Research Article

Commutators of Multilinear Singular Integrals on Weighted Herz Spaces

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Received 16 April 2015; Accepted 14 June 2015

Academic Editor: Rodolfo H. Torres

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Suppose $\vec{b} = (b_1, \dots, b_m) \in (\text{BMO})^m$, $T_{\vec{b}}$ is the iterated commutator of $\vec{b}$ and the m -linear Calderón-Zygmund operator T . The purpose of this paper is to discuss the boundedness properties of $T_{\vec{b}}$ on weighted Herz spaces with general Muckenhoupt weights.

1. Introduction and Results

In recent years, the study of multilinear integrals has received more attention, which is motivated by not only the generalization of the linear theory but also the natural appearance of singular integral theory. The initial work on the class of multilinear Calderón-Zygmund operators was done by Coifman and Meyer in [1] and was later systematically studied by Grafakos and Torres in [2–4]. More results on the multilinear Calderón-Zygmund operators and their commutators can be found in [5–10].

Let $\mathbb{R}^n$ be the n -dimensional Euclidean space, and let $(\mathbb{R}^n)^m = \mathbb{R}^n \times \dots \times \mathbb{R}^n$ be the m -fold product space ($m \in \mathbb{N}$). We denote by $\mathcal{S}(\mathbb{R}^n)$ the space of all Schwartz functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ its dual space, the set of all tempered distributions on $\mathbb{R}^n$.

We say that a locally integrable function $K(y_0, y_1, \dots, y_m)$ defined away from the diagonal $y_0 = y_1 = \dots = y_m$ in $(\mathbb{R}^n)^{m+1}$ is a kernel in the class $m\text{-CZK}(A, \epsilon)$ if it satisfies the size estimate

$$|K(y_0, y_1, \dots, y_m)| \leq \frac{A}{\left(\sum_{k,l=0}^m |y_l - y_k|\right)^{mn}} \quad (1)$$

for some $A > 0$ and all $(y_0, y_1, \dots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $y_l \neq y_k$ for some l, k . Moreover, assume that for some

$\epsilon > 0$ and all $0 \leq j \leq m$ it satisfies the smoothness estimates

$$\begin{aligned} & \left| K(y_0, \dots, y_j, \dots, y_m) - K(y_0, \dots, y'_j, \dots, y_m) \right| \\ & \leq \frac{A |y_j - y'_j|^\epsilon}{\left(\sum_{k,l=0}^m |y_l - y_k|\right)^{mn+\epsilon}} \end{aligned} \quad (2)$$

whenever

$$|y_j - y'_j| \leq \frac{1}{2} \max_{0 \leq k \leq m} |y_j - y_k|. \quad (3)$$

Let T be a multilinear operator initially defined on the m -fold product of Schwartz spaces and taking values into the space of tempered distributions,

$$T : \mathcal{S}(\mathbb{R}^n) \times \dots \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n). \quad (4)$$

We say that T is a m -linear Calderón-Zygmund operator if, for some $q_1, \dots, q_m \in [1, \infty)$ and $q \in (0, \infty)$ with $1/q = \sum_{i=1}^m 1/q_i$, it extends to a bounded multilinear operator from $L^{q_1}(\mathbb{R}^n) \times \dots \times L^{q_m}(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ and if there exists a kernel function $K(x, y_1, \dots, y_m)$ in the class $m\text{-CZK}(A, \epsilon)$, defined

away from the diagonal $x = y_1 = \dots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$\begin{aligned} T(\vec{f})(x) &= T(f_1, \dots, f_m)(x) \\ &= \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) \\ &\quad \cdot f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m \end{aligned} \quad (5)$$

for all $x \notin \bigcap_{j=1}^m \text{supp } f_j$ and $f_1, \dots, f_m$ are C^∞ functions with compact support.

Given a locally integrable vector function $\vec{b} = (b_1, \dots, b_m) \in (\text{BMO})^m$, the iterated commutator of $\vec{b}$ and the m -linear Calderón-Zygmund operator T , denoted by $T_{\Pi b}$, was introduced in [6] and is defined by

$$T_{\Pi b}(\vec{f}) = [b_1, \dots, [b_{m-1}, [b_m, T]_{m-1} \cdots]_1(\vec{f}). \quad (6)$$

To clarify the notation, if T is associated in the usual way with a Calderón-Zygmund kernel K , then at a formal level

$$\begin{aligned} T_{\Pi b}(\vec{f})(x) &= \int_{(\mathbb{R}^n)^m} \prod_{i=1}^m (b_i(x) - b_i(y_i)) \\ &\quad \cdot K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m. \end{aligned} \quad (7)$$

Suppose T is an m -linear Calderón-Zygmund operator. Let $\vec{\omega} \in A_{\vec{q}}$, let $\gamma_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{q/q_i}$ with $1/q = 1/q_1 + \dots + 1/q_m$, and let $\vec{b} \in (\text{BMO})^m$. In 2014, Pérez et al. [6] proved that the commutator $T_{\Pi b}$ is bounded from $L^{q_1}(\omega_1) \times \dots \times L^{q_m}(\omega_m)$ to $L^q(\gamma_{\vec{\omega}})$.

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$ for any $k \in \mathbb{Z}$. Denote $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, where χ_{C_k} is the characteristic function of the set C_k . The following weighted Herz space is introduced by Lu and Yang in [11].

Let $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, and ω_1, ω_2 be two weight functions on $\mathbb{R}^n$. The homogeneous weighted Herz space $\dot{K}_q^{\alpha, p}(\omega_1, \omega_2)(\mathbb{R}^n)$ is defined by

$$\begin{aligned} \dot{K}_q^{\alpha, p}(\omega_1, \omega_2)(\mathbb{R}^n) \\ = \left\{ f \in L_{\text{loc}}^q(\mathbb{R}^n \setminus \{0\}, \omega_2) : \|f\|_{\dot{K}_q^{\alpha, p}(\omega_1, \omega_2)} < \infty \right\}, \end{aligned} \quad (8)$$

where

$$\|f\|_{\dot{K}_q^{\alpha, p}(\omega_1, \omega_2)} = \left(\sum_{k=-\infty}^{\infty} \omega_1(B_k)^{\alpha p/n} \|f \chi_k\|_{L_{\omega_2}^q(\mathbb{R}^n)}^p \right)^{1/p}. \quad (9)$$

Obviously, if $\alpha = 0$, then $\dot{K}_q^{0, q}(\omega_1, \omega_2)(\mathbb{R}^n) = L_{\omega_2}^q(\mathbb{R}^n)$ for any $0 < q < \infty$. Thus, the weighted Herz spaces are generalizations of the weighted Lebesgue spaces.

In 2000, Lu et al. in [12] proved boundedness results for sublinear operators on weighted Herz spaces with general Muckenhoupt weights. Recently, many authors considered the boundedness of operators and their commutators on weighted Herz spaces. Wang in [13] proved that the intrinsic square functions are bounded on weighted Herz spaces and

in [14] he also obtained the boundedness of the intrinsic square functions on weighted Herz type Hardy spaces. In [15], Kuang considered the boundedness of generalized Hausdorff operators on weighted Herz spaces; Hu et al. in [16] established the weighted boundedness for the commutator of fractional integral operators on Herz spaces. The authors also have studied the boundedness of the multilinear integrals on weighted (unweighted) Herz spaces in [17–19]. In this paper, we focus on the boundedness of the iterated commutator of BMO and the m -linear Calderón-Zygmund operator on weighted Herz spaces. Our main results in this paper are formulated as follows.

Theorem 1. Suppose $m \geq 2$, $(b_1, \dots, b_m) \in (\text{BMO})^m$, T is an m -linear Calderón-Zygmund operator, $\gamma_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{q/q_i}$, and $\gamma \in A_r$ ($1 \leq r < \infty$). For $i = 1, \dots, m$, let $0 < p_i \leq \infty$, let $1 < q_i < \infty$, and let $\omega_i \in A_{r_i}$ ($1 \leq r_i < q_i$). If p, q, α satisfy

$$\begin{aligned} \frac{1}{p} &= \sum_{i=1}^m \frac{1}{p_i}, \\ \frac{1}{q} &= \sum_{i=1}^m \frac{1}{q_i}, \\ \alpha &= \sum_{i=1}^m \alpha_i, \end{aligned} \quad (10)$$

then $T_{\Pi b}$ is bounded from $\dot{K}_{q_1}^{\alpha_1, p_1}(\gamma, \omega_1)(\mathbb{R}^n) \times \dots \times \dot{K}_{q_m}^{\alpha_m, p_m}(\gamma, \omega_m)(\mathbb{R}^n)$ into $\dot{K}_q^{\alpha, p}(\gamma, \gamma_{\vec{\omega}})(\mathbb{R}^n)$ provided that

$$0 < \alpha_i r < n \left(1 - \frac{r_i}{q_i} \right) \quad (11)$$

for $i = 1, \dots, m$.

2. Some Preliminaries

A weight ω is a nonnegative, locally integrable function on $\mathbb{R}^n$. The symbol $B = B(x_0, r_B)$ denotes the ball with center x_0 , radius r_B and for any $\lambda > 0$, λB will stand for the ball $B(x_0, \lambda r_B)$. For a given weight function ω and a measurable set E , we also denote the Lebesgue measure of E by $|E|$ and set weighted measure $\omega(E) = \int_E \omega(x) dx$.

A weight ω is said to belong to A_q for $1 < q < \infty$, if there exists a constant C such that, for every ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{1-q'} dx \right)^{q-1} \leq C, \quad (12)$$

where q' is the dual of q such that $1/q + 1/q' = 1$. The class A_1 is defined by replacing the above inequality with

$$\frac{1}{|B|} \int_B \omega(y) dy \leq C \cdot \text{ess inf}_{x \in B} \omega(x) \quad (13)$$

for every ball $B \subset \mathbb{R}^n$.

A weight ω is said to belong to A_∞ if there are positive numbers C and δ so that

$$\frac{\omega(E)}{\omega(B)} \leq C \left(\frac{|E|}{|B|} \right)^\delta \quad (14)$$

for all balls B and all measurable $E \subset B$. It is well known that

$$A_\infty = \bigcup_{1 \leq q < \infty} A_q. \quad (15)$$

By (12), we have

$$\left(\int_B \omega(x) dx \right)^{1/q} \left(\int_B \omega(x)^{1-q'} dx \right)^{1/q'} \leq C |B| \quad (16)$$

for $1 < q < \infty$.

The classical A_q weight theory was first introduced by Muckenhoupt in the study of weighted L^q boundedness of Hardy-Littlewood maximal function in [20].

Lemma 2 (see [20, 21]). *Suppose $\omega \in A_q(\mathbb{R}^n)$ and the following statements hold.*

- (i) *For any $1 \leq q < \infty$, there are positive numbers C and δ such that*

$$\frac{\omega(B_k)}{\omega(B_j)} \leq C 2^{nq(k-j)} \quad \text{for } k > j, \quad (17)$$

$$\frac{\omega(B_k)}{\omega(B_j)} \leq C 2^{\delta(k-j)} \quad \text{for } k < j. \quad (18)$$

- (ii) $A_{q_1}(\mathbb{R}^n) \subset A_{q_2}(\mathbb{R}^n)$ for any $1 \leq q_1 < q_2 \leq \infty$.

- (iii) *For any $1 < q < \infty$, one has $\omega^{1-q'} \in A_{q'}(\mathbb{R}^n)$.*

A locally integrable function b is said to be in $BMO(\mathbb{R}^n)$ if

$$\sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |b(x) - b_B| dx = \|b\|_* < \infty, \quad (19)$$

where $b_B = |B|^{-1} \int_B b(y) dy$.

Lemma 3 (see [22]). *Suppose $\omega \in A_\infty$ and $b \in BMO$. Then, for any $1 \leq q < \infty$ and $r_1, r_2 > 0$, we have*

$$\begin{aligned} & \left(\frac{1}{\omega(B(x_0, r_1))} \int_{B(x_0, r_1)} |b(x) - b_{B(x_0, r_2)}|^q \omega(x) dx \right)^{1/q} \\ & \leq C \|b\|_* \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right). \end{aligned} \quad (20)$$

Now let us recall the definition of multiple weights. For m exponents $q_1, \dots, q_m$, we write $\vec{q} = (q_1, \dots, q_m)$. Let $q_1, \dots, q_m \in [1, \infty)$, and let $q \in (0, \infty)$ with $1/q = \sum_{k=1}^m 1/q_k$.

Given $\vec{\omega} = (\omega_1, \dots, \omega_m)$, set $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{q/q_i}$. We say that $\vec{\omega}$ satisfies the $A_{\vec{q}}$ condition if it satisfies

$$\begin{aligned} & \sup_B \left(\frac{1}{|B|} \int_B \nu_{\vec{\omega}}(x) dx \right)^{1/q} \\ & \cdot \prod_{i=1}^m \left(\frac{1}{|B|} \int_B \omega_i(x)^{1-q'_i}(x) dx \right)^{1/q'_i} < \infty. \end{aligned} \quad (21)$$

When $q_i = 1$, $(1/|B|) \int_B \omega_i(x)^{1-q'_i}(x) dx$ is understood as $(\inf_{x \in B} \omega_i(x))^{-1}$.

Let $q_1, \dots, q_m \in [1, \infty)$, and let $1/q = \sum_{k=1}^m 1/q_k$. By Hölder's inequality,

$$\nu_{\vec{\omega}}(B)^{1/q} \leq \prod_{i=1}^m \omega_i(B)^{1/q_i}. \quad (22)$$

Supposing $\omega_i \in A_{q_i}$, $i = 1, \dots, m$, then by (22) and the definition of multiple weights we have $\vec{\omega} \in A_{\vec{q}}$.

Lemma 4 (see [6]). *Let $q_1, \dots, q_m \in [1, \infty)$, and let $1/q = \sum_{k=1}^m 1/q_k$. Then $\vec{\omega} = (\omega_1, \dots, \omega_m) \in A_{\vec{q}}$ if and only if*

$$\begin{aligned} & \nu_{\vec{\omega}} \in A_{qm}, \\ & \omega_i^{1-q'_i} \in A_{mq'_i} \quad \text{for } i = 1, \dots, m, \end{aligned} \quad (23)$$

where $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{q/q_i}$ and the condition $\omega_i^{1-q'_i} \in A_{mq'_i}$ in the case $q_i = 1$ is understood as $\omega_i^{1/m} \in A_1$.

3. Proof of Theorem 1

Without loss of generality, we only prove the theorem for the case $m = 2$. The method can be extended for any m -linear case without any essential difficulty.

Let $\nu \in A_r$, and let $\omega_l \in A_{r_l}$, $l = 1, 2$. If $k > i, k > j$, then from (17) we have

$$\begin{aligned} & \frac{\nu(B_k)}{\nu(B_i)} \leq C 2^{(k-i)nr}, \\ & \frac{\nu(B_k)}{\nu(B_j)} \leq C 2^{(k-j)nr}, \\ & \frac{\omega_1(B_k)}{\omega_1(B_i)} \leq C 2^{(k-i)nr_1}, \\ & \frac{\omega_2(B_k)}{\omega_2(B_j)} \leq C 2^{(k-j)nr_2}. \end{aligned} \quad (24)$$

If $k < i, k < j$, then from (18) we have

$$\begin{aligned} \frac{\nu(B_k)}{\nu(B_i)} &\leq C 2^{(k-i)\delta}, \\ \frac{\nu(B_k)}{\nu(B_j)} &\leq C 2^{(k-j)\delta}, \\ \frac{\omega_1(B_k)}{\omega_1(B_i)} &\leq C 2^{(k-i)\delta_1}, \\ \frac{\omega_2(B_k)}{\omega_2(B_j)} &\leq C 2^{(k-j)\delta_2}. \end{aligned} \quad (25)$$

We have the following conclusions.

Theorem 5. Suppose $(b_1, b_2) \in (BMO)^2$, T is an 2-linear Calderón-Zygmund operator, and $\nu \in A_r$ ($1 \leq r < \infty$). For $i = 1, 2$, let $1 < q_i < \infty$, and let $\omega_i \in A_{r_i}$ ($1 \leq r_i < q_i$). If q, α satisfy $1/q = 1/q_1 + 1/q_2$ and $\alpha = \alpha_1 + \alpha_2$, then

$$\begin{aligned} &\nu(B_k)^{\alpha/n} \|T_{\Pi b}(f\chi_i, g\chi_j) \chi_k\|_{L^q(\nu_{\tilde{\omega}})} \\ &\leq C \|b_1\|_* D_1(k, i) \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \\ &\quad \times \|b_2\|_* D_2(k, j) \nu(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)}, \end{aligned} \quad (26)$$

where $\nu_{\tilde{\omega}} = \prod_{i=1}^2 \omega_i^{q/q_i}$, and

$$\begin{aligned} &D_l(k, i) \\ &= \begin{cases} (k-i) 2^{(k-i)(\alpha_1 r + nr_1/q_1 - n)} & \text{if } i \leq k-2; \\ 1 & \text{if } k-1 \leq i \leq k+1; \\ (i-k) 2^{(k-i)(\delta\alpha_1/n + \delta_1/q_1)} & \text{if } i \geq k+2 \end{cases} \end{aligned} \quad (27)$$

for $l = 1, 2$.

Proof. When $k-1 \leq i, j \leq k+1$, then, by the boundedness of $T_{\Pi b}$ from $L^{q_1}(\omega_1) \times L^{q_2}(\omega_2)$ to $L^q(\nu_{\tilde{\omega}})$ and $\nu(B_k) \sim \nu(B_i) \sim \nu(B_j)$, we obtain

$$\begin{aligned} &\nu(B_k)^{\alpha/n} \|T_{\Pi b}(f\chi_i, g\chi_j) \chi_k\|_{L^q(\nu_{\tilde{\omega}})} \\ &\leq C \|b_1\|_* \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \\ &\quad \times \|b_2\|_* \nu(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)}. \end{aligned} \quad (28)$$

This means $D_1(k, i) = D_2(k, j) = 1$ if $k-1 \leq i, j \leq k+1$.

In the other case, we see that

$$|x - y_1| + |x - y_2| \sim 2^{\max\{i, j, k\}} \quad (29)$$

for $x \in C_k, y_1 \in C_i, y_2 \in C_j$.

Taking

$$\begin{aligned} \lambda_{1i} &= (b_1)_{B_i} = \frac{1}{|B_i|} \int_{B_i} b_1(x) dx, \\ \lambda_{2j} &= (b_2)_{B_j} = \frac{1}{|B_j|} \int_{B_j} b_2(x) dx, \end{aligned} \quad (30)$$

then

$$\begin{aligned} &T_{\Pi b}(f\chi_i, g\chi_j)(x) \\ &= (b_1(x) - \lambda_{1i})(b_2(x) - \lambda_{2j}) T(f\chi_i, g\chi_j) \\ &\quad - (b_1(x) - \lambda_{1i}) T(f\chi_i, (b_2 - \lambda_{2j})g\chi_j) \\ &\quad - (b_2(x) - \lambda_{2j}) T((b_1 - \lambda_{1i})f\chi_i, g\chi_j)(x) \\ &\quad + T((b_1 - \lambda_{1i})f\chi_i, (b_2 - \lambda_{2j})g\chi_j)(x) \\ &= L_1(x) + L_2(x) + L_3(x) + L_4(x). \end{aligned} \quad (31)$$

Thus

$$\begin{aligned} &\nu(B_k)^{\alpha/n} \|T_{\Pi b}(f\chi_i, g\chi_j) \chi_k\|_{L^q(\nu_{\tilde{\omega}})} \\ &\leq \sum_{m=1}^4 \nu(B_k)^{\alpha/n} \|(L_m) \chi_k\|_{L^q(\nu_{\tilde{\omega}})} = \sum_{m=1}^4 J_m. \end{aligned} \quad (32)$$

Now, we will estimate each J_m ($m = 1, 2, 3, 4$), separately. Applying (29),

$$\begin{aligned} &|L_1(x)| \\ &= |(b_1(x) - \lambda_{1i})(b_2(x) - \lambda_{2j}) T(f\chi_i, g\chi_j)(x)| \\ &\leq C 2^{-2\max\{i, j, k\}n} |(b_1(x) - \lambda_{1i})(b_2(x) - \lambda_{2j})| \\ &\quad \cdot \|f\chi_i\|_{L^1} \|g\chi_j\|_{L^1}. \end{aligned} \quad (33)$$

By Hölder's inequality and (16), we have

$$\begin{aligned} &|L_1(x)| \leq C 2^{-2\max\{i, j, k\}n + in + jn} |b_1(x) - \lambda_{1i}| \\ &\quad \cdot |b_2(x) - \lambda_{2j}| \times \|f\chi_i\|_{L^{q_1}(\omega_1)} \|g\chi_j\|_{L^{q_2}(\omega_2)} \\ &\quad \cdot \left(\int_{B_i} \omega_1(x)^{1-q'_1} dx \right)^{1/q'_1} \left(\int_{B_j} \omega_2(x)^{1-q'_2} dx \right)^{1/q'_2} \\ &\leq C 2^{-2\max\{i, j, k\}n + in + jn} |b_1(x) - \lambda_{1i}| |b_2(x) - \lambda_{2j}| \\ &\quad \times \|f\chi_i\|_{L^{q_1}(\omega_1)} \|g\chi_j\|_{L^{q_2}(\omega_2)} \omega_1(B_i)^{-1/q_1} \omega_2(B_j)^{-1/q_2}. \end{aligned} \quad (34)$$

Then

$$\begin{aligned} &\nu(B_k)^{\alpha/n} \|(L_1) \chi_k\|_{L^q(\nu_{\tilde{\omega}})} \\ &\leq C 2^{-2\max\{i, j, k\}n + in + jn} \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \\ &\quad \cdot \nu(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)} \times \left(\frac{\nu(B_k)}{\nu(B_i)} \right)^{\alpha_1/n} \\ &\quad \cdot \left(\frac{\nu(B_k)}{\nu(B_j)} \right)^{\alpha_2/n} \omega_1(B_i)^{-1/q_1} \omega_2(B_j)^{-1/q_2} \end{aligned}$$

$$\begin{aligned} & \times \left(\int_{B_k} |(b_1(x) - \lambda_{1i})(b_2(x) - \lambda_{2j})|^q \right. \\ & \left. \cdot \nu_{\tilde{\omega}}(x) dx \right)^{1/q}. \end{aligned} \quad (35)$$

By Lemma 4 we know $\nu_{\tilde{\omega}} \in A_{\infty}$. If $q \geq 1/2$, then by Hölder's inequality, Lemma 3, and (22) we have

$$\begin{aligned} & \left(\int_{B_k} |(b_1(x) - \lambda_{1i})(b_2(x) - \lambda_{2j})|^q \nu_{\tilde{\omega}}(x) dx \right)^{1/q} \\ & \leq \left(\int_{B_k} |b_1(x) - \lambda_{1i}|^{2q} \nu_{\tilde{\omega}}(x) dx \right)^{1/2q} \\ & \cdot \left(\int_{B_k} |b_2(x) - \lambda_{2j}|^{2q} \nu_{\tilde{\omega}}(x) dx \right)^{1/2q} \leq C \|b_1\|_* \\ & \cdot \|b_2\|_* |k-i| |k-j| \nu_{\tilde{\omega}}(B_k)^{1/q} \leq C \|b_1\|_* \|b_2\|_* \\ & \cdot |k-i| |k-j| \omega_1(B_k)^{1/q_1} \omega_2(B_k)^{1/q_2}. \end{aligned} \quad (36)$$

If $0 < q < 1/2$, then by Hölder's and Lemma 3 we have

$$\begin{aligned} & \left(\int_{B_k} |(b_1(x) - \lambda_{1i})(b_2(x) - \lambda_{2j})|^q \nu_{\tilde{\omega}}(x) dx \right)^{1/q} \\ & \leq \left(\int_{B_k} |b_1(x) - \lambda_{1i}| \nu_{\tilde{\omega}}(x) dx \right) \\ & \cdot \left(\int_{B_k} |b_2(x) - \lambda_{2j}|^{q/(1-q)} \nu_{\tilde{\omega}}(x) dx \right)^{1/q-1} \\ & \leq C \|b_1\|_* |k-i| \nu_{\tilde{\omega}}(B_k) \\ & \cdot \left(\int_{B_k} |b_2(x) - \lambda_{2j}| \nu_{\tilde{\omega}}(x) dx \right) \nu_{\tilde{\omega}}(B_k)^{1/q-2} \\ & \leq C \|b_1\|_* \|b_2\|_* |k-i| |k-j| \omega_1(B_k)^{1/q_1} \omega_2(B_k)^{1/q_2}. \end{aligned} \quad (37)$$

Thus

$$\begin{aligned} & \nu(B_k)^{\alpha/n} \|(L_1) \chi_k\|_{L^q(\nu_{\tilde{\omega}})} \leq C \|b_1\|_* |k-i| \\ & \cdot 2^{-\max\{i,k\}n+in} \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \left(\frac{\nu(B_k)}{\nu(B_i)} \right)^{\alpha_1/n} \\ & \cdot \left(\frac{\omega_1(B_k)}{\omega_1(B_i)} \right)^{1/q_1} \times \|b_2\|_* |k-j| \\ & \cdot 2^{-\max\{j,k\}n+jn} \nu(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)} \left(\frac{\nu(B_k)}{\nu(B_j)} \right)^{\alpha_2/n} \\ & \cdot \left(\frac{\omega_2(B_k)}{\omega_2(B_j)} \right)^{1/q_2}. \end{aligned} \quad (38)$$

When $i \leq k-2, j \leq k-2$. Then by (38) and (24), we get

$$\begin{aligned} & \nu(B_k)^{\alpha/n} \|(L_1) \chi_k\|_{L^q(\nu_{\tilde{\omega}})} \leq C \|b_1\|_* (k-i) \\ & \cdot 2^{(k-i)(\alpha_1 r + nr_1/q_1 - n)} \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \times \|b_2\|_* \\ & \cdot (k-j) 2^{(k-j)(\alpha_2 r + nr_2/q_2 - n)} \nu(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)}. \end{aligned} \quad (39)$$

When $i \geq k+2, j \geq k+2$, by (38) and (25), we get

$$\begin{aligned} & \nu(B_k)^{\alpha/n} \|T_{\Pi b}(f\chi_i, g\chi_j) \chi_k\|_{L^q(\nu_{\tilde{\omega}})} \leq C \|b_1\|_* (i-k) \\ & \cdot 2^{(k-i)(\alpha_1 \delta/n + \delta_1/q_1)} \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \times \|b_2\|_* \\ & \cdot (j-k) 2^{(k-j)(\alpha_2 \delta/n + \delta_2/q_2)} \nu(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)}. \end{aligned} \quad (40)$$

When $i \leq k-2, j \geq k+2$. Then by (38) and (24)-(25) we get

$$\begin{aligned} & \nu(B_k)^{\alpha/n} \|T_{\Pi b}(f\chi_i, g\chi_j) \chi_k\|_{L^q(\nu_{\tilde{\omega}})} \leq C \|b_1\|_* (k-i) \\ & \cdot 2^{(k-i)(\alpha_1 r + nr_1/q_1 - n)} \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \times \|b_2\|_* \\ & \cdot (j-k) 2^{(k-j)(\alpha_2 \delta/n + \delta_2/q_2)} \nu(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)}. \end{aligned} \quad (41)$$

By symmetry, when $j \leq k-2, i \geq k+2$, we have

$$\begin{aligned} & \nu(B_k)^{\alpha/n} \|T_{\Pi b}(f\chi_i, g\chi_j) \chi_k\|_{L^q(\nu_{\tilde{\omega}})} \leq C \|b_1\|_* (i-k) \\ & \cdot 2^{(k-i)(\alpha_1 \delta/n + \delta_1/q_1)} \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \times \|b_2\|_* \\ & \cdot (k-j) 2^{(k-j)(\alpha_2 r + nr_2/q_2 - n)} \nu(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)}. \end{aligned} \quad (42)$$

When $i \leq k-2, k-1 \leq j \leq k+1$. Noting that $\nu(B_k) \sim \nu(B_j), \omega_2(B_k) \sim \omega_2(B_j)$, then

$$\begin{aligned} & \nu(B_k)^{\alpha/n} \|T_{\Pi b}(f\chi_i, g\chi_j) \chi_k\|_{L^q(\nu_{\tilde{\omega}})} \leq C \|b_1\|_* (k-i) \\ & \cdot 2^{(k-i)(\alpha_1 r + nr_1/q_1 - n)} \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \times \|b_2\|_* \\ & \cdot \nu(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)}. \end{aligned} \quad (43)$$

Similar to the estimates (43), we can verify (26) in the following case: $k-1 \leq i \leq k+1, j \leq k-2; k-1 \leq i \leq k+1, j > k+1$ and $i > k+1, k-1 \leq j \leq k+1$. Thus we obtain

$$\begin{aligned} & \nu(B_k)^{\alpha/n} \|(L_1) \chi_k\|_{L^q(\nu_{\tilde{\omega}})} \\ & \leq C \|b_1\|_* D_1(k, i) \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \\ & \times \|b_2\|_* D_2(k, j) \nu(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)}. \end{aligned} \quad (44)$$

Applying (29) and Hölder's inequality,

$$\begin{aligned}
 |L_2(x)| &= |(b_1(x) - \lambda_{1i}) T(f\chi_i, (b_2 - \lambda_{2j}) g\chi_j)| \\
 &\leq C 2^{-2 \max\{i,j,k\}n} |b_1(x) - \lambda_{1i}| \|f\chi_i\|_{L^1} \\
 &\quad \cdot \|(b_2 - \lambda_{2j}) g\chi_j\|_{L^1} \leq C 2^{-2 \max\{i,j,k\}n} |b_1(x) - \lambda_{1i}| \\
 &\quad \cdot \|f\chi_i\|_{L^{q_1}(\omega_1)} \left(\int_{B_i} \omega_1(x)^{1-q'_1} dx \right)^{1/q'_1} \times \|g\chi_j\|_{L^{q_2}(\omega_2)} \\
 &\quad \cdot \left(\int_{B_j} |b_2(x) - \lambda_{2j}|^{q'_2} \omega_2(x)^{1-q'_2} dx \right)^{1/q'_2}. \tag{45}
 \end{aligned}$$

By Lemma 2 we know $\omega_2^{1-q'_2} \in A_{q'_2}$. Then, by Lemma 3, we have

$$\begin{aligned}
 &\left(\int_{B_j} |b_2(x) - \lambda_{2j}|^{q'_2} \omega_2(x)^{1-q'_2} dx \right)^{1/q'_2} \\
 &\leq C \|b_2\|_* \left(\int_{B_j} \omega_2(x)^{1-q'_2} dx \right)^{1/q'_2}. \tag{46}
 \end{aligned}$$

Then, by (46) and (16), we have

$$\begin{aligned}
 |L_2(x)| &\leq C \|b_2\|_* 2^{-2 \max\{i,j,k\}n} |b_1(x) - \lambda_{1i}| \\
 &\quad \cdot \|f\chi_i\|_{L^{q_1}(\omega_1)} \left(\int_{B_i} \omega_1(x)^{1-q'_1} dx \right)^{1/q'_1} \times \|g\chi_j\|_{L^{q_2}(\omega_2)} \\
 &\quad \cdot \left(\int_{B_j} \omega_2(x)^{1-q'_2} dx \right)^{1/q'_2} \\
 &\leq C 2^{-\max\{i,k\}n+in} |b_1(x) - \lambda_{1i}| \|f\chi_i\|_{L^{q_1}(\omega_1)} \\
 &\quad \cdot \omega_1(B_i)^{-1/q_1} \times \|b_2\|_* |k-j| \\
 &\quad \cdot 2^{-\max\{j,k\}n+jn} \|g\chi_j\|_{L^{q_2}(\omega_2)} \omega_2(B_j)^{-1/q_2}. \tag{47}
 \end{aligned}$$

Note that, for $0 < q < \infty$,

$$\begin{aligned}
 &\left(\int_{B_k} |b_1(x) - \lambda_{1i}|^q v_{\tilde{\omega}}(x) dx \right)^{1/q} \\
 &\leq C \|b_1\|_* |k-i| v_{\tilde{\omega}}(B_k)^{1/q} \\
 &\leq C \|b_1\|_* |k-i| \omega_1(B_k)^{1/q_1} \omega_2(B_k)^{1/q_2}. \tag{48}
 \end{aligned}$$

Thus,

$$\begin{aligned}
 v(B_k)^{\alpha/n} \|(L_2) \chi_k\|_{L^q(v_{\tilde{\omega}})} &\leq C \|b_1\|_* |k-i| \\
 &\quad \cdot 2^{-\max\{i,k\}n+in} v(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \left(\frac{v(B_k)}{v(B_i)} \right)^{\alpha_1/n}
 \end{aligned}$$

$$\begin{aligned}
 &\cdot \left(\frac{\omega_1(B_k)}{\omega_1(B_i)} \right)^{1/q_1} \times \|b_2\|_* |k-j| \\
 &\quad \cdot 2^{-\max\{j,k\}n+jn} v(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)} \left(\frac{v(B_k)}{v(B_j)} \right)^{\alpha_2/n} \\
 &\quad \cdot \left(\frac{\omega_2(B_k)}{\omega_2(B_j)} \right)^{1/q_2}. \tag{49}
 \end{aligned}$$

Similar to the estimates of (44), we get

$$\begin{aligned}
 &v(B_k)^{\alpha/n} \|(L_2) \chi_k\|_{L^q(v_{\tilde{\omega}})} \\
 &\leq C \|b_1\|_* D_1(k, i) v(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \\
 &\quad \times \|b_2\|_* D_2(k, j) v(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)}. \tag{50}
 \end{aligned}$$

By symmetry, we also get

$$\begin{aligned}
 &v(B_k)^{\alpha/n} \|(L_3) \chi_k\|_{L^q(v_{\tilde{\omega}})} \\
 &\leq C \|b_1\|_* D_1(k, i) v(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \\
 &\quad \times \|b_2\|_* D_2(k, j) v(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)}. \tag{51}
 \end{aligned}$$

Finally, it remains to estimate J_4 . Applying (29) and Hölder's inequality,

$$\begin{aligned}
 |L_4(x)| &= |T((b_1 - \lambda_{1i}) f\chi_i, (b_2 - \lambda_{2j}) g\chi_j)| \\
 &\leq C 2^{-2 \max\{i,j,k\}n} \|(b_1 - \lambda_{1i}) f\chi_i\|_{L^1} \\
 &\quad \cdot \|(b_2 - \lambda_{2j}) g\chi_j\|_{L^1} \leq C 2^{-2 \max\{i,j,k\}n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \\
 &\quad \cdot \left(\int_{B_i} |b_1(x) - \lambda_{1i}|^{q'_1} \omega_1(x)^{1-q'_1} dx \right)^{1/q'_1} \\
 &\quad \times \|g\chi_j\|_{L^{q_2}(\omega_2)} \\
 &\quad \cdot \left(\int_{B_j} |b_2(x) - \lambda_{2j}|^{q'_2} \omega_2(x)^{1-q'_2} dx \right)^{1/q'_2}. \tag{52}
 \end{aligned}$$

By (46) and (16),

$$\begin{aligned}
 |L_4(x)| &\leq C \|b_1\|_* 2^{-\max\{i,k\}n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \\
 &\quad \cdot \left(\int_{B_i} \omega_1(x)^{1-q'_1} dx \right)^{1/q'_1} \times \|b_2\|_* \\
 &\quad \cdot 2^{-\max\{j,k\}n} \|g\chi_j\|_{L^{q_2}(\omega_2)} \left(\int_{B_j} \omega_2(x)^{1-q'_2} dx \right)^{1/q'_2} \\
 &\leq C \|b_1\|_* 2^{-\max\{i,k\}n+in} \|f\chi_i\|_{L^{q_1}(\omega_1)} \omega_1(B_i)^{-1/q_1} \\
 &\quad \times \|b_2\|_* 2^{-\max\{j,k\}n+jn} \|g\chi_j\|_{L^{q_2}(\omega_2)} \omega_2(B_j)^{-1/q_2}. \tag{53}
 \end{aligned}$$

Then

$$\begin{aligned}
& \nu(B_k)^{\alpha/n} \|(L_4) \chi_k\|_{L^q(\nu_{\tilde{\omega}})} \leq C \|b_1\|_* \\
& \cdot 2^{-\max\{i,k\}n+in} \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \left(\frac{\nu(B_k)}{\nu(B_i)} \right)^{\alpha_1/n} \\
& \cdot \left(\frac{\omega_1(B_k)}{\omega_1(B_i)} \right)^{1/q_1} \times \|b_2\|_* 2^{-\max\{j,k\}n+jn} \nu(B_j)^{\alpha_2/n} \\
& \cdot \|g\chi_j\|_{L^{q_2}(\omega_2)} \left(\frac{\nu(B_k)}{\nu(B_j)} \right)^{\alpha_2/n} \left(\frac{\omega_2(B_k)}{\omega_2(B_j)} \right)^{1/q_2} \\
& \leq C \|b_1\|_* |k-i| 2^{-\max\{i,k\}n+in} \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \\
& \cdot \left(\frac{\nu(B_k)}{\nu(B_i)} \right)^{\alpha_1/n} \left(\frac{\omega_1(B_k)}{\omega_1(B_i)} \right)^{1/q_1} \times \|b_2\|_* |k-j| \\
& \cdot 2^{-\max\{j,k\}n+jn} \nu(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)} \left(\frac{\nu(B_k)}{\nu(B_j)} \right)^{\alpha_2/n} \\
& \cdot \left(\frac{\omega_2(B_k)}{\omega_2(B_j)} \right)^{1/q_2}.
\end{aligned} \tag{54}$$

Thus

$$\begin{aligned}
& \nu(B_k)^{\alpha/n} \|(L_4) \chi_k\|_{L^q(\nu_{\tilde{\omega}})} \\
& \leq C \|b_1\|_* D_1(k, i) \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \\
& \times \|b_2\|_* D_2(k, j) \nu(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)}.
\end{aligned} \tag{55}$$

Now, we give the proof of Theorem 1.

If $q > 1$, by Minkowski's inequality and Theorem 5, we have

$$\begin{aligned}
& \nu(B_k)^{\alpha/n} \|T_{\Pi b}(f, g) \chi_k\|_{L^q(\nu_{\tilde{\omega}})} \\
& \leq C \nu(B_k)^{\alpha/n} \left\| \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} T_{\Pi b}(f\chi_i, g\chi_j) \chi_k \right\|_{L^q(\nu_{\tilde{\omega}})} \\
& \leq C \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \nu(B_k)^{\alpha/n} \|T_{\Pi b}(f\chi_i, g\chi_j) \chi_k\|_{L^q(\nu_{\tilde{\omega}})} \\
& \leq C \|b_1\|_* \|b_2\|_* \sum_{i=-\infty}^{\infty} D_1(k, i) \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \\
& \times \sum_{j=-\infty}^{\infty} D_2(k, j) \nu(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)} \\
& \leq C \|b_1\|_* \|b_2\|_* \sum_{i=-\infty}^{\infty} D_1(k, i)^{1-\epsilon} \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \\
& \times \sum_{j=-\infty}^{\infty} D_2(k, j)^{1-\epsilon} \nu(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)}
\end{aligned} \tag{56}$$

for any $0 < \epsilon < 1$, since $D_1(k, i) + D_2(k, j) \rightarrow 0$ whenever $i, j \rightarrow \pm\infty$.

If $0 < q \leq 1$, note the fact that

$$\begin{aligned}
& \sum_{i=-\infty}^{\infty} D_1(k, i)^{q\epsilon/(1-q)} < \infty, \\
& \sum_{j=-\infty}^{\infty} D_2(k, j)^{q\epsilon/(1-q)} < \infty,
\end{aligned} \tag{57}$$

and then by Theorem 5, the inequality $(\sum |a_i|)^q \leq \sum |a_i|^q$, and Hölder's inequality, we have

$$\begin{aligned}
& \nu(B_k)^{\alpha/n} \|T_{\Pi b}(f, g) \chi_k\|_{L^q(\nu_{\tilde{\omega}})} \\
& \leq C \left\{ \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \left[\nu(B_k)^{\alpha/n} \|T_{\Pi b}(f\chi_i, g\chi_j) \chi_k\|_{L^q(\nu_{\tilde{\omega}})} \right]^q \right\}^{1/q} \\
& \leq C \|b_1\|_* \|b_2\|_* \left\{ \sum_{i=-\infty}^{\infty} D_1(k, i)^q \nu(B_i)^{\alpha_1 q/n} \|f\chi_i\|_{L^{q_1}(\omega_1)}^q \right. \\
& \times \left. \sum_{j=-\infty}^{\infty} D_2(k, j)^q \nu(B_j)^{\alpha_2 q/n} \|g\chi_j\|_{L^{q_2}(\omega_2)}^q \right\}^{1/q} \leq C \|b_1\|_* \\
& \cdot \|b_2\|_* \sum_{i=-\infty}^{\infty} D_1(k, i)^{1-\epsilon} \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \\
& \cdot \left\{ \sum_{j=-\infty}^{\infty} D_1(k, j)^{q\epsilon/(1-q)} \right\}^{(1-q)/q} \\
& \times \sum_{j=-\infty}^{\infty} D_2(k, j)^{1-\epsilon} \nu(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)} \\
& \cdot \left\{ \sum_{j=-\infty}^{\infty} D_2(k, j)^{q\epsilon/(1-q)} \right\}^{(1-q)/q} \leq C \|b_1\|_* \|b_2\|_* \\
& \cdot \sum_{i=-\infty}^{\infty} D_1(k, i)^{1-\epsilon} \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \\
& \times \sum_{j=-\infty}^{\infty} D_2(k, j)^{1-\epsilon} \nu(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)}.
\end{aligned} \tag{58}$$

Since $0 < p < \infty$, by (56), (58) and Hölder's inequality,

$$\begin{aligned}
& \|T_{\Pi b}(f, g)\|_{\dot{K}_q^{\alpha, p}(\nu, \nu_{\tilde{\omega}})(\mathbb{R}^n)} \leq \left\{ \sum_{k=-\infty}^{\infty} \nu(B_k)^{\alpha p/n} \right. \\
& \cdot \left. \|T_{\Pi b}(f, g) \chi_k\|_{L^q(\nu_{\tilde{\omega}})}^p \right\}^{1/p} \leq C \|b_1\|_* \|b_2\|_* \\
& \cdot \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{i=-\infty}^{\infty} D_1(k, i)^{1-\epsilon} \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \right) \right.
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{j=-\infty}^{\infty} D_2(k, j)^{1-\epsilon} \nu(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)} \Big)^p \Big\}^{1/p} \leq C \|b_1\|_* \\
& \cdot \|b_2\|_* \\
& \cdot \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{i=-\infty}^{\infty} D_1(k, i)^{1-\epsilon} \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \right)^{p_1} \right\}^{1/p_1} \\
& \times \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{\infty} D_2(k, j)^{1-\epsilon} \nu(B_j)^{\alpha_2/n} \|g\chi_j\|_{L^{q_2}(\omega_2)} \right)^{p_2} \right\}^{1/p_2} \\
& = C \|b_1\|_* \|b_2\|_* N_1 \times N_2.
\end{aligned} \tag{59}$$

It is enough to show that $N_1 \leq C \|f\|_{\dot{K}_{q_1}^{\alpha_1, p_1}(\nu, \omega_1)(\mathbb{R}^n)}$ and $N_2 \leq C \|g\|_{\dot{K}_{q_2}^{\alpha_2, p_2}(\nu, \omega_2)(\mathbb{R}^n)}$. By the symmetry, we only give the estimate for N_1 . Consider

$$\begin{aligned}
N_1^{p_1} & \leq \sum_{k=-\infty}^{\infty} \left(\sum_{i=-\infty}^{k-2} D_1(k, i)^{1-\epsilon} \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \right)^{p_1} \\
& + \sum_{k=-\infty}^{\infty} \left(\sum_{i=k-1}^{k+1} D_1(k, i)^{1-\epsilon} \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \right)^{p_1} \\
& + \sum_{k=-\infty}^{\infty} \left(\sum_{i=k+2}^{\infty} D_1(k, i)^{1-\epsilon} \nu(B_i)^{\alpha_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)} \right)^{p_1} \\
& = N_{11} + N_{12} + N_{13}.
\end{aligned} \tag{60}$$

If $0 < p_1 \leq 1$, then, by the inequality $(\sum |a_i|)^{p_1} \leq \sum |a_i|^{p_1}$ and the fact that

$$\begin{aligned}
& \sum_{k=i+2}^{\infty} D_1(k, i)^{(1-\epsilon)p_1} \\
& = \sum_{k=i+2}^{\infty} (k-i)^{(1-\epsilon)p_1} 2^{(1-\epsilon)p_1(k-i)(\alpha_1 r + nr_1/q_1 - n)} < \infty,
\end{aligned} \tag{61}$$

we get

$$\begin{aligned}
N_{11} & \leq \sum_{i=-\infty}^{\infty} \nu(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)}^{p_1} \sum_{k=i+2}^{\infty} D_1(k, i)^{(1-\epsilon)p_1} \\
& \leq C \sum_{i=-\infty}^{\infty} \nu(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)}^{p_1} \\
& \leq C \|f\|_{\dot{K}_{q_1}^{\alpha_1, p_1}(\nu, \omega_1)(\mathbb{R}^n)}^{p_1}.
\end{aligned} \tag{62}$$

Note that

$$\begin{aligned}
& \sum_{i=-\infty}^{k-2} D_1(k, i)^{1-\epsilon} \\
& = \sum_{k=i+2}^{\infty} D_1(k, i)^{1-\epsilon} \\
& \leq C \sum_{k=i+2}^{\infty} (k-i)^{(1-\epsilon)} 2^{(1-\epsilon)(k-i)(\alpha_1 r + nr_1/q_1 - n)} < \infty.
\end{aligned} \tag{63}$$

Then, for $p_1 > 1$, we have

$$\begin{aligned}
N_{11} & \leq C \sum_{k=-\infty}^{\infty} \left(\sum_{i=-\infty}^{k-2} D_1(k, i)^{1-\epsilon} \nu(B_i)^{\alpha_1 p_1/n} \right. \\
& \cdot \|f\chi_i\|_{L^{q_1}(\omega_1)}^{p_1} \Big) \left(\sum_{i=-\infty}^{k-2} D_1(k, i)^{1-\epsilon} \right)^{p_1/p_1'} \\
& \leq C \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{k-2} D_1(k, i)^{1-\epsilon} \nu(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)}^{p_1} \\
& \leq C \sum_{i=-\infty}^{\infty} \nu(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)}^{p_1} \sum_{k=i+2}^{\infty} D_1(k, i)^{1-\epsilon} \\
& \leq C \|f\|_{\dot{K}_{q_1}^{\alpha_1, p_1}(\nu, \omega_1)(\mathbb{R}^n)}^{p_1}.
\end{aligned} \tag{64}$$

Since $D_1(k, i) = 1$ for $i = k-1, k, k+1$, then

$$\begin{aligned}
N_{12} & \leq C \sum_{k=-\infty}^{\infty} \sum_{i=k-1}^{k+1} \nu(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)}^{p_1} \\
& \leq C \sum_{i=-\infty}^{\infty} \sum_{k=i-1}^{i+1} \nu(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)}^{p_1} \\
& \leq C \sum_{i=-\infty}^{\infty} \nu(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)}^{p_1} \\
& \leq C \|f\|_{\dot{K}_{q_1}^{\alpha_1, p_1}(\nu, \omega_1)(\mathbb{R}^n)}^{p_1}.
\end{aligned} \tag{65}$$

Thus, we have obtained that $N_{12} \leq C \|f\|_{\dot{K}_{q_1}^{\alpha_1, p_1}(\nu, \omega_1)(\mathbb{R}^n)}^{p_1}$ holds for any $0 < p_1 < \infty$.

Let us now turn to estimate the last term N_{13} . If $0 < p_1 \leq 1$, then

$$\begin{aligned}
N_{13} & \leq C \sum_{k=-\infty}^{\infty} \sum_{i=k+2}^{\infty} D_1(k, i)^{(1-\epsilon)p_1} \nu(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)}^{p_1} \\
& \leq C \sum_{i=-\infty}^{\infty} \nu(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)}^{p_1} \sum_{k=i-2}^{\infty} D_1(k, i)^{(1-\epsilon)p_1}.
\end{aligned} \tag{66}$$

But

$$\begin{aligned} & \sum_{k=-\infty}^{i-2} D_1(k, i)^{(1-\epsilon)p_1} \\ &= \sum_{k=-\infty}^{i-2} (i-k)^{(1-\epsilon)p_1} 2^{(1-\epsilon)p_1(k-i)(\delta\alpha_1/n+\delta_1/q_1)} < \infty. \end{aligned} \quad (67)$$

Then

$$\begin{aligned} N_{13} &\leq C \sum_{i=-\infty}^{\infty} \nu(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)}^{p_1} \\ &\leq C \|f\|_{\dot{K}_{q_1}^{\alpha_1, p_1}(\nu, \omega_1)(\mathbb{R}^n)}^{p_1}. \end{aligned} \quad (68)$$

Note that

$$\begin{aligned} & \sum_{k=-\infty}^{i-2} D_1(k, i)^{1-\epsilon} \\ &= \sum_{i=k+2}^{\infty} D_1(k, i)^{1-\epsilon} \\ &\leq C \sum_{i=k+2}^{\infty} (i-k)^{1-\epsilon} 2^{(1-\epsilon)(k-i)(\delta\alpha_1/n+\delta_1/q_1)} < \infty. \end{aligned} \quad (69)$$

Then, for $p > 1$, we have

$$\begin{aligned} N_{13} &\leq C \sum_{k=-\infty}^{\infty} \left(\sum_{i=k+2}^{\infty} D_1(k, i)^{1-\epsilon} \nu(B_i)^{\alpha_1 p_1/n} \right. \\ &\quad \cdot \|f\chi_i\|_{L^{q_1}(\omega_1)}^{p_1} \left. \right)^{p_1/p'_1} \\ &\leq C \sum_{k=-\infty}^{\infty} \sum_{i=k+2}^{\infty} D_1(k, i)^{1-\epsilon} \nu(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)}^{p_1} \\ &\leq C \sum_{i=-\infty}^{\infty} \nu(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{L^{q_1}(\omega_1)}^{p_1} \sum_{k=-\infty}^{i-2} D_1(k, i)^{1-\epsilon} \\ &\leq C \|f\|_{\dot{K}_{q_1}^{\alpha_1, p_1}(\nu, \omega_1)(\mathbb{R}^n)}^{p_1}. \end{aligned} \quad (70)$$

This completes the proof of Theorem 1. $\square$

Remark 6. Using Theorem 5 and the methods in [18], we can easily get the boundedness of the iterated commutator $T_{\Pi b}$ on weighted Herz-Morrey spaces.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors are very grateful to the anonymous referees and the editor for their insightful comments and suggestions.

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