Research Article

Uniform Extendibility of the Bergman Kernel for Generalized Minimal Balls

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We consider a class of convex domains which contains non-Reinhardt domains with nonsmooth boundary. We show that the domains of this class satisfy the condition **Q**.

1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$. The Bergman space $A^2(\Omega)$ is the space of holomorphic functions on Ω which are square integrable. The Bergman space has the Bergman kernel $K(z, w)$, which is uniquely determined by the properties that it is an element of $A^2(\Omega)$ in z , is conjugate symmetric, and reproduces $A^2(\Omega)$. The Bergman projection P of $L^2(\Omega)$ onto $A^2(\Omega)$ is given by

$$(Pf)(w) = \int_{\Omega} f(z) \overline{K(z, w)} d\nu(z), \quad (1)$$

where $f \in L^2(\Omega)$ and $w \in \Omega$.

A bounded domain Ω is said to satisfy the condition **Q** if the Bergman projection maps $C_0^\infty(\Omega)$ into the space $\mathcal{O}(\overline{\Omega})$ of all holomorphic functions on Ω that can be extended holomorphically to a neighborhood of $\overline{\Omega}$.

It was also known that the condition **Q** holds on a domain Ω whenever the $\bar{\partial}$ -Neumann problem is globally real analytic hypoelliptic on Ω [1–3]. When a domain is a smoothly bounded pseudoconvex domain with real analytic boundary, there are many well-known examples on which the globally real analytic hypoellipticity of the $\bar{\partial}$ -Neumann problem holds:

- (i) strictly pseudoconvex domains [4–6],
- (ii) smooth bounded pseudoconvex circular domains with $\sum_{j=1}^n z_j(\partial r/\partial z_j) \neq 0$ near the boundary, where $r(z)$ is the defining function for domains [7],
- (iii) Reinhardt domains [8].

In this paper, we consider a class of convex circular domains which neither are Reinhardt nor have smooth boundaries and prove that the condition **Q** holds on each domain in the class defined above. Recently, there are many results of computing the Bergman kernels for various domains explicitly. See [9–14].

Let us restrict our attention to circular domains which are not Reinhardt. For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, consider the norm

$$N_*(z) = \sqrt{\frac{|z|^2 + |z \bullet z|}{2}}, \quad (2)$$

where $z \bullet w = \sum_{j=1}^n z_j w_j$ and $|z|^2 = z \bullet \bar{z}$. The norm N_* is the smallest norm in $\mathbb{R}^n$, under certain restriction [15]. The minimal ball $\mathbb{B}_* := \{z \in \mathbb{C}^n : N_*(z) < 1\}$ is the first known bounded domain in $\mathbb{C}^n$ which is neither Reinhardt nor homogeneous and for which the Bergman kernel can be computed explicitly as the closed form [16].

For any positive integers p and q , we denote by $M_{p,q}(\mathbb{C})$ the pq -dimensional complex vector space of all $(p \times q)$ -matrices with complex coefficients. If $Z = (z_{jk})_{1 \leq j \leq p; 1 \leq k \leq q}$ is an element of $M_{p,q}(\mathbb{C})$, we set

$$\|Z\|_* := \left\{ \sum_{j=1}^p \left(\sum_{k=1}^q |z_{jk}|^2 + \left| \sum_{k=1}^q z_{jk}^2 \right| \right) \right\}^{1/2}. \quad (3)$$

Let d be a positive integer and let $m = (m_1, \dots, m_d)$ and $n = (n_1, \dots, n_d)$ be multi-indices with $m_j \geq 1$ and $n_j \geq 1$ for all

$j = 1, \dots, d$. We denote by j_0 the number of those n_j 's that are equal to 1. If $j_0 \geq 1$, we may assume without loss of generality that $n_1 = \dots = n_{j_0} = 1$ and $n_j \geq 2$ for $j = j_0 + 1, \dots, d$. Let $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ with $a_j \geq 1$, and consider the function $\rho_{a,m,n}(Z)$ defined for $Z = (Z(1), \dots, Z(d)) \in M_{m_1, n_1}(\mathbb{C}) \times \dots \times M_{m_d, n_d}(\mathbb{C})$ by

$$\rho_{a,m,n}(Z) := \sum_{j=1}^d \|Z(j)\|_*^{2a_j}, \quad (4)$$

and define the *generalized minimal balls* $\Omega_{a,m,n} := \{Z : \rho_{a,m,n}(Z) < 1\}$, where

$$Z(j) := \begin{pmatrix} Z_1(j) \\ \vdots \\ Z_{m_j}(j) \end{pmatrix}, \quad (5)$$

$$Z_l(j) := (Z_{l,1}(j), \dots, Z_{l,n_j}(j)) \in \mathbb{C}^{n_j}.$$

Note that

$$\begin{aligned} \|Z(j)\|_* &= \sqrt{\sum_{l=1}^{m_j} \left(\sum_{r=1}^{n_j} |Z_{l,r}(j)|^2 + \left| \sum_{r=1}^{n_j} Z_{l,r}(j)^2 \right| \right)} \\ &= \sqrt{\sum_{l=1}^{m_j} N(Z_l(j))^2}, \end{aligned} \quad (6)$$

where $N(\cdot) := \sqrt{2}N_*(\cdot)$.

Remark 1. (i) Note that, for $n = (1, \dots, 1)$, the domain $\Omega_{a,m,n}$ is a complex ellipsoid, which is a smoothly bounded Reinhardt domain and clearly satisfies the condition **Q** [8].

(ii) Note also that, for $d = 1$ and $a = m = (1)$, the domain $\Omega_{a,m,n}$ is the minimal ball with radius $1/\sqrt{2}$, which is not Reinhardt but satisfies the condition **Q** [16]. If at least one of n_j 's is greater than 1, then $\Omega_{a,m,n}$ is neither smooth nor Reinhardt.

In 2002, Youssfi [17] computed the Bergman kernel for the generalized minimal ball $\Omega_{a,m,n}$ using the transformation formula of the Bergman kernel under proper holomorphic mappings [18–20]. See the Bergman kernel for $\Omega_{a,m,n}$ in Theorem 4. The Bergman kernel for $\Omega_{a,m,n}$ contains an infinite series $\mathcal{H}_{a,m,n}(x)$ defined by

$$\begin{aligned} &\sum_{k_1=0}^{\infty} \dots \sum_{k_d=0}^{\infty} \left(\left(\Gamma \left(1 + \sum_{j=1}^{j_0} \frac{k_j + m_j}{a_j} \right. \right. \right. \\ &\quad \left. \left. \left. + \sum_{j=j_0+1}^d \frac{k_j + m_j n_j - m_j}{a_j} \right) \right) \right. \\ &\quad \left. \cdot \left(\prod_{j=1}^{j_0} \Gamma \left(\frac{k_j + m_j}{a_j} \right) \right) \right) \end{aligned}$$

$$\begin{aligned} &\cdot \prod_{j=j_0+1}^d \Gamma \left(\frac{k_j + m_j n_j - m_j}{a_j} \right) \Big)^{-1} \\ &\cdot x_1^{k_1} \dots x_d^{k_d}. \end{aligned} \quad (7)$$

It is the crucial part to investigate the infinite series $\mathcal{H}_{a,m,n}(x)$, when we study the Bergman kernel for $\Omega_{a,m,n}$.

In [16], it was proved that the minimal ball satisfies the condition **Q**. In this paper, we generalize this result to the generalized minimal balls $\Omega_{a,m,n}$ as follows.

Theorem 2. *The condition **Q** holds on each domain*

$$\Omega_{a,m,n} = \{Z : \rho_{a,m,n}(Z) < 1\}, \quad (8)$$

where $\rho_{a,m,n}(Z)$ is defined as in (4) and all a_j 's are positive integers.

Similarly as in [16], we obtain the properties of the proper holomorphic mappings on $\Omega_{a,m,n}$.

Corollary 3. *Let $D \subset \mathbb{C}^{m_1 n_1 + \dots + m_d n_d}$ be any bounded circular domain which contains the origin.*

(i) *If $f : \Omega_{a,m,n} \rightarrow D$ is a proper holomorphic mapping, then f extends holomorphically to a neighborhood of $\overline{\Omega_{a,m,n}}$.*

(ii) *If D is smooth then there is no proper holomorphic mapping from $\Omega_{a,m,n}$ into D .*

In Section 2, we review the notation and the Bergman kernel for $\Omega_{a,m,n}$ following [17]. In Section 3, we prove the theorem using the characterization theorem (Lemma 6). In the final section, the explicit formulas of $\mathcal{H}_{a,m,n}(x)$ have been obtained in some cases and we show that the minimal ball satisfies the condition **Q** using Theorem 2.

2. Bergman Kernel Function for $\Omega_{a,m,n}$

In this section we review the Bergman kernel for $\Omega_{a,m,n}$ following the notations which were introduced in [17]. For $u = (u_{j,l})_{l=1, \dots, m_j; j=1, \dots, d}$ with $u_{j,l} \in \mathbb{C}$, we define

$$tu := (t_{j,l} u_{j,l})_{l=1, \dots, m_j; j=1, \dots, d}, \quad \text{set } t^2 = tt. \quad (9)$$

Consider the differential operator $\mathcal{D} = \mathcal{D}(a, m, n)$ acting on functions $f(t)$ for $t = (t_{j,l})$ by

$$\begin{aligned} &(\mathcal{D}f)(t) \\ &:= a_1 \dots a_d (\mathcal{D}_{1,1} \dots \mathcal{D}_{1,m_1}) \dots (\mathcal{D}_{d,1} \dots \mathcal{D}_{d,m_d})(f(t)), \end{aligned} \quad (10)$$

where

$$(\mathcal{D}_{j,l}f)(t) := \begin{cases} 4f(t) & \text{if } j \leq j_0, \\ \frac{2}{(n_j - 1)!} \left(2t_{j,l} \frac{df}{dt_{j,l}} + (n_j - 1)f(t) \right) & \text{if } j > j_0. \end{cases} \quad (11)$$

We also consider the differential operator $\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}(a, m, n)$ acting on functions $g(x)$ for $x = (x_1, \dots, x_d)$ defined by

$$(\widetilde{\mathcal{D}}g)(x) := \widetilde{\mathcal{D}}_1 \cdots \widetilde{\mathcal{D}}_d \left(\prod_{j=1}^{j_0} x_j^{m_j-1} \prod_{j=j_0+1}^d x_j^{m_j n_j - m_j - 1} g(x) \right), \quad (12)$$

where

$$\widetilde{\mathcal{D}}_j := \begin{cases} \frac{d^{m_j-1}}{dx_j^{m_j-1}} & \text{if } j \leq j_0, \\ \frac{d^{m_j n_j - m_j - 1}}{dx_j^{m_j n_j - m_j - 1}} & \text{if } j > j_0. \end{cases} \quad (13)$$

Note that j_0 depends on n . Since $n_j \geq 2$ for $j > j_0$, then we have $m_j n_j - m_j - 1 \geq m_j - 1 \geq 0$.

We also consider the product group

$$\Lambda := \{-1, 1\}^{m_{j_0+1}} \times \cdots \times \{-1, 1\}^{m_d} \quad (14)$$

and extend each element $\varepsilon = (\varepsilon_{j,l})$, $l = 1, \dots, m_j$, $j = j_0 + 1, \dots, d$, of Λ to an element $\tilde{\varepsilon}$ of $\{-1, 1\}^m$ by

$$\tilde{\varepsilon}_{j,l} := \begin{cases} 1 & \text{if } j \leq j_0, \\ \varepsilon_{j,l} & \text{if } j > j_0. \end{cases} \quad (15)$$

If $f(t)$ is a smooth function defined on the region of $t = (t_{j,l})$ with $\sum_{j=1}^d \sum_{l=1}^{m_j} |t_{j,l}| < 1$, we consider the function $\tilde{f}$ on the region of (t, u) with $\sum_{j=1}^d \sum_{l=1}^{m_j} (|t_{j,l}| + |u_{j,l}|) < 1$ in $\mathbb{C}^m \times \mathbb{C}^m$ by

$$\tilde{f}(t, u) := \sum_{\varepsilon \in \Lambda} \frac{f(t + \tilde{\varepsilon}u)}{\prod_{j=j_0+1}^d \prod_{l=1}^{m_j} \varepsilon_{j,l} u_{j,l}}, \quad (16)$$

where $\mathbb{C}^m = \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_d}$. Then define $\mathcal{L}f(t, u) := \tilde{f}(t, \sqrt{u})$. For example, we consider $d = 2$ and $j_0 = 1$. Then,

$$\begin{aligned} \tilde{f}(t, u) &= \sum_{\varepsilon \in \Lambda} \frac{f(t_{1,1} + u_{1,1}, t_{2,1} + \varepsilon_{2,1} u_{2,1})}{\varepsilon_{2,1} u_{2,1}} \\ &= \frac{f(t_{1,1} + u_{1,1}, t_{2,1} + u_{2,1})}{u_{2,1}} \\ &\quad - \frac{f(t_{1,1} + u_{1,1}, t_{2,1} - u_{2,1})}{u_{2,1}}. \end{aligned} \quad (17)$$

Consider the series defined for $x = (x_1, \dots, x_d) \in \mathbb{C}^d$ by

$$G(x) := \widetilde{\mathcal{D}}(\mathcal{H}_{a,m,n})(x), \quad (18)$$

where $\mathcal{H}_{a,m,n}(x)$ is defined in (7). The series $G(x)$ converges for $|x| < 1$. Finally, we set

$$R(t) := G\left(\sum_{l=1}^{m_1} t_{1,l}, \dots, \sum_{l=1}^{m_d} t_{d,l}\right), \quad (19)$$

$$H(t, u) := (\mathcal{L}(\mathcal{D}R))(t, u).$$

We extend the operation $\bullet$ to a $\mathbb{C}^m$ -valued bilinear mapping defined for $Z, W \in M_{m_1, n_1}(\mathbb{C}) \times \cdots \times M_{m_d, n_d}(\mathbb{C})$ by

$$Z \bullet W := (Z_l(j) \bullet W_l(j))_{l=1, \dots, m_j; j=1, \dots, d} \quad (20)$$

if $Z = (Z(1), \dots, Z(d))$, $W = (W(1), \dots, W(d))$, and

$$Z(j) = \begin{pmatrix} Z_1(j) \\ \vdots \\ Z_{m_j}(j) \end{pmatrix}, \quad W(j) = \begin{pmatrix} W_1(j) \\ \vdots \\ W_{m_j}(j) \end{pmatrix}. \quad (21)$$

In 2002, Youssfi [17] computed the Bergman kernel for $\Omega_{a,m,n}$ as follows.

Theorem 4 ([17], Theorem A). *The Bergman kernel function for the domain $\Omega = \Omega_{a,m,n}$ is given by the formula*

$$K_\Omega(Z, W) = \frac{1}{v(\Omega)} \cdot \frac{H(t, u)}{H(0, 0)}, \quad \text{for } Z, W \in \Omega, \quad (22)$$

where $t = Z \bullet \bar{W}$ and $u = (Z \bullet Z)(\overline{W \bullet W})$. The latter product is understood in the sense of (9). More precisely, one has $t = (t_{j,l})_{l=1, \dots, m_j; j=1, \dots, d}$ and $u = (u_{j,l})_{l=1, \dots, m_j; j=1, \dots, d}$, where

$$\begin{aligned} t_{j,l} &= Z_l(j) \bullet \bar{W}_l(j) = \sum_{r=1}^{n_j} Z_{l,r}(j) \bar{W}_{l,r}(j), \\ u_{j,l} &= (Z_l(j) \bullet Z_l(j)) (\overline{W_l(j) \bullet W_l(j)}) \\ &= \sum_{r=1}^{n_j} Z_{l,r}(j)^2 \sum_{r=1}^{n_j} \overline{W_{l,r}(j)}^2. \end{aligned} \quad (23)$$

Remark 5. In fact, the weighted Bergman kernel and Szegő kernel have been obtained in [17].

3. Proof of Theorem 2

For the study of $\mathcal{H}_{a,m,n}(x)$, we need to introduce Appell's hypergeometric functions. Recall that Appell's hypergeometric function [21] is defined by

$$\begin{aligned} F_A^{(n)}(\alpha; \beta_1, \dots, \beta_n; \gamma_1, \dots, \gamma_n; \zeta_1, \dots, \zeta_n) \\ = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \frac{(\alpha)_{k_1+\dots+k_n} (\beta_1)_{k_1} \cdots (\beta_n)_{k_n}}{k_1! \cdots k_n! (\gamma_1)_{k_1} \cdots (\gamma_n)_{k_n}} \zeta_1^{k_1} \cdots \zeta_n^{k_n}, \end{aligned} \quad (24)$$

where $(a)_m = \Gamma(a+m)/\Gamma(a)$.

Now we prove the main theorem of this paper. At first we need to represent $\mathcal{H}_{a,m,n}(x)$ in terms of Appell hypergeometric functions $F_A^{(d)}$ when a_j 's are positive integers. If we write

$$h_j = \begin{cases} m_j, & j \leq j_0; \\ m_j(n_j - 1), & j > j_0, \end{cases} \quad (25)$$

then we have

$$\begin{aligned} \mathcal{H}_{a,m,n}(x) &= \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \frac{\Gamma(1 + \sum_{j=1}^d ((k_j + h_j)/a_j))}{\prod_{j=1}^d \Gamma((k_j + h_j)/a_j)} x^k \\ &= \prod_{j=1}^d x_j^{1-h_j} \sum_{k_1=h_1-1}^{\infty} \cdots \sum_{k_d=h_d-1}^{\infty} \frac{\Gamma(1 + \sum_{j=1}^d ((k_j + 1)/a_j))}{\prod_{j=1}^d \Gamma((k_j + 1)/a_j)} x^k \\ &= \prod_{j=1}^d x_j^{1-h_j} (I(a; x) + P_0(x)), \end{aligned} \quad (26)$$

where $P_0(x) = P_0(x_1, \dots, x_d)$ is a polynomial such that there exists a j with $1 \leq j \leq d$ satisfying that the degree in x_j is strictly less than $h_j - 1$ and

$$I(a; x) := \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \frac{\Gamma(1 + \sum_{j=1}^d ((k_j + 1)/a_j))}{\prod_{j=1}^d \Gamma((k_j + 1)/a_j)} x^k. \quad (27)$$

Thus we have

$$\begin{aligned} G(x) &= \mathcal{D}_1 \cdots \mathcal{D}_d \left(\prod_{j=1}^d x_j^{h_j-1} \mathcal{H}_{a,m,n}(x) \right) \\ &= \frac{\partial^{h_1-1}}{\partial x_1^{h_1-1}} \cdots \frac{\partial^{h_d-1}}{\partial x_d^{h_d-1}} I(a; x). \end{aligned} \quad (28)$$

If a_j 's are positive integers, then there are positive integers p_j and q_j such that $k_j = a_j p_j + q_j$ and $0 \leq q_j \leq a_j - 1$. It follows that

$$\begin{aligned} I(a; x) &= \sum_{q_1=0}^{a_1-1} \cdots \sum_{q_d=0}^{a_d-1} \sum_{p_1=0}^{\infty} \cdots \sum_{p_d=0}^{\infty} \left(\Gamma \left(1 + \sum_{j=1}^d p_j + \sum_{j=1}^d \frac{q_j + 1}{a_j} \right) \right. \\ &\quad \left. \cdot \left(\prod_{j=1}^d \Gamma \left(p_j + \frac{q_j + 1}{a_j} \right) \right)^{-1} \right) \end{aligned}$$

$$\begin{aligned} &\cdot \prod_{j=1}^d x_j^{a_j p_j + q_j} \\ &= \sum_{q_1=0}^{a_1-1} \cdots \sum_{q_d=0}^{a_d-1} \prod_{j=1}^d c(p_j, q_j, a_j) x_j^{q_j} \\ &\quad \times F_A^{(d)} \left(1 + \sum_{j=1}^d p_j + \sum_{j=1}^d \frac{q_j + 1}{a_j}; \right. \\ &\quad \left. \mathbf{1}; \frac{\mathbf{q} + \mathbf{1}}{\mathbf{a}}; \mathbf{x}^{\mathbf{a}} \right), \end{aligned} \quad (29)$$

where

$$c(p_j, q_j, a_j) := \frac{\Gamma(1 + \sum_{j=1}^d p_j + \sum_{j=1}^d ((q_j + 1)/a_j))}{\prod_{j=1}^d \Gamma((q_j + 1)/a_j)}. \quad (30)$$

Here we used the notation $\mathbf{1} = (\underbrace{1, \dots, 1}_d)$ and $(\mathbf{q} + \mathbf{1})/\mathbf{a} = ((q_1 + 1)/a_1, \dots, (q_d + 1)/a_d)$.

Note that Appell's hypergeometric functions $F_A^{(n)}(\alpha; \beta; \gamma; x)$ converge for $|x| < 1$ and diverge for $|x| > 1$. Thus there exist a nonnegative constant C_j and an analytic function $P_1(x)$ such that

$$I(a; x) = \frac{P_1(x)}{\prod_{j \in \{1, \dots, d\}} (1 - \sum_{j \in J} x_j^{a_j})^{C_j}}. \quad (31)$$

From (28), there exist a nonnegative constant C_j^* and an analytic function $P_2(x)$ such that

$$G(x) = \frac{P_2(x)}{\prod_{j \in \{1, \dots, d\}} (1 - \sum_{j \in J} x_j^{a_j})^{C_j^*}}. \quad (32)$$

From (19), there exists an analytic function $P_3(x)$ such that

$$\mathcal{D}R(t) = \prod_{j=1}^d \left(a_j \prod_{l=1}^{m_j} \mathcal{D}_{j,l} \right) R(t) = \frac{P_3(t)}{P_4(t)}, \quad (33)$$

where

$$P_4(t) := \prod_{j \in \{1, \dots, d\}} \left(1 - \sum_{j \in J} \left(\sum_{l=1}^{m_j} t_{j,l} \right)^{a_j} \right)^{C_j^*}. \quad (34)$$

The following lemma is useful when we show that a bounded domain satisfies the condition **Q**.

Lemma 6 ([22, 23]). *For a bounded domain Ω , the condition **Q** holds on Ω if and only if, for each compact subset L of Ω , there exists an open set U_L containing $\bar{\Omega}$ such that*

- (i) *for each fixed $w_0 \in LK(z, w_0)$ is holomorphic in z on U_L ,*
- (ii) *$K(z, w)$ is continuous on $U_L \times L$.*

The above property for smoothly bounded domains was proved by Chen [22] and duplicated for general bounded domains by Thomas [23].

At first we need to study the singular points of $\tilde{f}(t, u)$ defined as in (16).

Lemma 7. *Let $f(t)$ be a smooth function on a region of $t = (t_{j,l})$ with $\sum_{j=1}^d \sum_{l=1}^{m_j} |t_{j,l}| < 1$. Denote*

$$P_f := \{t = (t_{j,l}) \in \mathbb{C}^m : f(t) \text{ has a pole at } t\}. \quad (35)$$

If $\tilde{f}(t, u)$ is defined as in (16), then there exists a function $F(t, u)$ such that

$$(i) P_F := \{(t, u) \in \mathbb{C}^m \times \mathbb{C}^m : F(t, u) \text{ has a pole at } (t, u)\} \text{ equal to } \{(t + \varepsilon u) \in P_f : \varepsilon \in \Lambda\},$$

$$(ii) \tilde{f}(t, u) = F(t, u) \times 2^{\sum_{j=j_0+1}^d m_j} \text{ for } (t, u) \notin P_F.$$

Proof. If we define $a_{j,l}^+ = t_{j,l} + u_{j,l}$ and $a_{j,l}^- = t_{j,l} - u_{j,l}$, then $t + \varepsilon u$ is equal to one of

$$\left(\begin{pmatrix} a_{1,1}^+ \\ \vdots \\ a_{1,m_1}^+ \end{pmatrix}, \dots, \begin{pmatrix} a_{j_0,1}^+ \\ \vdots \\ a_{j_0,m_{j_0}}^+ \end{pmatrix}, \begin{pmatrix} a_{j_0+1,1}^+ \\ \vdots \\ a_{j_0+1,m_{j_0+1}}^+ \end{pmatrix}, \dots, \begin{pmatrix} a_{d,1}^+ \\ \vdots \\ a_{d,m_d}^+ \end{pmatrix} \right), \quad (36)$$

where $a_{j,l}^\pm$ is one of either $a_{j,l}^+$ or $a_{j,l}^-$.

Let N_0 be the number of $a_{j,l}^-$'s. For fixed j and l with $j_0 + 1 \leq j \leq d$, $1 \leq l \leq m_j$, we have

$$\begin{aligned} \tilde{f}(t, u) &= \frac{1}{\prod_{j=j_0+1}^d \prod_{l=1}^{m_j} u_{j,l}} \sum_{\varepsilon \in \Lambda} (-1)^{N_0} f(t + \varepsilon u) \\ &= \frac{1}{\prod_{j=j_0+1}^d \prod_{l=1}^{m_j} u_{j,l}} \\ &\quad \cdot \left(\sum_{a_{j,l}^+} (-1)^{N_0} f(t + \varepsilon u) + \sum_{a_{j,l}^-} (-1)^{N_0+1} f(t + \varepsilon u) \right) \\ &= \frac{1}{\prod_{j=j_0+1}^d \prod_{l=1}^{m_j} u_{j,l}} \\ &\quad \cdot \left(\sum_{a_{j,l}^+} (-1)^{N_0} f(t + \varepsilon u) - \sum_{a_{j,l}^-} (-1)^{N_0} f(t + \varepsilon u) \right), \end{aligned} \quad (37)$$

where $\sum_{a_{j,l}^+} (\sum_{a_{j,l}^-}, \text{ resp.})$ means summation running over $\varepsilon \in \Lambda$ which contains $a_{j,l}^+ (\sum_{a_{j,l}^-}, \text{ resp.})$.

From the above formula, we see that a numerator has a factor $a_{j,l}^+ - a_{j,l}^-$. Repeating this process for all $j_0 + 1 \leq j \leq d$

and $1 \leq l \leq m_j$, we obtain that $\tilde{f}(t, u)$ is equal to certain function $F(t, u)$ times

$$\begin{aligned} \prod_{j=j_0+1}^d \prod_{l=1}^{m_j} (a_{j,l}^+ - a_{j,l}^-) &= \prod_{j=j_0+1}^d \prod_{l=1}^{m_j} 2u_{j,l} \\ &= 2^{\sum_{j=j_0+1}^d m_j} \prod_{j=j_0+1}^d \prod_{l=1}^{m_j} u_{j,l}, \end{aligned} \quad (38)$$

which means that F satisfies condition (ii). $\square$

Let $t_{j,l}$ and $u_{j,l}$ be defined as in (23). By Lemma 7 the Bergman kernel of $\Omega_{a,m,n}$ has poles at $(t, u) \in \mathbb{C}^m \times \mathbb{C}^m$ with

$$\sum_{j \in J} \sum_{l=1}^{m_j} (t_{j,l} \pm u_{j,l}^{1/2})^{a_j} = 1 \quad (39)$$

for some $J \subset \{1, \dots, d\}$. Write $t_{j,l}$ and $u_{j,l}$ as

$$t_{j,l} = \sum_{r=1}^{n_j} a_r \bar{b}_r, \quad u_{j,l} = \sum_{r=1}^{n_j} a_r^2 \sum_{r=1}^{n_j} \bar{b}_r^2, \quad (40)$$

where $a_r = Z_{l,r}(j)$ and $b_r = W_{l,r}(j)$. Then it follows that

$$N(Z_l(j)) = \sqrt{\sum_{r=1}^{n_j} |a_r|^2 + \sum_{r=1}^{n_j} a_r^2}, \quad (41)$$

$$N(W_l(j)) = \sqrt{\sum_{r=1}^{n_j} |b_r|^2 + \sum_{r=1}^{n_j} b_r^2}.$$

The following lemma is needed when we estimate the function $P_4(x)$.

Lemma 8. *For $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ with $a_r, b_r \in \mathbb{C}$, let*

$$t = \sum_{r=1}^n a_r \bar{b}_r, \quad u = \sum_{r=1}^n a_r^2 \cdot \sum_{r=1}^n \bar{b}_r^2. \quad (42)$$

Then one has

$$|t \pm \sqrt{u}| \leq \sqrt{(\sum |a_r|^2 + \sum a_r^2)(\sum |b_r|^2 + \sum b_r^2)}. \quad (43)$$

Proof. By the Schwartz inequality, we have

$$\begin{aligned} |t \pm \sqrt{u}| &\leq \sum |a_r b_r| + \sqrt{\sum a_r^2 \cdot \sum \bar{b}_r^2} \\ &\leq \sqrt{(\sum |a_r|^2)(\sum |b_r|^2)} + \sqrt{|\sum a_r^2 \cdot \sum \bar{b}_r^2|} \\ &\leq \sqrt{(\sum |a_r|^2 + \sum a_r^2)(\sum |b_r|^2 + \sum b_r^2)}. \end{aligned} \quad (44)$$

$\square$

Then, by using Lemma 8, the identity (41), and the inequality $(xy)^{(1/2)a} \leq (1/2)(x^a + y^a)$ for $x, y > 0$, we have

$$\begin{aligned}
 & \left| 1 - \sum_{j \in J} \sum_{l=1}^{m_j} (t_{j,l} \pm u_{j,l}^{1/2})^{a_j} \right| \\
 & \geq 1 - \sum_{j \in J} \sum_{l=1}^{m_j} |t_{j,l} \pm u_{j,l}^{1/2}|^{a_j} \\
 & \geq 1 - \sum_{j \in J} \sum_{l=1}^{m_j} \{N(Z_l(j))^2 \cdot N(W_l(j))^2\}^{(1/2)a_j} \\
 & \geq 1 - \frac{1}{2} \sum_{j \in J} \sum_{l=1}^{m_j} N(Z_l(j))^{2a_j} + N(W_l(j))^{2a_j} \\
 & = 1 - \frac{1}{2} \sum_{j \in J} \left(\sum_{l=1}^{m_j} N(Z_l(j))^{2a_j} + \sum_{l=1}^{m_j} N(W_l(j))^{2a_j} \right) \\
 & \geq 1 - \frac{1}{2} \sum_{j \in J} \left\{ \left(\sum_{l=1}^{m_j} N(Z_l(j))^2 \right)^{a_j} + \left(\sum_{l=1}^{m_j} N(W_l(j))^2 \right)^{a_j} \right\} \\
 & \geq 1 - \frac{1}{2} \{\rho(Z) + \rho(W)\}.
 \end{aligned} \tag{45}$$

Thus by Lemma 6, each domain $\Omega_{a,m,n}$ satisfies the condition Q.

4. Examples

In fact, it is difficult to write $\mathcal{H}_{a,m,n}(x)$ explicitly. In the final section, we give some explicit formulas for $\mathcal{H}_{a,m,n}(x)$.

Lemma 9. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_j \in \mathbb{N} \cup \{0\}$ and $t = (t_1, \dots, t_n) \in \mathbb{C}^n$. For any $s > 0$ and $t \in \mathbb{C}^n$ with $\sum_{j=1}^n |t_j| < 1$, one has

$$\sum_{\alpha} \frac{\Gamma(\sum_{j=1}^n \alpha_j + s)}{\Gamma(s) \Gamma(\alpha_j + 1)} \prod_{j=1}^n t_j^{\alpha_j} = \frac{1}{(1 - \sum_{j=1}^n t_j)^s}. \tag{46}$$

Proof. Consider the Taylor expansion of the function in the right hand side of (46). See the details in [11] or [13]. $\square$

Lemma 10. Let $a(k)$ be a polynomial of degree r with variable k . Then there exist unique constants c_l for $0 \leq l \leq r$ such that

$$\sum_{k=0}^{\infty} a(k) x^k = \sum_{l=0}^r \frac{c_l}{(1-x)^{l+1}}. \tag{47}$$

Proof. For a given polynomial $a(k)$, there exist unique constants b_l for $1 \leq l \leq r+1$ such that

$$\begin{aligned}
 a(k) &= b_0 + b_1(k+1) + b_2(k+1)(k+2) \\
 &\quad + \dots + b_r(k+1) \dots (k+r).
 \end{aligned} \tag{48}$$

Note that, for $l \geq 1$,

$$\begin{aligned}
 \frac{1}{(1-x)^{l+1}} &= \frac{1}{l!} \frac{d^l}{dx^l} \left(\frac{1}{1-x} \right) \\
 &= \frac{1}{l!} \frac{d^l}{dx^l} \left(\sum_{k=0}^{\infty} x^k \right) \\
 &= \frac{1}{l!} \sum_{k=0}^{\infty} (k+1) \dots (k+l) x^k.
 \end{aligned} \tag{49}$$

If we write $c_l = b_l/l!$, then we obtain (47). $\square$

Example 11. The complex d -dimensional ball with radius $1/\sqrt{2}$: if $d \geq 1$ and $a = m = n = (1, \dots, 1)$, then using Lemma 9, we have

$$\mathcal{H}_{a,m,n}(x) = \sum_{h \in \mathbb{N}_0^d} \frac{1 + \sum_{j=1}^d (h_j + 1)}{\prod_{j=1}^d \Gamma(h_j + 1)} x^h = \frac{\Gamma(d+1)}{(1 - \sum_{j=1}^d x_j)^{d+1}}. \tag{50}$$

Example 12. The Thullen domain $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2p} < 1\}$ with a positive integer p : if $d = 2$, $a = (1, p)$, and $n = m = (1, 1)$, then

$$\Omega_{a,m,n} = \{(z_1, z_2) \in \mathbb{C}^2 : 2|z_1|^2 + 2^p|z_2|^{2p} < 1\} \tag{51}$$

is linearly biholomorphic to the Thullen domain $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2p} < 1\}$. In this case, we have

$$\mathcal{H}_{a,m,n}(x) = \sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \frac{\Gamma(2 + h_1 + (h_2 + 1)/p)}{\Gamma(h_1 + 1) \Gamma((h_2 + 1)/p)} x_1^{h_1} x_2^{h_2}. \tag{52}$$

If we apply Lemma 9 to a variable x_1 , then we obtain

$$\begin{aligned}
 & \mathcal{H}_{a,m,n}(x) \\
 &= \sum_{h_2=0}^{\infty} \frac{\Gamma(2 + (h_2 + 1)/p)}{\Gamma((h_2 + 1)/p)} \frac{x_2^{h_2}}{(1 - x_1)^{2+(h_2+1)/p}} \\
 &= \sum_{h_2=0}^{\infty} \left(\frac{h_2 + 1}{p} \right) \left(1 + \frac{h_2 + 1}{p} \right) \left(\frac{x_2}{(1 - x_1)^{1/p}} \right)^{h_2} \\
 &\quad \cdot \frac{1}{(1 - x_1)^{2+1/p}} \\
 &= \frac{1}{p^2} \sum_{h_2=0}^{\infty} (h_2 + 1)(h_2 + 1 + p) \left(\frac{x_2}{(1 - x_1)^{1/p}} \right)^{h_2} \\
 &\quad \cdot \frac{1}{(1 - x_1)^{2+1/p}}.
 \end{aligned} \tag{53}$$

By Lemma 10, there exist constants b_l such that

$$\begin{aligned}\mathcal{H}_{a,m,n}(x) &= \frac{1}{p^2} \sum_{h_2=0}^{\infty} \sum_{l=0}^2 \frac{c_l}{((1-x_1)^{1/p} - x_2)^{l+1} (1-x_1)^{2-l/p}} \\ &= \frac{1}{p^2} \sum_{h_2=0}^{\infty} \sum_{l=0}^2 \frac{c_l \cdot g_p((1-x_1)^{1/p}, x_2)}{((1-x_1) - x_2^p)^{l+1} (1-x_1)^{2-l/p}},\end{aligned}\quad (54)$$

where $g_p(x_1, x_2) = (x_1^p - x_2^p)/(x_1 - x_2)$ is a polynomial for a positive integer p . Note that $2 - l/p \geq 2 - 2/p > 0$.

Example 13. A complex ellipsoid $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^4 + |z_2|^4 < 1\}$: if $d = 2$, $a = (2, 2)$, and $m = n = (1, 1)$, then the following formula was proved in [13]:

$$\begin{aligned}\mathcal{H}_{a,m,n}(x) &= \frac{x(\pi + 2 \arcsin x)f(x^2, y^2)}{4\pi(1-x^2)^{3/2}(1-x^2-y^2)^3} \\ &\quad + \frac{y(\pi + 2 \arcsin y)f(y^2, x^2)}{4\pi(1-y^2)^{3/2}(1-x^2-y^2)^3} \\ &\quad + \frac{2xy}{(1-x^2-y^2)^3} \\ &\quad + \frac{g(x^2, y^2)}{2\pi(1-x^2)(1-y^2)(1-x^2-y^2)^2},\end{aligned}\quad (55)$$

where $f(a, b) = 3 - 6a + 6b + 3a^2 - 6ab - b^2$ and $g(a, b) = 2 - a - b - a^2 + 2ab - b^2$.

Example 14. The minimal ball with radius $1/\sqrt{2}$: if $d = 1$, $a = m = (1)$, and $n \geq 2$, the infinite series $\mathcal{H}_{a,m,n}(x)$ is equal to

$$\begin{aligned}\sum_{h=0}^{\infty} \frac{\Gamma(h+n)}{\Gamma(h+n-1)} x^h &= \sum_{h=0}^{\infty} (h+n-1)x^h \\ &= \sum_{h=0}^{\infty} (h+1)x^h + (n-2) \sum_{h=0}^{\infty} x^h \\ &= \frac{(n-1) - (n-2)x}{(1-x)^2} \\ &= x^{2-n} \left\{ \frac{(n-1)x^{n-2} - (n-2)x^{n-1}}{(1-x)^2} \right\}.\end{aligned}\quad (56)$$

Since $n \geq 2$, we have $(n-1)x^{n-2} - (n-2)x^{n-1}$ which is analytic in x and this shows that the minimal ball satisfies the condition **Q**.

Example 15. Let $d = 2$, $m = (m_1, m_2)$, and $n = (1, n_2)$ with $n_2 \geq 2$ and $a = (1, 1)$. Then the infinite series $\mathcal{H}_{a,m,n}(x)$ is equal to

$$\begin{aligned}&\sum_h \frac{\Gamma(1+h_1+m_1+h_2+m_2n_2-m_2)}{\Gamma(h_1+m_1)\Gamma(h_2+m_2n_2-m_2)} x^h \\ &= \sum_{h_1=0}^{\infty} \frac{x_1^{h_1}}{\Gamma(h_1+m_1)} \\ &\quad \cdot \sum_{h_2=0}^{\infty} \frac{\Gamma(h_2+1+h_1+m_1+m_2n_2-m_2)}{\Gamma(h_2+m_2n_2-m_2)} x_2^{h_2} \\ &= \sum_{h_1=0}^{\infty} \frac{x_1^{h_1}}{\Gamma(h_1+m_1)} x_2^{1-m_2n_2+m_2} \\ &\quad \cdot \sum_{k_2=m_2n_2-m_2-1}^{\infty} \frac{\Gamma(k_2+2+h_1+m_1)}{\Gamma(k_2+1)} x_2^{k_2} \\ &= x_2^{1-m_2n_2+m_2} \sum_{k_2=m_2n_2-m_2-1}^{\infty} \frac{x_2^{k_2}}{\Gamma(k_2+1)} \\ &\quad \cdot \sum_{h_1=0}^{\infty} \frac{\Gamma(h_1+k_2+2+m_1)}{\Gamma(h_1+m_1)} x_1^{h_1} \\ &= x_1^{1-m_1} x_2^{1-m_2n_2+m_2} \\ &\quad \cdot \sum_{k_1=m_1-1}^{\infty} \sum_{k_2=m_2n_2-m_2-1}^{\infty} \frac{\Gamma(k_1+k_2+3)}{\Gamma(k_1+1)\Gamma(k_2+1)} x_1^{k_1} x_2^{k_2}.\end{aligned}\quad (57)$$

Next, for nonnegative integers l_1 and l_2 , we need to compute the infinite series

$$\sum_{k_1=l_1}^{\infty} \sum_{k_2=l_2}^{\infty} \frac{\Gamma(k_1+k_2+3)}{\Gamma(k_1+1)\Gamma(k_2+1)} x_1^{k_1} x_2^{k_2}. \quad (58)$$

It is interesting to compare this series (58) with the left hand side in (46). We can express the infinite series (58) in terms of finite series using Lemma 9. Precisely, the series (58) is equal to

$$\begin{aligned}&\frac{2}{(1-x_1-x_2)^3} - \sum_{k_1=0}^{l_1-1} \frac{x_1^{k_1}}{\Gamma(k_1+1)} \frac{\Gamma(k_1+3)}{(1-x_2)^{k_1+3}} \\ &\quad - \sum_{k_2=0}^{l_2-1} \frac{x_2^{k_2}}{\Gamma(k_2+1)} \frac{\Gamma(k_2+3)}{(1-x_1)^{k_2+3}} \\ &\quad + \sum_{k_1=0}^{l_1-1} \sum_{k_2=0}^{l_2-1} \frac{\Gamma(k_1+k_2+3)}{\Gamma(k_1+1)\Gamma(k_2+1)} x_1^{k_1} x_2^{k_2}.\end{aligned}\quad (59)$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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