exposed face of B_X provided that there exists $f \in S_{X^*}$ such that $C = C_f$, where $C_f := f^{-1}(1) \cap B_X$.

It is immediate that every exposed face is a proper face, and every proper face must be contained in the unit sphere. Also notice that $J_X(x) = C_x$ for every $x \in S_X$; that is, $J_X(x)$ is

an exposed face of B_{X^*} . We will refer to these exposed faces as w^* -exposed faces.

A face, an exposed face, or a w^* -exposed face, which is a singleton, is called an extreme point, an exposed point, or a w^* -exposed point, respectively.

Remark 3. Observe the following, since we will use it in upcoming sections (see [1, 4]).

- (1) A point $x \in \text{smo}(B_X)$ if and only if C_x is a singleton, and, in this situation, $C_x = \{J_X(x)\}$. So with this notation if x is a smooth point, then $J_X(x)$ is a w^* -exposed point.
- (2) Also note that $\text{int}_{S_X}(C) \subseteq \text{smo}(B_X)$ for every proper face C .
- (3) Assume now that $T : X \rightarrow Y$ is a surjective linear isometry between the real or complex normed spaces X and Y . It is not difficult to check that if $f \in S_{X^*}$, then

$$T(C_f) = C_{(T^{-1})^*(f)}. \quad (1)$$

As mentioned before, in a smooth space if $(x_n)_{n \in \mathbb{N}}$ converges to x , then the sequence $(J_X(x_n))_{n \in \mathbb{N}}$ is w^* -convergent to $J_X(x)$ (see [1]). The main result in this paper is the converse to the previous statement (see Theorem 12).

Theorem 4. *Let X be a separable real Banach space with $\dim(X) > 1$. Consider $x \in S_X$. Then one has the following.*

- (1) $\text{lbd}_{S_{X^*}}(J_X(x)) \subseteq J_X^*(x)$.
- (2) If X is reflexive and X^* has the Radon-Riesz property, then $J_X^*(x) \subseteq \text{bd}_{S_{X^*}}(J_X(x))$.
- (3) If X is infinite dimensional, then there exists an equivalent norm on X such that $\text{int}_{S_{X^*}}(J_X(x)) \subseteq J_X^*(x)$.

More precisely, for a given real or complex Banach space X and $x \in S_X$ we will define the set $J_X^*(x)$ of all $x^* \in B_{X^*}$ for which there exist two sequences $(x_n)_{n \in \mathbb{N}} \subseteq S_X \setminus \{x\}$ and $(x_n^*)_{n \in \mathbb{N}} \subseteq S_{X^*}$ such that $(x_n)_{n \in \mathbb{N}}$ converges to x , $(x_n^*)_{n \in \mathbb{N}}$ has a subnet w^* -convergent to x^* , and $x_n^*(x_n) = 1$ for all $n \in \mathbb{N}$. As expected, $\text{bd}_{S_{X^*}}(J_X(x))$ and $\text{int}_{S_{X^*}}(J_X(x))$ denote the boundary and the interior, respectively, of $J_X(x)$ relative to S_{X^*} . The set $\text{lbd}_{S_{X^*}}(J_X(x))$ denotes the localized boundary of $J_X(x)$ (see Definition 9), that is, the set of all $x^* \in J_X(x)$ for which there exists a dense subset D of $X \setminus \mathbb{R}x$ with $\text{card}(D) = \text{den}(X)$ and satisfying that $x^*|_{Y_d} \in \text{bd}_{S_{Y_d^*}}(J_{Y_d}(x))$ for all $d \in D$, where $Y_d := \text{span}\{x, d\}$ (recall that $\text{den}(X)$ stands for the density character of X).

To conclude, we draw the reader's attention to the fact that if X is real and $\dim(X) = 1$, then $J_X^*(x) = \emptyset$ for all $x \in S_X$.

2. The Geometric Tools

Note that no complex Banach space admits proper faces with nonempty interior relative to the unit sphere in its unit ball (see [5, Theorem 2.1]). This is the reason for considering real Banach spaces only. On our journey to proving the main

result, we will utilize several technical lemmas and remarks. The first lemma involves the intersections of w^* -exposed faces and the disjointness of their interiors.

Lemma 5. *Let X be a real Banach space and consider $x \neq y \in S_X$. Then one has the following.*

- (1) $J_X(x) \cap \text{int}_{S_{X^*}}(J_X(y)) = \emptyset$.
- (2) $J_X(x) \cap J_X(y) \neq \emptyset$ if and only if $[x, y] \subset S_X$.
- (3) If $z \in S_X \cap (x, y)$, then $J_X(x) \cap J_X(y) = J_X(z)$.

Proof. (1) Assume that there exists $z^* \in J_X(x) \cap \text{int}_{S_{X^*}}(J_X(y))$. As mentioned in the introduction, we have that $\text{int}_{S_{X^*}}(J_X(y)) \subseteq \text{smo}(B_{X^*})$; therefore $x = J_{X^*}(z) = y$ which contradicts the hypothesis of the lemma.

(2) If $y^* \in J_X(x) \cap J_X(y)$, then $y^*(tx + (1-t)y) = 1$ and $\|tx + (1-t)y\| = 1$ for all $t \in [0, 1]$. Conversely, if $[x, y] \subset S_X$, then the Hahn-Banach Separation Theorem assures the existence of $y^* \in S_{X^*}$ such that

$$\sup y^*(U_X) \leq \inf y^*([x, y]), \quad (2)$$

which immediately implies that $y^*(y) = y^*(x) = 1$.

(3) The proof uses a similar argument as in the previous item. $\square$

Remark 6. No extreme point can lie in the interior of a proper face relative to the unit sphere. Therefore,

$$\text{ext}(B_{X^*}) \cap J_X(x) = \text{ext}(J_X(x)) \subseteq \text{bd}_{S_{X^*}}(J_X(x)). \quad (3)$$

Every w^* -exposed face is w^* -closed and thus w^* -compact; therefore the Krein-Milman Theorem allows the existence of extreme points in every w^* -exposed face (even more, every w^* -exposed face is the w^* -closed convex hull of its extreme points).

The second technical lemma is a slight generalization of the well-known fact that the dual map is $\|\cdot\|$ - w^* continuous in smooth spaces.

Lemma 7. *Let X be a real Banach space with $\dim(X) > 1$ and fix an arbitrary element $x \in S_X$. Then $\emptyset \neq J_X^*(x) \subseteq J_X(x)$. As a consequence, if $x \in \text{smo}(B_X)$, then $J_X^*(x) = J_X(x)$.*

Proof. Let $x^* \in J_X^*(x)$. We can find two sequences $(x_n)_{n \in \mathbb{N}} \subseteq S_X \setminus \{x\}$ and $(x_n^*)_{n \in \mathbb{N}} \subseteq S_{X^*}$ such that $(x_n)_{n \in \mathbb{N}}$ converges to x , $(x_n^*)_{n \in \mathbb{N}}$ has a subnet $(x_{n_i}^*)_{i \in I}$ w^* -convergent to x^* , and $x_n^*(x_n) = 1$ for all $n \in \mathbb{N}$. Next observe that for all $i \in I$ we have

$$\begin{aligned} |1 - x^*(x)| &= |x_{n_i}^*(x_{n_i}) - x^*(x)| \\ &\leq |x_{n_i}^*(x_{n_i}) - x_{n_i}^*(x)| + |x_{n_i}^*(x) - x^*(x)| \\ &\leq \|x_{n_i} - x\| + |x_{n_i}^*(x) - x^*(x)|, \end{aligned} \quad (4)$$

which implies that $x^*(x) = 1$ and thus $x^* \in J_X(x)$. In order to show that $J_X^*(x) \neq \emptyset$, we simply start off with two sequences $(x_n)_{n \in \mathbb{N}} \subseteq S_X \setminus \{x\}$ and $(x_n^*)_{n \in \mathbb{N}} \subseteq S_{X^*}$ such that $(x_n)_{n \in \mathbb{N}}$ converges to x and $x_n^*(x_n) = 1$ for all $n \in \mathbb{N}$. The

w^* -compactness of B_{X^*} assures that $(x_n^*)_{n \in \mathbb{N}}$ has a subnet $(x_{n_i}^*)_{i \in I}$ w^* -convergent to x^* . By definition $x^* \in J_X(x)$. $\square$

In general, no convexity properties are verified by $J_X^*(x)$ even in finite dimensions. Indeed, if $X := \ell_\infty^2$ and $x := (1, 1)$, then $J_X^*(x) = \{(1, 0), (0, 1)\}$.

We will finish this section with a characterization of a smooth point in terms of localizing (linearly and topologically).

Proposition 8. *Let X be a real Banach space. For a given point $x \in S_X$, the following are equivalent.*

- (1) $x \in \text{smo}(B_X)$.
- (2) For every $d \in X$, x is a smooth point of the unit ball of $\text{span}\{x, d\}$.
- (3) For every dense subset D of X , x is a smooth point of the unit ball of $\text{span}\{x, d\}$ for all $d \in D$.
- (4) There exists a dense subset D of X with $\text{card}(D) = \text{den}(X)$ such that x is a smooth point of the unit ball of $\text{span}\{x, d\}$ for all $d \in D$.

Proof. (1) $\Rightarrow$ (2) Let $x^*, y^* \in J_{\text{span}\{x, d\}}(x)$. By the Hahn-Banach Extension Theorem we may assume that $x^*, y^* \in J_X(x)$, which implies that $x^* = y^*$.

(2) $\Rightarrow$ (3) Immediate.

(3) $\Rightarrow$ (4) Immediate.

(4) $\Rightarrow$ (1) Let $x^*, y^* \in J_X(x)$. For every $d \in D$ we have that

$$x^*|_{\text{span}\{x, d\}}, y^*|_{\text{span}\{x, d\}} \in J_{\text{span}\{x, d\}}(x), \quad (5)$$

thus by hypothesis $x^*|_{\text{span}\{x, d\}} = y^*|_{\text{span}\{x, d\}}$ for all $d \in D$. As a consequence, $x^*|_D = y^*|_D$ and so $x^* = y^*$ due to the density of D in X . $\square$

In [6, Remark 2.9(1)] it is shown that if K is a Hausdorff locally compact topological space with more than one point, then the constant function **1** is a uniformly nonsmooth point of $B_{\mathcal{C}(K)}$ (recall that $x \in S_X$ is said to be a uniformly nonsmooth point provided that x is not a smooth point of the unit ball of $\text{span}\{x, y\}$ for all $y \in S_X \setminus \{x\}$).

We refer the reader to [7, page 168] where a precise description of the smooth points of $B_{\mathcal{C}(K)}$ is given, for K is a Hausdorff locally compact topological space.

3. The Main Result

We first define a new boundary for the w^* -exposed faces other than the regular topological boundary.

Definition 9. Let X be a real Banach space and consider $x \in S_X$. We define the localized boundary of $J_X(x)$ as the set $\text{lbd}_{S_{X^*}}(J_X(x))$ of all $x^* \in J_X(x)$ for which there exists a dense subset D of $X \setminus \mathbb{R}x$ with $\text{card}(D) = \text{den}(X)$ and satisfying that $x^*|_{Y_d} \in \text{bd}_{S_{Y_d^*}}(J_{Y_d}(x))$ for all $d \in D$, where $Y_d := \text{span}\{x, d\}$.

Note that the localized boundary of a w^* -exposed face is always contained in the topological boundary of that face relative to the dual unit sphere.

Proposition 10. *Let X be a 2-dimensional real Banach space. For every $x \in S_X$ we have that $\text{lbd}_{S_{X^*}}(J_X(x)) = \text{bd}_{S_{X^*}}(J_X(x)) \subseteq J_X^*(x)$.*

Proof. The 2-dimensionality of X assures that

$$\text{lbd}_{S_{X^*}}(J_X(x)) = \text{bd}_{S_{X^*}}(J_X(x)) = \{x_1^*, x_2^*\} \quad (6)$$

for some $x_1^* \neq x_2^* \in S_{X^*}$. It is clear that we assume that x is not a smooth point in virtue of Lemma 7. In accordance to Mazur's Theorem (see [8]), the separability of X allows for the density of the smooth points in the unit sphere; therefore

$$\begin{aligned} \text{smo}(B_X) \cap S_X \cap (x_1^* - x_2^*)^{-1}((0, +\infty)) &\text{ is dense in} \\ S_X \cap (x_1^* - x_2^*)^{-1}((0, +\infty)), & \\ \text{smo}(B_X) \cap S_X \cap (x_1^* - x_2^*)^{-1}((-\infty, 0)) &\text{ is dense in} \\ S_X \cap (x_1^* - x_2^*)^{-1}((-\infty, 0)). & \end{aligned} \quad (7)$$

We can then find two sequences

$$\begin{aligned} (x_n)_{n \in \mathbb{N}} &\subset \text{smo}(B_X) \cap S_X \cap (x_1^* - x_2^*)^{-1}((0, +\infty)), \\ (y_n)_{n \in \mathbb{N}} &\subset \text{smo}(B_X) \cap S_X \cap (x_1^* - x_2^*)^{-1}((-\infty, 0)) \end{aligned} \quad (8)$$

both converging to x . Because of the compactness of S_{X^*} we can assume without loss of generality that the two sequences $(J_X(x_n))_{n \in \mathbb{N}}$ and $(J_X(y_n))_{n \in \mathbb{N}}$ are, respectively, convergent to some $a^* \neq b^* \in \{x_1^*, x_2^*\}$ (recall the proof of Lemma 7). $\square$

Remark 11. In Proposition 10 we have proved something more. Indeed, under the hypotheses of Proposition 10, if $x^* \in \text{bd}_{S_{X^*}}(J_X(x))$, then a sequence $(x_n)_{n \in \mathbb{N}} \subset \text{smo}(B_X) \setminus \{x\}$ exists such that $(J_X(x_n))_{n \in \mathbb{N}}$ w^* -converges to x^* .

Prior to stating and proving the main result, we recall the following fact: a real or complex Banach space X is said to have the Radon-Riesz property provided that the following condition holds: if $(x_n)_{n \in \mathbb{N}}$ is w -convergent to x and $(\|x_n\|)_{n \in \mathbb{N}}$ is convergent to $\|x\|$, then $(x_n)_{n \in \mathbb{N}}$ converges to x .

Theorem 12. *Let X be a separable real Banach space with $\dim(X) > 1$. Consider $x \in S_X$. Then one has the following.*

- (1) $\text{lbd}_{S_{X^*}}(J_X(x)) \subseteq J_X^*(x)$.
- (2) If X is reflexive and X^* has the Radon-Riesz property, then $J_X^*(x) \subseteq \text{bd}_{S_{X^*}}(J_X(x))$.
- (3) If X is infinite dimensional, then there exists an equivalent norm on X such that $\text{int}_{S_{X^*}}(J_X(x)) \subseteq J_X^*(x)$.

Proof. (1) Let x^* be an element in $\text{lbd}_{S_{X^*}}(J_X(x))$. By hypothesis there exists a dense sequence $(d_n)_{n \in \mathbb{N}} \subset X \setminus \mathbb{R}x$ such that $x^*|_{Y_n} \in \text{bd}_{S_{Y_n^*}}(J_{Y_n}(x))$ for all $n \in \mathbb{N}$, where $Y_n := \text{span}\{x, d_n\}$. By Proposition 10 for every $n \in \mathbb{N}$, we have that $x^*|_{Y_n} \in J_{Y_n}^*(x)$, so we can find $x_n \in S_{Y_n} \setminus \{x\}$ and $x_n^* \in S_{Y_n^*}$ such that $x_n^*(x_n) = 1$, $\|x - x_n\| < 1/n$ and $|x^*(d_n) - x_n^*(d_n)| < 1/n$. By the Hahn-Banach Extension Theorem we may assume

that $x_n^* \in S_{X^*}$ for all $n \in \mathbb{N}$. It only remains to show that $(x_n^*)_{n \in \mathbb{N}}$ has a subnet w^* -convergent to x^* . In fact, we will show more; we will prove that the whole sequence $(x_n^*)_{n \in \mathbb{N}}$ is w^* -convergent to x^* . Fix an arbitrary $y \in X$. Notice that for every $n \in \mathbb{N}$ we have that

$$\begin{aligned} |x^*(y) - x_n^*(y)| &= |x^*(y) - x^*(d_n)| + |x^*(d_n) - x_n^*(d_n)| \\ &\quad + |x_n^*(d_n) - x_n^*(y)| \\ &\leq 2\|y - d_n\| + \frac{1}{n}. \end{aligned} \quad (9)$$

At this stage, it is easy to understand that $(x_n^*(y_n))_{n \in \mathbb{N}}$ converges to $x^*(y)$.

(2) Assume to the contrary that there exists $x^* \in J_X^*(x)$ which is also in the interior of $J_X(x)$ relative to S_{X^*} . Notice that in this case $x \notin \text{smo}(B_X)$ since the interior of $J_X(x)$ relative to S_{X^*} is not empty (which implies that $J_X(x)$ cannot be a singleton). Now consider two sequences $(x_n)_{n \in \mathbb{N}} \subseteq S_X$ and $(x_n^*)_{n \in \mathbb{N}} \subseteq S_{X^*}$ such that $(x_n)_{n \in \mathbb{N}}$ converges to x , $(x_n^*)_{n \in \mathbb{N}}$ has a subnet w^* -convergent to x^* , and $x_n^*(x_n) = 1$ for all $n \in \mathbb{N}$. Observe that $x^* \in \overline{\{x_n^* : n \in \mathbb{N}\}}^{w^*}$, so since the w^* topology is metrizable in B_{X^*} (due to the fact that X is separable), there exists a subsequence $(x_{n_k}^*)_{k \in \mathbb{N}}$ of $(x_n^*)_{n \in \mathbb{N}}$ which is w^* -convergent to x^* . Since X is reflexive, we have that the w^* and the w topologies coincide on X^* ; therefore $(x_{n_k}^*)_{k \in \mathbb{N}}$ is w -convergent to x^* . Since X^* has the Radon-Riesz property, $(x_{n_k}^*)_{k \in \mathbb{N}} \subseteq S_{X^*}$, and $x^* \in S_{X^*}$, we deduce that $(x_{n_k}^*)_{k \in \mathbb{N}}$ is convergent to x^* . This means that there exists $m \in \mathbb{N}$ such that $(x_{n_k}^*)_{k \geq m} \subset \text{int}_{S_{X^*}}(J_X(x))$. This is impossible by Lemma 5.

(3) Fix an arbitrary $x^* \in S_{X^*}$ such that $x^*(x) = 1$. Observe that X can be equivalently renormed so that $X = \mathbb{R}x \oplus \ker(x^*)$ and $X^* = \mathbb{R}x^* \oplus \ker(x)$, so we will assume from now on that X is already endowed in such a way. Since X is infinite dimensional we have that S_{X^*} is w^* -dense in B_{X^*} and so is $\text{NA}(X) \cap S_{X^*}$ since X is complete, where $\text{NA}(X)$ denotes the set of norm-attaining functionals on X . We refer the reader to [9, Theorem 3.11] in order to take into consideration the fact that

$$\text{int}_{S_{X^*}}(J_X(x)) = \{x^* + m^* : m^* \in U_{\ker(x)}\}. \quad (10)$$

Therefore, fix $m^* \in \ker(x)$ which is also in the open unit ball of X^* and denote $y^* := x^* + m^*$. We can find a sequence $(m_n^*)_{n \in \mathbb{N}} \subseteq \text{NA}(X) \cap S_{\ker(x)}$ which is w^* -convergent to m^* (remember that the w^* -topology is metrizable in B_{X^*} due to the separability of X). For every $n \in \mathbb{N}$ there exists $m_n \in S_{\ker(x^*)}$ such that $m_n^*(m_n) = 1$. Finally, consider for every $n \in \mathbb{N}$ the elements $x_n := (1 - 2^{-n})x + 2^{-n}m_n$ and $x_n^* := x^* + m_n^*$. By construction we have that $(x_n)_{n \in \mathbb{N}} \subseteq S_X$ is convergent to x and $(x_n^*)_{n \in \mathbb{N}} \subseteq S_{X^*}$ is w^* -convergent to y^* . As a consequence, $y^* \in J_X^*(x)$. $\square$

Note that if a (nonnecessarily convex) subset E of S_{X^*} verifies the extremal condition (see Definition 2) and $E \cap \text{int}_{S_X}(C) \neq \emptyset$ for some proper face C of B_X , then $C \subseteq E$.

Corollary 13. *Let X be an infinite dimensional separable real Banach space. For every $x \in S_X$ there exists an equivalent norm on X satisfying that $J_X^*(x)$ does not verify the extremal condition.*

By combining Proposition 10 and (2) of Theorem 12 we obtain the following corollary.

Corollary 14. *Let X be a 2-dimensional real Banach space. For every $x \in S_X$, $J_X^*(x) = bd_{S_{X^*}}(J_X(x))$.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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