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Research Article

New Fixed Point Results for Fractal Generation in Jungck Noor Orbit with *s*-Convexity

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We establish new fixed point results in the generation of fractals (Julia sets, Mandelbrot sets, and Tricorns and Multicorns for linear or nonlinear dynamics) by using Jungck Noor iteration with *s*-convexity.

1. Introduction

The fractal geometry in mathematics has presented some attractive complex graphs and objects to computer graphics. Fractal is a Latin word, derived from the word "Fractus" which means "Broken." The term "fractal" was first used by a young mathematician, Julia [1], when he was studying Cayley's problem related to the behavior of Newton's method in complex plane. Julia introduced the concept of iterative function system (IFS) and, by using it, he derived the Julia set in 1919. After that, in 1982, Mandelbrot [2] extended the work of Gaston Julia and introduced the Mandelbrot set, a set of all connected Julia sets. The fractal structures of Mandelbrot and Julia sets have been demonstrated for quadratic, cubic, and higher degree polynomials, by using Picard orbit which is an application of one-step feedback process [3].

Julia and Mandelbrot sets have been studied under the effect of noises [1–4] arising in the objects. In 1982, Mandelbrot [2] introduced superior iterates (a two-step feedback process) in the study of fractal theory and created superior Julia and Mandelbrot sets. Rani et al. [5–7] generated and analyzed superior Julia and superior Mandelbrot sets for quadratic, cubic, and nth degree complex polynomials. After creation of superior Mandelbrot sets, Negi and Rani [5] collected the properties of midgets of quadratic superior Mandelbrot sets. Negi and Rani [6] simulated the behavior of Julia sets using switching processes. Chauhan et al. [4] obtained new Julia and Mandelbrot sets via Ishikawa iterates (an example of three-step feedback process). Kang et al. [8] introduced Julia and Mandelbrot sets in Jungck Mann and Jungck Ishikawa orbits.

In 1994, Hudzik and Maligranda [9] discussed a few results connecting with s-convex functions in second sense and some new results about Hadamard's inequality for s-convex functions are discussed in [10, 11]. In 1915, Bernstein and Doetsch [12] proved a variant of Hermite-Hadamard's inequality for s-convex functions in second sense. Takahashi [13] first introduced a notion of convex metric space, which is more general space, and each linear normed space is a special example of the space. Recently, Ojha and Mishra [14] discussed an application of fixed point theorem for s-convex function.

It is a well known fact that s-convexity and Ishikawa iteration play a vital role in the development of geometrical picturesque of fractal sets. The applications of fractal sets are in cryptography and other useful areas in our modern era. Our aim is to deal with generalization of s-convexity, approximate convexity, and results of Bernstein and Doetsch [12]. The concept of s-convexity and rational s-convexity was introduced by Breckner and Orbán [15] in 1978.

In this paper, we establish some new fixed point results in the generation of fractals (Julia sets, Mandelbrot sets, and Tricorns and Multicorns for linear or nonlinear dynamics) by using Jungck Noor iteration with s-convexity. We define the

Jungck Noor orbit and escape criterions for quadratic, cubic, and nth degree complex polynomials by using Jungck Noor iteration with s-convexity.

2. Preliminaries

Definition 1 (Mandelbrot set [2, 6]). The Mandelbrot set M for the quadratic polynomial $Q_c(z) = z^2 + c$ is defined as the collection of all $c \in C$ for which the orbit of the point 0 is bounded; that is,

$$M = \{c \in C : \{Q_c^n(0)\};\ n = 0, 1, 2, \ldots\} \tag{1}$$

is bounded. An equivalent formulation is

$$\begin{aligned} &M \\ &\quad = \{c \in C : Q_c^n(0) \text{ does not tend to } \infty \text{ as } n \longrightarrow \infty\}. \end{aligned} \tag{2}$$

We choose the initial point 0, as 0 is the only critical point of Q_c.

Definition 2 (Julia set [1]). The attractor basin of infinity is never all of C, since f_c has fixed points $z_f = 1/2 \pm \sqrt{1/4 + c}$ (and also points of period n, which satisfy a polynomial equation of degree 2^n; namely, $f^n(z) = z$). The nonempty compact boundary of the attractor basin of infinity is called the Julia set of f_c:

$$J_c = \partial A_\infty(c). \tag{3}$$

Definition 3 (filled Julia set [1, 3, 7]). The filled Julia set of the function f is denoted by $K(f)$ and is defined as

$$K(f) = \left\{z \in C : f^k(z) \longrightarrow \infty\right\}. \tag{4}$$

Definition 4 (see [1, 3, 7]). The Julia set of the function f is defined to be the boundary of filled Julia set $K(f)$. That is,

$$J(f) = \partial K(f). \tag{5}$$

Definition 5 (see [3]). Let $\{z_n : n = 1, 2, 3, 4, \ldots\}$ be a sequence of complex numbers. Then, one says $\lim_{n\to\infty} z_n = \infty$ if, for given $M > 0$, there exists $N > 0$, such that, for all $n > N$, one must have $|z_n| > M$. Thus all the values of z_n lie outside a circle of radius M, for sufficiently large values of n. Let

$$Q(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z^1 + a_n z^0; \quad a_0 \neq 0 \tag{6}$$

be a polynomial of degree n, where $n \geq 2$. The coefficients are allowed to be complex numbers. In other words, it follows that $Q_c(z) = z^2 + c$.

Definition 6 (Picard orbit [3]). Let X be a nonempty set and $f : X \to X$. For any point $x_0 \in X$, Picard's orbit is defined as the set of iterates of a point x_0; that is,

$$O(f, x_0) = \{x_n; x_n = f(x_{n-1}),\ n = 1, 2, 3, \ldots\}. \tag{7}$$

Definition 7 (Jungck Noor orbit [8]). Let us consider the sequence $\{x_n\}$ of iterates for any initial point $x_0 \in X$ such that

$$\begin{aligned} Sx_{n+1} &= (1-\alpha)^s Sx_n + \alpha^s Ty_n; \\ Sy_n &= (1-\beta)^s Sz_n + \beta^s Tu_n, \\ Su_n &= (1-\gamma)^s Sz_n + \gamma^s Tz_n, \end{aligned} \tag{8}$$

where $\alpha, \beta, \gamma, s \in (0, 1)$ for $n = 0, 1, 2, \ldots$. The above sequence of iterates is called Jungck three-step orbit or Jungck Noor orbit with s-convexity, denoted by JNO, which is a function of six tuples (T, x_0, α, β, γ, and s).

Remark 8. The JNO reduces to the following:

(1) The Jungck Ishikawa orbit when $\gamma = 0$, $s = 1$.

(2) The Jungck Mann orbit when $\beta = \gamma = 0$, $s = 1$.

(3) The Jungck orbit when $\beta = \gamma = 0$ and $s, \alpha = 1$.

In nonlinear dynamics, we have two different types of points. Points that leave the interval after a finite number are in stable set of infinity. Points that never leave the interval after any number of iterations have bounded orbits. So, an orbit is bounded if there exists a positive real number, such that the modulus of every point in the orbit is less than this number. The collection of points that are bounded (i.e., there exists M, such that $|Q^n(z)| \leq M$, for all n) is called a prisoner set, while the collection of points that are in the stable set of infinity is called the escape set. Hence, the boundary of the prisoner set is simultaneously the boundary of escape set and that is Mandelbrot set for Q.

3. Escape Criterions for the Complex Polynomials in Jungck Noor Orbit

Now we prove the escape criterions of Julia and Mandelbrot sets for quadratic, cubic, and the higher degree complex polynomials in Jungck Noor orbit with s-convexity.

3.1. Escape Criterion for the Quadratic Complex Polynomials. For quadratic complex polynomial $P_c z = z^2 - az + c$, we will choose $Tz = z^2 + c$ and $Sz = az$, where a and c are complex numbers.

Theorem 9. *Assume that* $|z| \geq |c| > 2(1 + |a|)/s\alpha$, $|z| \geq |c| > 2(1+|a|)/s\beta$, *and* $|z| \geq |c| > 2(1+|a|)/s\gamma$, *where* $0 < \alpha, \beta, \gamma, s < 1$ *and* c *is a complex parameter. Define*

$$\begin{aligned} Sz_1 &= (1-\alpha)^s Sz + \alpha^s Ty, \\ &\vdots \\ Sz_n &= (1-\alpha)^s Sz_{n-1} + \alpha^s Ty_{n-1}; \\ Sy &= (1-\beta)^s Sz + \beta^s Tu, \\ &\vdots \end{aligned}$$

$$\begin{aligned} Sy_{n-1} &= (1-\beta)^s Sz_{n-1} + \beta^s Tu_{n-1}, \\ Su &= (1-\beta)^s Sz + \beta^s Tz, \\ &\vdots \\ Su_{n-1} &= (1-\beta)^s Sz_{n-1} + \beta^s Tz_{n-1}, \end{aligned} \tag{9}$$

where Sz is injective, Tz is a quadratic polynomial, and $n = 2, 3, 4, \ldots$. Then, $|z_n| \to \infty$ as $n \to \infty$.

Proof. Let $Tz = z^2 + c$ and for $z_0 = z$, $y_0 = y$, and $u_0 = u$, we have considered that

$$Su_{n-1} = (1-\gamma)^s Sz_{n-1} + \gamma^s Tz_{n-1} \tag{10}$$

implies

$$\begin{aligned} |Su| &= \left|(1-\gamma)^s Sz + \gamma^s Tz\right| \\ &= \left|(1-\gamma)^s az + \gamma^s \left(z^2 + c\right)\right| \\ &= \left|(1-\gamma)^s az + (1-(1-\gamma))^s \left(z^2 + c\right)\right|. \end{aligned} \tag{11}$$

Using binomial's series up to linear terms of α and $(1-\alpha)$, we get that

$$\begin{aligned} |Su| &= \left|(1-s\gamma)\, az + (1-s(1-\gamma))\left(z^2+c\right)\right| \\ &\geq (1-s(1-\gamma))\left|z^2+c\right| - \left|(1-s\gamma)\, az\right| \\ &\geq (s-s(1-\gamma))\left|z^2+c\right| - \left|(1-s\gamma)\, az\right|, \\ &\qquad \text{because } s<1 \\ &\geq s\gamma\left|z^2\right| - s\gamma|c| - |az| + |s\gamma az| \\ &\geq s\gamma\left|z^2\right| - s\gamma|z| - |az| + |s\gamma az|, \\ &\qquad \text{because } |z| \geq |c| \\ &\geq s\gamma\left|z^2\right| - s\gamma|z| - |az|, \\ &\qquad \text{because } |a| \geq 0 \end{aligned} \tag{12}$$

gives us

$$\begin{aligned} |au| &\geq s\gamma\left|z^2\right| - |z| - |az|, \\ &\qquad \text{because } s\gamma < 1 \\ &\geq s\gamma\left|z^2\right| - |a|\,|z| - |z| \\ &= s\gamma\left|z^2\right| - (1+|a|)\,|z| \\ &= |z|\left(s\gamma|z| - (1+|a|)\right). \end{aligned} \tag{13}$$

Thus

$$|u| \geq |z|\left(\frac{s\gamma|z|}{1+|a|} - 1\right). \tag{14}$$

Also, we have

$$\begin{aligned} |Sy| &= \left|(1-\beta)^s Sz + \beta^s Tu\right| \\ &= \left|(1-\beta)^s az + \beta^s\left(u^2+c\right)\right| \\ &\geq s\beta\left|u^2\right| - s\beta|c| - |az| + |s\beta az|, \\ &\qquad \text{because } s<1 \\ &\geq s\beta\left|u^2\right| - s\beta|z| - |az|, \\ &\qquad \text{because } |a| \geq 0,\ |z| \geq |c|, \\ &\geq s\beta\left|u^2\right| - (1+|a|)\,|z|, \\ &\qquad \text{because } s\beta < 1. \end{aligned} \tag{15}$$

Since $|z| > 2(1+|a|)/s\gamma$, this implies $|z|^2(s\gamma|z|/(1+|a|)-1)^2 > |z|^2$. Hence $|u|^2 > |z|^2(s\gamma|z|/(1+|a|)-1)^2 > |z|^2 > s\gamma|z|^2$; this implies

$$ay \geq s^2\beta\gamma\left|z^2\right| - (1+|a|)\,|z|. \tag{16}$$

Thus

$$|y| \geq |z|\left(\frac{s^2\beta\gamma|z|}{1+|a|} - 1\right). \tag{17}$$

For $z_0 = z$ and $y_0 = y$, consider

$$\left|Sz_n\right| = \left|(1-\alpha)^s Sz_{n-1} + \alpha^s Ty_{n-1}\right|, \tag{18}$$

which implies that

$$\left|Sz_1\right| = \left|(1-\alpha)^s Sz + \alpha^s Ty\right| \tag{19}$$

yields

$$\begin{aligned} \left|az_1\right| &= \left|(1-\alpha)^s Sz + \alpha^s Ty\right| \\ &= \left|(1-\alpha)^s az + (1-(1-\alpha))^s\left(y^2+c\right)\right| \\ &\geq (1-s(1-\alpha))\left|y^2+c\right| - \left|(1-s\alpha)\, az\right| \\ &\geq s\alpha\left|y^2\right| - (1+|a|)\,|z| \\ &\geq s^3\alpha\beta\gamma\left|z^2\right| - (1+|a|)\,|z| \\ &\geq |z|\left(s^3\alpha\beta\gamma|z| - (1+|a|)\right). \end{aligned} \tag{20}$$

Hence

$$\left|z_1\right| \geq |z|\left(\frac{s^3\alpha\beta\gamma|z|}{1+|a|} - 1\right), \tag{21}$$

since $|z| \geq |c| > 2(1+|a|)/s\alpha$, $|z| \geq |c| > 2(1+|a|)/s\beta$, and $|z| \geq |c| > 2(1+|a|)/s\gamma$, so that $|z| > 2(1+|a|)/s^3\alpha\beta\gamma$. Therefore there exist $\lambda > 0$, such that $s^3\alpha\beta\gamma|z|/(1+|a|) - 1 > 1+\lambda$. Consequently $|z_1| > (1+\lambda)|z|$. In particular, $|z_n| > |z|$. So we may apply the same argument repeatedly to find $|z_n| > (1+\lambda)^n|z|$. Thus, the orbit of z tends to infinity. This completes the proof. □

Corollary 10. *Suppose that* $|c| > 2(1+|a|)/s\alpha$, $|c| > 2(1+|a|)/s\beta$, *and* $|c| > 2(1+|a|)/s\gamma$; *then the orbit of Jungck Noor JNO*$(T_c, 0, \alpha, \beta, \gamma, s)$ *escapes to infinity.*

In the proof of theorem we used the facts that $|z| \geq |c| > 2(1+|a|)/s\alpha$, $|z| \geq |c| > 2(1+|a|)/s\beta$, and $|z| \geq |c| > 2(1+|a|)/s\gamma$. Hence the following corollary is the refinement of the escape criterion discussed in the above theorem.

Corollary 11 (escape criterion). *Let*

$$|z| > \max\left\{|c|, \frac{2(1+|a|)}{s\alpha}, \frac{2(1+|a|)}{s\beta}, \frac{2(1+|a|)}{s\gamma}\right\}; \quad (22)$$

then $|z_n| > (1+\lambda)^n |z|$ *and* $|z_n| \to \infty$ *as* $n \to \infty$.

Corollary 12. *Suppose that*

$$|z_k| > \max\left\{|c|, \frac{2(1+|a|)}{s\alpha}, \frac{2(1+|a|)}{s\beta}, \frac{2(1+|a|)}{s\gamma}\right\}, \quad (23)$$

for some $k \geq 0$. *Then* $|z_{k+1}| > (1+\lambda)^n |z_k|$ *and* $|z_{k+1}| \to \infty$ *as* $n \to \infty$.

This corollary gives us an algorithm in the generation of fractals (Julia sets, Mandelbrot sets, and Tricorns and Multicorns for linear or nonlinear dynamics) for $P_c z$. Given any point $|z| \leq |c|$, we have computed the orbit "JNO" of z. If, for some n, $|z_n|$ lies outside the circle of radius

$$\max\left\{|c|, \frac{2(1+|a|)}{s\alpha}, \frac{2(1+|a|)}{s\beta}, \frac{2(1+|a|)}{s\gamma}\right\}, \quad (24)$$

we guarantee that the orbit escapes. Hence, z is not in the Julia sets and also is not in the Mandelbrot sets. On the other hand, if $|z_n|$ never exceeds this bound, then by definition of the Julia sets and the Mandelbrot sets, we can make extensive use of this algorithm in the next section.

3.2. Escape Criterion for the Cubic Complex Polynomials. For cubic complex polynomial $P_c z = z^3 - az + c$, we will choose $Tz = z^3 + c$ and $Sz = az$, where a and c are complex numbers.

Theorem 13. *Assume that* $|z| \geq |c| > (2(1+|a|)/s\alpha)^{1/2}$, $(2(1+|a|)/s\beta)^{1/2}$ *and* $|z| \geq |c| > (2(1+|a|)/s\gamma)^{1/2}$, *where* $0 < \alpha, \beta, \gamma, s < 1$ *and* c *is a complex parameter. Define*

$$\begin{aligned}
Sz_1 &= (1-\alpha)^s Sz + \alpha^s Ty, \\
&\vdots \\
Sz_n &= (1-\alpha)^s Sz_{n-1} + \alpha^s Ty_{n-1}; \\
Sy &= (1-\beta)^s Sz + \beta^s Tu, \\
&\vdots \\
Sy_{n-1} &= (1-\beta)^s Sz_{n-1} + \beta^s Tu_{n-1}, \\
Su &= (1-\beta)^s Sz + \beta^s Tz, \\
&\vdots \\
Su_{n-1} &= (1-\beta)^s Sz_{n-1} + \beta^s Tz_{n-1},
\end{aligned} \quad (25)$$

where Sz *is injective,* Tz *is a cubic polynomial, and* $n = 2, 3, 4, \ldots$. *Then,* $|z_n| \to \infty$ *as* $n \to \infty$.

Proof. Let $Tz = z^3 + c$ and for $z_0 = z$, $y_0 = y$, and $u_0 = u$, we have considered that

$$Su_{n-1} = (1-\gamma)^s Sz_{n-1} + \gamma^s Tz_{n-1} \quad (26)$$

implies

$$\begin{aligned}
|Su| &= \left|(1-\gamma)^s Sz + \gamma^s Tz\right| \\
&= \left|(1-\gamma)^s az + \gamma^s \left(z^3 + c\right)\right| \\
&= \left|(1-\gamma)^s az + (1-(1-\gamma))^s \left(z^3 + c\right)\right|.
\end{aligned} \quad (27)$$

Using binomial's series up to linear terms of α and $(1-\alpha)$, we get that

$$\begin{aligned}
|Su| &= \left|(1-s\gamma)\, az + (1-s(1-\gamma))\left(z^3 + c\right)\right| \\
&\geq (1-s(1-\gamma))\left|z^3 + c\right| - \left|(1-s\gamma)\, az\right| \\
&\geq (s - s(1-\gamma))\left|z^3 + c\right| - \left|(1-s\gamma)\, az\right|, \\
&\qquad\qquad \text{because } s < 1 \\
&\geq s\gamma\left|z^3\right| - s\gamma|c| - |az| + |s\gamma az| \\
&\geq s\gamma\left|z^3\right| - s\gamma|z| - |az| + |s\gamma az|, \\
&\qquad\qquad \text{because } |z| \geq |c| \\
&\geq s\gamma\left|z^3\right| - s\gamma|z| - |az|, \\
&\qquad\qquad \text{because } |a| \geq 0
\end{aligned} \quad (28)$$

gives us

$$\begin{aligned}
|au| &\geq s\gamma\left|z^3\right| - |z| - |az|, \\
&\qquad\qquad \text{because } s\gamma < 1 \\
&\geq s\gamma\left|z^3\right| - |a|\,|z| - |z| \\
&= s\gamma\left|z^3\right| - (1+|a|)\,|z| \\
&= |z|\left(s\gamma\left|z^2\right| - (1+|a|)\right).
\end{aligned} \quad (29)$$

Thus

$$|u| \geq |z|\left(\frac{s\gamma\left|z^2\right|}{1+|a|} - 1\right). \quad (30)$$

Also, we have

$$\begin{aligned}|Sy| &= \left|(1-\beta)^s Sz + \beta^s Tu\right| \\ &= \left|(1-\beta)^s az + \beta^s\left(u^3+c\right)\right| \\ &\geq s\beta\left|u^3\right| - s\beta|c| - |az| + |s\beta az|, \\ &\qquad \text{because } s<1 \\ &\geq s\beta\left|u^3\right| - s\beta|z| - |az|, \\ &\qquad \text{because } |a|\geq 0,\ |z|\geq|c|, \\ &\geq s\beta\left|u^3\right| - (1+|a|)|z|, \\ &\qquad \text{because } s\beta<1.\end{aligned} \tag{31}$$

Since $|z| > (2(1+|a|)/s\gamma)^{1/2}$, this implies $|z|^3(s\gamma|z|^2/(1+|a|)-1)^3 > |z|^3$. Hence $|u|^3 > |z|^3(s\gamma|z|^2/(1+|a|)-1)^3 > |z|^3 > s\gamma|z|^3$; this implies

$$ay \geq s^2\beta\gamma\left|z^3\right| - (1+|a|)|z|. \tag{32}$$

Thus

$$|y| \geq |z|\left(\frac{s^2\beta\gamma\left|z^2\right|}{1+|a|} - 1\right). \tag{33}$$

For $z_0 = z$ and $y_0 = y$, consider

$$\left|Sz_n\right| = \left|(1-\alpha)^s Sz_{n-1} + \alpha^s Ty_{n-1}\right|, \tag{34}$$

which implies that

$$\left|Sz_1\right| = \left|(1-\alpha)^s Sz + \alpha^s Ty\right| \tag{35}$$

yields

$$\begin{aligned}\left|az_1\right| &= \left|(1-\alpha)^s Sz + \alpha^s Ty\right| \\ &= \left|(1-\alpha)^s az + (1-(1-\alpha))^s\left(y^3+c\right)\right| \\ &\geq (1-s(1-\alpha))\left|y^3+c\right| - |(1-s\alpha)az| \\ &\geq s\alpha\left|y^3\right| - (1+|a|)|z| \\ &\geq s^3\alpha\beta\gamma\left|z^3\right| - (1+|a|)|z| \\ &\geq |z|\left(s^3\alpha\beta\gamma\left|z^2\right| - (1+|a|)\right).\end{aligned} \tag{36}$$

Hence

$$\left|z_1\right| \geq |z|\left(\frac{s^3\alpha\beta\gamma|z|^2}{1+|a|} - 1\right), \tag{37}$$

since $|z| \geq |c| > (2(1+|a|)/s\alpha)^{1/2}, (2(1+|a|)/s\beta)^{1/2}$, and $|z| \geq |c| > (2(1+|a|)/s\gamma)^{1/2}$, so that $|z| > (2(1+|a|)/s^3\alpha\beta\gamma)^{1/2}$. Therefore there exist $\lambda > 0$, such that $s^3\alpha\beta\gamma|z^2|/(1+|a|)-1 > 1+\lambda$. Consequently $|z_1| > (1+\lambda)|z|$. In particular, $|z_n| > |z|$. So we may apply the same argument repeatedly to find $|z_n| > (1+\lambda)^n|z|$. Thus, the orbit of z tends to infinity. This completes the proof. □

Corollary 14. *Suppose that $|z| \geq |c| > (2(1+|a|)/s\alpha)^{1/2}, (2(1+|a|)/s\beta)^{1/2}$ and $|z| \geq |c| > (2(1+|a|)/s\gamma)^{1/2}$; then the orbit of Jungck Noor JNO$(T_c, 0, \alpha, \beta, \gamma, s)$ escapes to infinity.*

In the proof of theorem we used the facts that $|z| \geq |c| > (2(1+|a|)/s\alpha)^{1/2}, (2(1+|a|)/s\beta)^{1/2}$ and $|z| \geq |c| > (2(1+|a|)/s\gamma)^{1/2}$. Hence the following corollary is the refinement of the escape criterion discussed in the above theorem.

Corollary 15 (escape criterion). *Let*

$$|z| > \max\left\{|c|, \left(\frac{2(1+|a|)}{s\alpha}\right)^{1/2}, \left(\frac{2(1+|a|)}{s\beta}\right)^{1/2}, \left(\frac{2(1+|a|)}{s\gamma}\right)^{1/2}\right\}; \tag{38}$$

then $|z_n| > (1+\lambda)^n|z|$ and $|z_n| \to \infty$ as $n \to \infty$.

Corollary 16. *Suppose that*

$$\left|z_k\right| > \max\left\{|c|, \left(\frac{2(1+|a|)}{s\alpha}\right)^{1/2}, \left(\frac{2(1+|a|)}{s\beta}\right)^{1/2}, \left(\frac{2(1+|a|)}{s\gamma}\right)^{1/2}\right\}, \tag{39}$$

for some $k \geq 0$. Then $|z_{k+1}| > (1+\lambda)^n|z_k|$ and $|z_{k+1}| \to \infty$ as $n \to \infty$.

This corollary gives us an algorithm in the generation of fractals (Julia sets, Mandelbrot sets, and Tricorns and Multicorns for linear or nonlinear dynamics) for $P_c z$.

3.3. Escape Criterion for Higher Degree Complex Polynomials. For higher degree complex polynomial $P_c z = z^n - az + c$, we will choose $Tz = z^n + c$ and $Sz = az$, where $n = 2, 3, 4, \ldots$ and a and c are complex numbers.

Theorem 17. *Assume that $|z| \geq |c| > (2(1+|a|)/s\alpha)^{1/(n-1)}$, $|z| \geq |c| > (2(1+|a|)/s\beta)^{1/(n-1)}$, and $|z| \geq |c| > (2(1+|a|)/s\gamma)^{1/(n-1)}$, where $0 < \alpha, \beta, \gamma, s < 1$ and c is a complex parameter. Define*

$$\begin{gathered}Sz_1 = (1-\alpha)^s Sz + \alpha^s Ty, \\ \vdots \\ Sz_n = (1-\alpha)^s Sz_{n-1} + \alpha^s Ty_{n-1}; \\ Sy = (1-\beta)^s Sz + \beta^s Tu, \\ \vdots\end{gathered}$$

$$\begin{aligned} Sy_{n-1} &= (1-\beta)^s Sz_{n-1} + \beta^s Tu_{n-1}, \\ Su &= (1-\beta)^s Sz + \beta^s Tz, \\ &\vdots \\ Su_{n-1} &= (1-\beta)^s Sz_{n-1} + \beta^s Tz_{n-1}, \end{aligned} \tag{40}$$

where Sz is injective, $Tz = z^n + c$, and $n = 2, 3, 4, \ldots$. Then, $|z_n| \to \infty$ as $n \to \infty$.

Proof. To prove the theorem, we follow the mathematical induction. For $n = 2$, $Tz = z^2 + c$, so the escape criterion is $|z| > \max\{|c|, 2(1+|a|)/s\alpha, 2(1+|a|)/s\beta, 2(1+|a|)/s\gamma\}$. For $n = 3$, $Tz = z^3 + c$, so escape criterion is

$$|z| > \max\left\{|c|, \left(\frac{2(1+|a|)}{s\alpha}\right)^{1/2}, \left(\frac{2(1+|a|)}{s\beta}\right)^{1/2}, \left(\frac{2(1+|a|)}{s\gamma}\right)^{1/2}\right\}. \tag{41}$$

Hence the theorem is true for $n = 2, 3, \ldots$. Now suppose that theorem is true for any n. Let $Tz = z^{n+1} + c$, $z_0 = z$, $y_0 = y$, $|z| \geq |c| > (2(1+|a|)/s\alpha)^{1/n}$, $|z| \geq |c| > (2(1+|a|)/s\beta)^{1/n}$, and $|z| \geq |c| > (2(1+|a|)/s\gamma)^{1/n}$ exist; then we have considered that

$$Su_{n-1} = (1-\gamma)^s Sz_{n-1} + \gamma^s Tz_{n-1} \tag{42}$$

implies

$$\begin{aligned} |Su| &= \left|(1-\gamma)^s Sz + \gamma^s Tz\right| \\ &= \left|(1-\gamma)^s az + \gamma^s\left(z^{n+1} + c\right)\right| \\ &= \left|(1-\gamma)^s az + (1-(1-\gamma))^s\left(z^{n+1} + c\right)\right|. \end{aligned} \tag{43}$$

Using binomial's series up to linear terms of α and $(1-\alpha)$, we get that

$$\begin{aligned} |Su| &= \left|(1-s\gamma)az + (1-s(1-\gamma))\left(z^{n+1}+c\right)\right| \\ &\geq (1-s(1-\gamma))\left|z^{n+1}+c\right| - \left|(1-s\gamma)az\right| \\ &\geq (s-s(1-\gamma))\left|z^{n+1}+c\right| - \left|(1-s\gamma)az\right|, \\ &\qquad\qquad \text{because } s < 1 \\ &\geq s\gamma\left|z^{n+1}\right| - s\gamma|c| - |az| + |s\gamma az| \\ &\geq s\gamma\left|z^{n+1}\right| - s\gamma|z| - |az| + |s\gamma az|, \\ &\qquad\qquad \text{because } |z| \geq |c| \\ &\geq s\gamma\left|z^{n+1}\right| - s\gamma|z| - |az|, \\ &\qquad\qquad \text{because } |a| \geq 0 \end{aligned} \tag{44}$$

gives us

$$\begin{aligned} |au| &\geq s\gamma\left|z^{n+1}\right| - |z| - |az|, \\ &\qquad\qquad \text{because } s\gamma < 1 \\ &\geq s\gamma\left|z^{n+1}\right| - |a||z| - |z| \\ &= s\gamma\left|z^{n+1}\right| - (1+|a|)|z| \\ &= |z|\left(s\gamma\left|z^n\right| - (1+|a|)\right). \end{aligned} \tag{45}$$

Thus

$$|y| \geq |z|\left(\frac{s^2\beta\gamma|z^n|}{1+|a|} - 1\right). \tag{46}$$

For $z_0 = z$ and $y_0 = y$, consider

$$|Sz_n| = \left|(1-\alpha)^s Sz_{n-1} + \alpha^s Ty_{n-1}\right|, \tag{47}$$

which implies that

$$|Sz_1| = \left|(1-\alpha)^s Sz + \alpha^s Ty\right| \tag{48}$$

yields

$$\begin{aligned} |az_1| &= \left|(1-\alpha)^s Sz + \alpha^s Ty\right| \\ &= \left|(1-\alpha)^s az + (1-(1-\alpha))^s\left(y^{n+1}+c\right)\right| \\ &\geq (1-s(1-\alpha))\left|y^{n+1}+c\right| - |(1-s\alpha)az| \\ &\geq s\alpha\left|y^{n+1}\right| - (1+|a|)|z| \\ &\geq s^3\alpha\beta\gamma\left|z^{n+1}\right| - (1+|a|)|z| \\ &\geq |z|\left(s^3\alpha\beta\gamma|z^n| - (1+|a|)\right). \end{aligned} \tag{49}$$

Hence

$$|z_1| \geq |z|\left(\frac{s^3\alpha\beta\gamma|z^n|}{1+|a|} - 1\right), \tag{50}$$

since $|z| \geq |c| > (2(1+|a|)/s\alpha)^{1/n}$, $|z| \geq |c| > (2(1+|a|)/s\beta)^{1/n}$, and $|z| \geq |c| > (2(1+|a|)/s\gamma)^{1/n}$, so that $|z| > (2(1+|a|)/s^3\alpha\beta\gamma)^{1/n}$. Therefore there exist $\lambda > 0$, such that $s^3\alpha\beta|z^n|/(1+|a|) - 1 > 1 + \lambda$. Consequently $|z_1| > (1+\lambda)|z|$. In particular, $|z_n| > |z|$. So we may apply the same argument repeatedly to find $|z_n| > (1+\lambda)^n|z|$. Thus, the orbit of z tends to infinity. This completes the proof. □

Corollary 18. *Suppose that $|z| \geq |c| > (2(1+|a|)/s\alpha)^{1/(n-1)}$, $|z| \geq |c| > (2(1+|a|)/s\beta)^{1/(n-1)}$, and $|z| \geq |c| > (2(1+|a|)/s\gamma)^{1/(n-1)}$; then the orbit $JNO(T_c, 0, \alpha, \beta, \gamma, s)$ escapes to infinity.*

Corollary 19 (escape criterion). *Suppose that*

$$|z_k| > \max\left\{|c|, \left(\frac{2(1+|a|)}{s\alpha}\right)^{1/(n-1)}, \left(\frac{2(1+|a|)}{s\beta}\right)^{1/(n-1)}, \left(\frac{2(1+|a|)}{s\gamma}\right)^{1/(n-1)}\right\}, \tag{51}$$

for some $k \geq 0$. *Then* $|z_{k+1}| > (1+\lambda)^n |z_k|$ *and* $|z_{k+1}| \to \infty$ *as* $n \to \infty$.

This corollary gives us an algorithm in the generation of fractals (Julia sets, Mandelbrot sets, and Tricorns and Multicorns for linear or nonlinear dynamics) for $P_c z$.

4. Conclusions

In this paper, new fixed point results for Jungck Noor iteration with s-convexity have been introduced in the generation of fractals (Julia sets, Mandelbrot sets, and Tricorns and Multicorns for linear or nonlinear dynamics). The new escape criterions for complex quadratic, cubic, and nth degree polynomials have been established. If we take $s = 1$, it provides previous existing results in the relative literature [8].

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors read and approved the final paper.

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References

[1] G. Julia, "Sur l'iteration des functions rationnelles," *Journal de Mathématiques Pures et Appliquées*, vol. 8, pp. 737–747, 1918.

[2] B. B. Mandelbrot, *The Fractal Geometry of Nature*, W. H. Freeman, New York, NY, USA, 1982.

[3] R. L. Devaney, *A First Course in Chaotic Dynamical Systems: Theory and Experiment*, Addison-Wesley, 1992.

[4] Y. S. Chauhan, R. Rana, and A. Negi, "Complex dynamics of Ishikawa iterates for non integer values," *International Journal of Computer Applications*, vol. 9, no. 2, pp. 9–16, 2010.

[5] A. Negi and M. Rani, "Midgets of superior Mandelbrot set," *Chaos, Solitons and Fractals*, vol. 36, no. 2, pp. 237–245, 2008.

[6] A. Negi and M. Rani, "A new approach to dynamic noise on superior Mandelbrot set," *Chaos, Solitons & Fractals*, vol. 36, no. 4, pp. 1089–1096, 2008.

[7] M. Rani and R. Agarwal, "Effect of noise on Julia sets generated by logistic map," in *Proceedings of the 2nd International Conference on Computer and Automation Engineering (ICCAE '10)*, pp. 55–59, IEEE, Singapore, February 2010.

[8] S. M. Kang, A. Rafiq, M. Tanveer, F. Ali, and Y. C. Kwun, "Julia and Mandelbrot sets in modified Jungck three-step orbit," *Wulfenia Journal (Klagentrut, Austria)*, vol. 22, no. 1, pp. 167–185, 2015.

[9] H. Hudzik and L. Maligranda, "Some remarks on s-convex functions," *Aequationes Mathematicae*, vol. 48, no. 1, pp. 100–111, 1994.

[10] M. Alomari and M. Darus, "Hadamard-type inequalities for s-convex functions," *International Mathematical Forum*, vol. 3, no. 40, pp. 1965–1975, 2008.

[11] U. S. Kirmaci, M. Klaričić Bakula, M. E. Özdemir, and J. Pečarić, "Hadamard-type inequalities for s-convex functions," *Applied Mathematics and Computation*, vol. 193, no. 1, pp. 26–35, 2007.

[12] F. Bernstein and G. Doetsch, "Zur Theorie der konvexen Funktionen," *Mathematische Annalen*, vol. 76, no. 4, pp. 514–526, 1915.

[13] W. Takahashi, "A convexity in metric space and nonexpansive mappings. I," *Kodai Mathematical Seminar Reports*, vol. 22, no. 2, pp. 142–149, 1970.

[14] D. B. Ojha and M. K. Mishra, "An application of fixed point theorem for s-convex function," *International Journal of Engineering Science and Technology*, vol. 2, no. 8, pp. 3371–3375, 2010.

[15] W. W. Breckner and G. Orbán, *Continuity Properties of Rationally S-Convex Mappings with Values in Ordered Topological Liner Space*, Babeş-Bolyai University, Kolozsvár, Romania, 1978.