Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2016, Article ID 1035253, 9 pages
http://dx.doi.org/10.1155/2016/1035253

*Research Article*

# On a Kantorovich Variant of $(p, q)$-Szász-Mirakjan Operators

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Received 23 October 2015; Accepted 31 December 2015

Academic Editor: Pasquale Vetro



We propose a Kantorovich variant of $(p, q)$-analogue of Szász-Mirakjan operators. We establish the moments of the operators with the help of a recurrence relation that we have derived and then prove the basic convergence theorem. Next, the local approximation and weighted approximation properties of these new operators in terms of modulus of continuity are studied.

## 1. Introduction and Notations

Approximation theory has been an established field of mathematics in the past century. The approximation of functions by positive linear operators is an important research topic in general mathematics and it also provides powerful tools to application areas such as computer-aided geometric design, numerical analysis, and solution of differential equations.

During the last two decades, the applications of $q$-calculus emerged as a new area in the field of approximation theory. The rapid development of $q$-calculus has led to the discovery of various generalizations of Bernstein polynomials involving $q$-integers. Several researchers introduced and studied many positive linear operators based on $q$-integers, $q$-Bernstein basis, $q$-beta basis, $q$-derivative, $q$-integrals, and so forth. Using $q$-integers, Lupaş [1] introduced the first $q$-Bernstein operators [2] and investigated their approximating and shape-preserving properties. Another $q$-analogue of the Bernstein polynomials is due to Phillips [3]. Since then several generalizations of well-known positive linear operators based on $q$-integers have been introduced and their approximation properties studied. Aral [4] and Aral and Gupta [5] proposed and studied some $q$-analogue of Szász-Mirakjan operators [6], defined on positive real axis. Also Mahmudov [7] introduced $q$-parametric Szász-Mirakjan operators and studied their convergence properties. Recently, approximation properties for King's type $q$-Bernstein-Kantorovich operators have been studied in [8].

Very recently, Mursaleen et al. applied $(p, q)$-calculus in approximation theory and introduced the $(p, q)$-analogue of Bernstein operators [9], $(p, q)$-Bernstein-Stancu operators [10], and $(p, q)$-Bernstein-Schurer operators [11] and investigated their approximation properties. Also Acar [12] has introduced $(p, q)$ parametric generalization of Szász-Mirakjan operators. In the present work, we define a Kantorovich variant of Szász-Mirakjan operators and establish the moments with the help of a recurrence relation that we have derived and then prove the basic convergence theorem. Next, the local approximation as well as weighted approximation properties of these new operators in terms of modulus of continuity are studied.

The $(p, q)$-integer was introduced in order to generalize or unify several forms of $q$-oscillator algebras well known in the earlier physics literature related to the representation theory of single parameter quantum algebras [13]. Let us recall certain notations of $(p, q)$-calculus.

The $(p, q)$-integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} := \frac{p^n - q^n}{p - q}, \quad n = 0, 1, 2, \ldots, \ 0 < q < p \leq 1. \quad (1)$$

The $(p,q)$-facorial and $(p,q)$-Binomial coefficients are defined by

$$[n]_{p,q}! := \begin{cases} [n]_{p,q}[n-1]_{p,q}\cdots[1]_{p,q}, & n\in\mathbb{N}; \\ 1, & n=0, \end{cases}$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \tag{2}$$

respectively. Further, the $(p,q)$-binomial expansions are given as

$$(ax+by)_{p,q}^{n} := \sum_{k=0}^{n} p^{\binom{n-k}{2}} q^{\binom{k}{2}} a^{n-k} b^{k} x^{n-k} y^{k},$$
$$(x-y)_{p,q}^{n} := (x-y)(px-qy) \cdot \left(p^{2}x-q^{2}y\right)\cdots\left(p^{n-1}x-q^{n-1}y\right). \tag{3}$$

Let $m$ and $n$ be two nonnegative integers. Then the following assertion is valid:

$$(x-y)_{p,q}^{m+n} := (x-y)_{p,q}^{m}\left(p^{m}x-q^{m}y\right)_{p,q}^{n}. \tag{4}$$

Also, the $(p,q)$-derivative of a function $f$, denoted by $D_{p,q}f$, is defined by

$$\left(D_{p,q}f\right)(x) := \frac{f(px)-f(qx)}{(p-q)x}, \quad x\neq 0,$$
$$\left(D_{p,q}f\right)(0) := f'(0), \tag{5}$$

provided that $f$ is differentiable at 0. The $(p,q)$-derivative fulfills the following product rules:

$$D_{p,q}(f(x)g(x)) := f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x),$$
$$D_{p,q}(f(x)g(x)) := f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x). \tag{6}$$

Moreover,

$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) := \frac{g(qx)D_{p,q}f(x)-f(qx)D_{p,q}g(x)}{g(px)g(qx)},$$
$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) := \frac{g(px)D_{p,q}f(x)-f(px)D_{p,q}g(x)}{g(px)g(qx)}. \tag{7}$$

We consider the $(p,q)$-exponential functions in the following forms:

$$e_{p,q}(x) = \sum_{n=0}^{\infty} p^{n(n-1)/2}\frac{x^{n}}{[n]_{p,q}!},$$
$$E_{p,q}(x) = \sum_{n=0}^{\infty} q^{n(n-1)/2}\frac{x^{n}}{[n]_{p,q}!}, \tag{8}$$

which satisfy the equality $e_{p,q}(x)E_{p,q}(-x) = 1$. The definite integrals of the function $f$ are defined by

$$\int_0^a f(x)\,d_{p,q}x = (q-p)\,a\sum_{k=0}^{\infty}\frac{p^{k}}{q^{k+1}} f\left(\frac{p^{k}}{q^{k+1}}a\right), \quad \text{when } \left|\frac{p}{q}\right| < 1,$$
$$\int_0^a f(x)\,d_{p,q}x = (p-q)\,a\sum_{k=0}^{\infty}\frac{q^{k}}{p^{k+1}} f\left(\frac{q^{k}}{p^{k+1}}a\right), \quad \text{when } \left|\frac{p}{q}\right| > 1. \tag{9}$$

Details on $(p,q)$-calculus can be found in [13, 14]. For $p = 1$, all the notions of $(p,q)$-calculus are reduced to $q$-calculus.

## 2. Operators and Estimation of Moments

Now we set the $(p,q)$-Szász-Mirakjan basis function as

$$s_n(p,q;x) =: E_{p,q}\left(-[n]_{p,q}x\right)\sum_{k=0}^{\infty} q^{k(k-1)/2}\frac{\left([n]_{p,q}x\right)^{k}}{[k]_{p,q}!}. \tag{10}$$

For $q\in(0,p)$, $p\in(q,1]$, and $x\in[0,\infty)$, $s_n(p,q;x)\geq 0$. We can easily check that

$$\sum_{k=0}^{\infty} s_n(p,q;x) =: E_{p,q}\left(-[n]_{p,q}x\right)\sum_{k=0}^{\infty} q^{k(k-1)/2}\frac{\left([n]_{p,q}x\right)^{k}}{[k]_{p,q}!} = 1. \tag{11}$$

For $0 < q < p \leq 1$, the $(p,q)$-Szász-Mirakjan operators are defined as

$$S_n(f,p,q;x) = [n]_{p,q}\sum_{k=0}^{\infty} p^{-k}q^{k}s_{n,k}(p,q;x)\, f\left(\frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}}\right), \quad x\in[0,\infty). \tag{12}$$

From the definition of the $(p,q)$-Szász-Mirakjan operators we derive the following formulas.

**Lemma 1.** *Let $0 < q < p \leq 1$. One has*

(i) $S_n(1,p,q;x) = 1$;

(ii) $S_n(t,p,q;x) = x$;

(iii) $S_n(t^2,p,q;x) = px^2/q + x/[n]_{p,q}$;

(iv) $S_n(t^3,p,q;x) = (p^3/q^3)x^3 + ((p^2+2pq)/q[n]_{p,q})x^2 + (q^2/[n]_{p,q}^2)x$;

(v) $S_n(t^4,p,q;x) = (p^6/q^6)x^4 + (p^3(p^2+2q+3q^2)/q^4[n]_{p,q})x^3 + (p(p^2+3pq+3q^2)/q[n]_{p,q})x^2 + (q^3/[n]_{p,q}^3)x$.

Now we propose our Kantorovich variant of $(p,q)$-Szász-Mirakjan operators (12) as follows.

For $f \in C[0,\infty)$, $0 < q < p \le 1$, and each positive integer $n$,

$$K_n(f,p,q;x) = [n]_{p,q} \cdot \sum_{k=0}^{\infty} p^{-k} q^k s_{n,k}(p,q;x) \int_{[k]_{p,q}/q^{k-1}[n]_{p,q}}^{[k+1]_{p,q}/q^k[n]_{p,q}} f(t)\, d_{p,q}t, \tag{13}$$

where $f$ is a nondecreasing function. We will derive the recurrence formula for $K_n(t^m,p,q;x)$ and calculate $K_n(t^m,p,q;x)$ for $m = 0,1,2$.

**Lemma 2.** *For the operators $K_n$, one has*

$$K_n(t^m,p,q;x) = \frac{1}{[m+1]_{p,q}} \sum_{j=0}^{m} \sum_{i=0}^{j} \frac{p^i}{q^i [n]_{p,q}^{j-i}} \binom{j}{i} S_n\left(t^{m+i-j},p,q;x\right). \tag{14}$$

*Proof.* Using the expansion $a^{m+1} - b^{m+1} = (a-b)(a^m + a^{m-1}b + \cdots + ab^{m-1} + b^m)$, we have

$$\int_{[k]_{p,q}/q^{k-1}[n]_{p,q}}^{[k+1]_{p,q}/q^k[n]_{p,q}} t^m d_{p,q}t = \frac{1}{[m+1]_{p,q}} \left\{ \left( \frac{[k+1]_{p,q}}{q^k [n]_{p,q}} \right)^{m+1} - \left( \frac{[k]_{p,q}}{q^{k-1} [n]_{p,q}} \right)^{m+1} \right\}. \tag{15}$$

Using $[k+1]_{p,q} = p^k + q[k]_{p,q}$ and also $[k+1]_{p,q} = q^k + p[k]_{p,q}$, we have

$$\begin{aligned}
\int_{[k]_{p,q}/q^{k-1}[n]_{p,q}}^{[k+1]_{p,q}/q^k[n]_{p,q}} t^m d_{p,q}t &= \frac{1}{[m+1]_{p,q}} \frac{p^k}{q^k [n]_{p,q}} \cdot \sum_{j=0}^{m} \left( \frac{[k+1]_{p,q}}{q^k [n]_{p,q}} \right)^j \left( \frac{[k]_{p,q}}{q^{k-1} [n]_{p,q}} \right)^{m-j} \\
&= \frac{1}{[m+1]_{p,q}} \frac{p^k}{q^k [n]_{p,q}} \cdot \sum_{j=0}^{m} \left( \frac{q^k + p[k]_{p,q}}{q^k [n]_{p,q}} \right)^j \left( \frac{[k]_{p,q}}{q^{k-1} [n]_{p,q}} \right)^{m-j} \\
&= \frac{1}{[m+1]_{p,q}} \frac{p^k}{q^k [n]_{p,q}} \cdot \sum_{j=0}^{m} \sum_{i=0}^{j} \binom{j}{i} \frac{p^i [k]_{p,q}^i q^{k(j-i)}}{q^{kj} [n]_{p,q}^j} \frac{[k]_{p,q}^{m-j}}{q^{(k-1)(m-j)} [n]_{p,q}^{m-j}} \\
&= \frac{1}{[m+1]_{p,q}} \frac{p^k}{q^k [n]_{p,q}} \cdot \sum_{j=0}^{m} \sum_{i=0}^{j} \binom{j}{i} \frac{p^i [k]_{p,q}^{m+i-j}}{q^{ki} [n]_{p,q}^m q^{(k-1)(m-j)}}.
\end{aligned} \tag{16}$$

Writing this in the definition of $K_n(t^m,p,q;x)$, we get

$$\begin{aligned}
K_n(t^m,p,q;x) &= [n]_{p,q} \cdot \sum_{k=0}^{\infty} p^{-k} q^k s_{n,k}(p,q;x) \int_{[k]_{p,q}/q^{k-1}[n]_{p,q}}^{[k+1]_{p,q}/q^k[n]_{p,q}} t^m d_{p,q}t \\
&= \frac{1}{[m+1]_{p,q}} \cdot \sum_{j=0}^{m} \sum_{k=0}^{\infty} s_{n,k}(p,q;x) \sum_{i=0}^{j} \frac{p^i}{q^i [n]_{p,q}^{j-i}} \binom{j}{i} \frac{[k]_{p,q}^{m+i-j}}{q^{(k-1)(m+i-j)} [n]_{p,q}^{m+i-j}} \\
&= \frac{1}{[m+1]_{p,q}} \cdot \sum_{j=0}^{m} \sum_{i=0}^{j} \frac{p^i}{q^i [n]_{p,q}^{j-i}} \binom{j}{i} \sum_{k=0}^{\infty} \frac{[k]_{p,q}^{m+i-j}}{q^{(k-1)(m+i-j)} [n]_{p,q}^{m+i-j}} s_{n,k}(p,q;x) \\
&= \frac{1}{[m+1]_{p,q}} \sum_{j=0}^{m} \sum_{i=0}^{j} \frac{p^i}{q^i [n]_{p,q}^{j-i}} \binom{j}{i} S_n\left(t^{m+i-j},p,q;x\right).
\end{aligned} \tag{17}$$

Using recurrence formula (14), we may easily calculate $K_n(t^m,p,q;x)$ for $m = 0,1,2$. □

**Lemma 3.** *One has*

(i) $K_n(1,p,q;x) = 1$;

(ii) $K_n(t,p,q;x) = \dfrac{1}{q}x + \dfrac{1}{[2]_{p,q}[n]_{p,q}}$;

(iii) $K_n(t^2,p,q;x) = \dfrac{p}{q^3}x^2 + \left( \dfrac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}} + \dfrac{1}{q^2[n]_{p,q}} \right) x + \dfrac{1}{[3]_{p,q}[n]_{p,q}^2}$;

(iv) $K_n(t^3,p,q;x) = \dfrac{p^3}{q^6}x^3 + \left( \dfrac{p^2+2pq}{q^4[n]_{p,q}} + \dfrac{p(3p^2+2pq+q^2)}{q^3[4]_{p,q}[n]_{p,q}} \right) x^2 + \left( \dfrac{1}{q[n]_{p,q}^2} + \dfrac{3p^2+2pq+q^2}{q^2[4]_{p,q}[n]_{p,q}^2} + \dfrac{3p+q}{q[4]_{p,q}[n]_{p,q}^2} \right) x + \dfrac{1}{[4]_{p,q}[n]_{p,q}^3}$;

(v) $K_n(t^4,p,q;x) = \dfrac{p^6}{q^{10}}x^4 + \left( \dfrac{p^3(p^2+2q+3q^2)}{q^8[n]_{p,q}} + \dfrac{p^3(4p^3+3p^2q+2pq^2+q^3)}{q^6[5]_{p,q}[n]_{p,q}} \right) x^3$

$$+\left(\frac{p\left(p^2+3pq+3q^2\right)}{q^5\,[n]_{p,q}^2}+\frac{\left(p^2+2pq\right)\left(4p^3+3p^2q+2pq^2+q^3\right)}{q^4\,[5]_{p,q}\,[n]_{p,q}^2}+\frac{p\left(6p^2+3pq+q^2\right)}{q^3\,[5]_{p,q}\,[n]_{p,q}^2}\right)x^2+\left(\frac{1}{q\,[n]_{p,q}^3}+\frac{4p^3+3p^2q+2pq^2+q^3}{q\,[5]_{p,q}\,[n]_{p,q}^3}+\frac{6p^2+3pq+q^2}{q^2\,[5]_{p,q}\,[n]_{p,q}^3}+\frac{4p+q}{q\,[5]_{p,q}\,[n]_{p,q}^3}\right)x+\frac{1}{[5]_{p,q}\,[n]_{p,q}^4};$$

(vi) $$K_n\left((t-x),p,q;x\right)=\frac{1-q}{q}x+\frac{1}{[2]_{p,q}\,[n]_{p,q}}; \tag{18}$$

(vii) $$K_n\left((t-x)^2,p,q;x\right)=\left(\frac{p}{q^3}-\frac{2}{q}+1\right)x^2+\left(\frac{p+[2]_{p,q}}{q\,[3]_{p,q}\,[n]_{p,q}}+\frac{1}{q^2\,[n]_{p,q}}-\frac{2}{[2]_{p,q}\,[n]_{p,q}}\right)x+\frac{1}{[3]_{p,q}\,[n]_{p,q}^2}; \tag{19}$$

(viii) $$K_n\left((t-x)^4,p,q;x\right)=x^4\left(\frac{p^6}{q^{10}}-\frac{4p^3}{q^6}+\frac{6p}{q^3}-\frac{4}{q}+1\right)+\frac{x^3}{[n]_{p,q}}\left(\frac{p^3\left(p^2+2q+3q^2\right)}{q^8}+\frac{p^3\left(4p^3+3p^2q+2pq^2+q^3\right)}{q^6\,[5]_{p,q}}-\frac{4\left(p^2+2pq\right)}{q^4}-\frac{4p\left(3p^2+2pq+q^2\right)}{q^3\,[4]_{p,q}}+\frac{6\left(2p+q\right)}{q\,[3]_{p,q}}+\frac{6}{q^2}-\frac{4}{[2]_{p,q}}\right)+\frac{x^2}{[n]_{p,q}^2}\left(\frac{p\left(p^2+3pq+3q^2\right)}{q^5}+\frac{\left(p^2+2pq\right)\left(4p^3+3p^2q+2pq^2+q^3\right)}{q^4\,[5]_{p,q}}+\frac{p\left(6p^2+3pq+q^2\right)}{q^3\,[5]_{p,q}}-\frac{4\left(3p^2+2pq+q^2\right)}{q^2}-\frac{4}{q}+\frac{6}{[3]_{p,q}}\right)+\frac{x}{[n]_{p,q}^3}\left(\frac{1}{q}+\frac{4p^3+3p^2q+2pq^2+q^3}{q\,[5]_{p,q}}+\frac{6p^2+3pq+q^2}{q^2\,[5]_{p,q}}+\frac{4p+q}{q\,[5]_{p,q}}\right). \tag{20}$$

*Proof.* Obviously, with the help of Lemma 1, we can get

$$\begin{aligned}
K_n\left(t,p,q;x\right)&=\frac{1}{[2]_{p,q}}\left\{\left(1+\frac{p}{q}\right)S_n\left(t,p,q;x\right)+\frac{1}{[n]_{p,q}}S_n\left(1,p,q;x\right)\right\}=\frac{1}{q}x+\frac{1}{[2]_{p,q}\,[n]_{p,q}},\\
K_n\left(t^2,p,q;x\right)&=\frac{1}{[3]_{p,q}}\left\{\left(1+\frac{p}{q}+\frac{p^2}{q^2}\right)S_n\left(t^2,p,q;x\right)+\left(\frac{1}{[n]_{p,q}}+\frac{2p}{q\,[n]_{p,q}}\right)S_n\left(t,p,q;x\right)+\frac{1}{[n]_{p,q}^2}S_n\left(1,p,q;x\right)\right\}=\frac{1}{q^2}S_n\left(t^2,p,q;x\right)\\
&\quad+\frac{p+[2]_{p,q}}{q\,[3]_{p,q}\,[n]_{p,q}}S_n\left(t,p,q;x\right)+\frac{1}{[3]_{p,q}\,[n]_{p,q}^2}\cdot S_n\left(1,p,q;x\right)=\frac{p}{q^3}x^2+\left(\frac{p+[2]_{p,q}}{q\,[3]_{p,q}\,[n]_{p,q}}+\frac{1}{q^2\,[n]_{p,q}}\right)x+\frac{1}{[3]_{p,q}\,[n]_{p,q}^2}.
\end{aligned} \tag{21}$$

Using the linearity of the operators, we can have

$$\begin{aligned}
K_n\left((t-x)^2,p,q;x\right)&=K_n\left(t^2,p,q;x\right)-2xK_n\left(t,p,q;x\right)+x^2K_n\left(1,p,q;x\right)\\
&=\frac{p}{q^3}x^2+\left(\frac{p+[2]_{p,q}}{q\,[3]_{p,q}\,[n]_{p,q}}+\frac{1}{q^2\,[n]_{p,q}}\right)x+\frac{1}{[3]_{p,q}\,[n]_{p,q}^2}-2x\left(\frac{1}{q}x+\frac{1}{[2]_{p,q}\,[n]_{p,q}}\right)+x^2\\
&=\left(\frac{p}{q^3}-\frac{2}{q}+1\right)x^2+\left(\frac{p+[2]_{p,q}}{q\,[3]_{p,q}\,[n]_{p,q}}+\frac{1}{q^2\,[n]_{p,q}}-\frac{2}{[2]_{p,q}\,[n]_{p,q}}\right)x+\frac{1}{[3]_{p,q}\,[n]_{p,q}^2}.
\end{aligned} \tag{22}$$

□

*Remark 4.* For $q\in(0,1)$ and $p\in(q,1]$, it is obvious that (i) when $p=1$, $\lim_{n\to\infty}[n]_{p,q}=\lim_{n\to\infty}((1-q^n)/(1-q))=1/(1-q)$, and (ii) when $p<1$, $\lim_{n\to\infty}[n]_{p,q}=\lim_{n\to\infty}((p^n-q^n)/(p-q))=0$. In order to reach convergence

results of the operator $K_n$, we take sequences $q_n \in (0,1)$ and $p_n \in (q_n, 1]$ such that $\lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} q_n = 1$. So we get that $\lim_{n\to\infty} [n]_{p_n,q_n} = \infty$.

Thus the above remark provides an example that such a sequence can always be constructed. If we choose for $a > b > 0$, $q_n = n/(n+a) < n/(n+b) = p_n$ such that $0 < q_n < p_n \le 1$, it can be easily seen that $\lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} p_n^n = e^{-b}$, $\lim_{n\to\infty} q_n^n = e^{-a}$. Hence we guarantee that $\lim_{n\to\infty} [n]_{p_n,q_n} = \infty$.

## 3. Direct Approximation Results

In this section we study Korovkin's approximation property of the Kantorovich variant of $(p,q)$-Szász operators.

**Theorem 5.** *Let $0 < q_n < p_n \le 1$ and $A > 0$. Then for each $f \in C_m[0,\infty) := \{f \in C[0,\infty) : |f(x)| \le M_f(1+x^m)$, for some $M_f > 0$ depending on $f$, $m > 0\}$ where $C_m[0,\infty)$ be endowed with the norm $\|f\|_m = \sup_{x\in[0,\infty)} |f(x)|/(1+x^m)$, the sequence of operators $K_n(f, p_n, q_n; x)$ converges to $f$ uniformly on $[0, A]$ if and only if $\lim_{n\to\infty} p_n = 1$ and $\lim_{n\to\infty} q_n = 1$.*

*Proof.* First, we assume that $\lim_{n\to\infty} p_n = 1$ and $\lim_{n\to\infty} q_n = 1$. Now, we have to show that $K_n(f, p_n, q_n; x)$ converges to $f$ uniformly on $[0, A]$.

From Lemma 3, we see that

$$\begin{aligned} K_n(1, p_n, q_n; x) &\longrightarrow 1, \\ K_n(t, p_n, q_n; x) &\longrightarrow x, \\ K_n(t^2, p_n, q_n; x) &\longrightarrow x^2 \end{aligned} \tag{23}$$

uniformly on $[0, A]$ as $n \to \infty$.

Therefore, the well-known property of the Korovkin theorem implies that $K_n(f, p_n, q_n; x)$ converges to $f$ uniformly on $[0, A]$ provided $f \in C_m[0,\infty)$.

We show the converse part by contradiction. Assume that $p_n$ and $q_n$ do not converge to 1. Then they must contain subsequences $p_{n_k} \in (0,1)$, $q_{n_k} \in (0,1)$, $p_{n_k} \to a \in [0,1)$, and $q_{n_k} \to b \in [0,1)$ as $k \to \infty$, respectively.

Thus,

$$\frac{1}{[n_k]_{p_{n_k},q_{n_k}}} = \frac{p_{n_k} - q_{n_k}}{(p_{n_k})^{n_k} - (q_{n_k})^{n_k}} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty \tag{24}$$

and we get

$$\begin{aligned} K_n(t, p_{n_k}, q_{n_k}; x) - x &= \frac{1}{q_{n_k}} x + \frac{1}{[2]_{p_{n_k},q_{n_k}} [n]_{p_{n_k},q_{n_k}}} \\ &\quad - x \longrightarrow \frac{x}{b} - x \neq 0. \end{aligned} \tag{25}$$

This leads to a contradiction. Thus $p_n \to 1$ and $q_n \to 1$ as $n \to \infty$. □

**Theorem 6.** *Let $f \in C_2[0,\infty)$, $q = q_n \in (0,1)$, and $p = p_n \in (q, 1]$ such that $p_n \to 1$, $q_n \to 1$ as $n \to \infty$ and let $\omega_{a+1}(f,\delta)$ be the modulus of continuity on the finite interval $[0, a+1] \subset [0,\infty)$, where $a > 0$. Then*

$$\begin{aligned} &|K_n(f, p, q; x) - f(x)| \\ &\quad \le 4M_f(1+a^2)\delta_n^2(x) + 2\omega_{a+1}(f, \delta_n(x)), \end{aligned} \tag{26}$$

*where $\delta_n(x) = \sqrt{K_n((t-x)^2, p, q; x)}$, given by (19).*

*Proof.* For $x \in [0,a]$ and $t > a+1$, since $t - x > 1$, we have

$$\begin{aligned} |f(t) - f(x)| &\le M_f(2 + x^2 + t^2) \\ &\le M_f(2 + 3x^2 + 2(t-x)^2) \\ &\le M_f(4 + 3x^2)(t-x)^2 \\ &\le 4M_f(1+a^2)(t-x)^2. \end{aligned} \tag{27}$$

For $x \in [0,a]$ and $t \le a+1$, we have

$$\begin{aligned} |f(t) - f(x)| &\le \omega_{a+1}(f, |t-x|) \\ &\le \left(1 + \frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta) \end{aligned} \tag{28}$$

with $\delta > 0$.

From (27) and (28), we may write

$$\begin{aligned} |f(t) - f(x)| &\le 4M_f(1+a^2)(t-x)^2 \\ &\quad + \left(1 + \frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta), \end{aligned} \tag{29}$$

for $x \in [0,a]$ and $t \ge 0$. Thus, by applying Cauchy-Schwarz's inequality, we have

$$\begin{aligned} &|K_n(f, p, q; x) - f(x)| \le K_n(|f(t) - f(x)|, p, q; x) \\ &\quad \le 4M_f(1+a^2) K_n((t-x)^2, p, q; x) \\ &\qquad + \left(1 + \frac{1}{\delta}\sqrt{K_n((t-x)^2, p, q; x)}\right)\omega_{a+1}(f,\delta) \\ &\quad \le 4M_f(1+a^2)\delta_n^2(x) + 2\omega_{a+1}(f, \delta_n(x)) \end{aligned} \tag{30}$$

on choosing $\delta := \delta_n(x)$. This completes the proof of the theorem. □

## 4. Local Approximation

In this section we establish local approximation theorem for the Kantorovich variant of $(p,q)$-Szász operators. Let $C_B[0,\infty)$ be the space of all real-valued continuous bounded functions $f$ on $[0,\infty)$, endowed with the norm $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$. Peetre's $K$-functional is defined by

$$K_2(f,\delta) = \inf_{g\in C^2[0,\infty)} \{\|f - g\| + \delta\|g''\|\}, \tag{31}$$

where $C_B^2[0,\infty) := \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By [2, p.177, Theorem 2.4], there exists an absolute constant $M > 0$ such that

$$K_2(f,\delta) \le M\omega_2\left(f, \sqrt{\delta}\right), \tag{32}$$

where $\delta > 0$ and the second-order modulus of smoothness is defined as

$$\begin{aligned}&\omega_2(f,\delta)\\ &\quad = \sup_{0<h\le\delta}\sup_{x\in[0,\infty)} \left|f(x+2h) - 2f(x+h) + f(x)\right|,\end{aligned} \tag{33}$$

where $f \in C_B[0,\infty)$ and $\delta > 0$.

**Theorem 7.** *Let $f \in C_B[0,\infty)$ and $0 < q < p \le 1$. Then, for every $x \in [0,\infty)$, one has*

$$\begin{aligned}&\left|K_n(f,p,q;x) - f(x)\right|\\ &\quad \le M\omega_2\left(f, \sqrt{\delta_n(x)}\right)\\ &\qquad + \omega\left(f, \frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1-q}{q}x\right),\end{aligned} \tag{34}$$

*where $M$ is an absolute constant and*

$$\begin{aligned}\delta_n(x) &= K_n\left((t-x)^2, p, q; x\right)\\ &\quad + \left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1-q}{q}x\right)^2.\end{aligned} \tag{35}$$

*Proof.* For $x \in [0,\infty)$, we consider the auxiliary operators $K_n^*$ defined by

$$\begin{aligned}K_n^*(f,p,q;x) &= K_n(f,p,q;x)\\ &\quad - f\left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x\right) + f(x).\end{aligned} \tag{36}$$

From Lemma 3, we observe that the operators $K_n^*(f,p,q;x)$ are linear and reproduce the linear functions. Hence

$$\begin{aligned}K_n^*((t-x),p,q;x) &= K_n((t-x),p,q;x)\\ &\quad - \left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x - x\right)\\ &= K_n(t,p,q;x)\\ &\quad - xK_n(1,p,q;x)\\ &\quad - \left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x\right) + x\\ &= 0.\end{aligned} \tag{37}$$

Let $x \in [0,\infty)$ and $g \in C_B^2[0,\infty)$. Using Taylor's formula,

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)\, g''(u)\, du. \tag{38}$$

Applying $K_n^*$ to both sides of the above equation and using (37), we have

$$\begin{aligned}&K_n^*(g,p,q;x) - g(x) = K_n^*\left((t-x)\, g'(x), p, q; x\right)\\ &\quad + K_n^*\left(\int_x^t (t-u)\, g''(u)\, du, p, q; x\right) = g'(x)\\ &\quad \cdot K_n^*((t-x), p, q; x)\\ &\quad + K_n^{(p,q)}\left(\int_x^t (t-u)\, g''(u)\, du, p, q; x\right)\\ &\quad - \int_x^{1/[2]_{p,q}[n]_{p,q}+(1/q)x} \left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x - u\right)\\ &\quad \cdot g''(u)\, du = K_n\left(\int_x^t (t-u)\, g''(u)\, du, p, q; x\right)\\ &\quad - \int_x^{1/[2]_{p,q}[n]_{p,q}+(1/q)x} \left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x - u\right)\\ &\quad \cdot g''(u)\, du.\end{aligned} \tag{39}$$

On the other hand, since

$$\begin{aligned}&\left|\int_x^t (t-u)\, g''(u)\, du\right| \le \int_x^t |t-u| \left|g''(u)\right| du \le \left\|g''\right\|\\ &\quad \cdot \int_x^t |t-u|\, du \le (t-x)^2 \left\|g''\right\|,\\ &\left|\int_x^{1/[2]_{p,q}[n]_{p,q}+(1/q)x} \left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x - u\right)\right.\\ &\quad \left.\cdot g''(u)\, du\right| \le \left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x - u\right)^2 \left\|g''\right\|,\end{aligned} \tag{40}$$

we conclude that

$$\begin{aligned}&\left|K_n^*(g,p,q;x) - g(x)\right|\\ &\quad = \left|K_n\left(\int_x^t (t-u)\, g''(u)\, du, p, q; x\right)\right.\\ &\quad - \int_x^{1/[2]_{p,q}[n]_{p,q}+(1/q)x} \left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x - u\right)\\ &\quad \left.\cdot g''(u)\, du\right| \le \left\|g''\right\| K_n\left((t-x)^2, p, q; x\right)\\ &\quad + \left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}x - x\right)^2 \left\|g''\right\| = \delta_n(x) \left\|g''\right\|.\end{aligned} \tag{41}$$

Now, taking into account boundedness of $K_n^*$ by (36), we have

$$\left|K_n^*(f,p,q;x)\right| \le \left|K_n(f,p,q;x)\right| + 2\left\|f\right\| \le 3\left\|f\right\|. \tag{42}$$

Using (41) and (42) in (36), we obtain

$$\begin{aligned}
&\left|K_n(f,p,q;x)-f(x)\right| \\
&\quad\le \left|K_n^*(f,p,q;x)-f(x)\right| \\
&\qquad+\left|f(x)-f\left(\frac{1}{[2]_{p,q}[n]_{p,q}}+\frac{1}{q}x\right)\right| \\
&\quad\le \left|K_n^*(f-g,p,q;x)-(f-g)(x)\right| \\
&\qquad+\left|f(x)-f\left(\frac{1}{[2]_{p,q}[n]_{p,q}}+\frac{1}{q}x\right)\right| \\
&\qquad+\left|K_n^*(g,p,q;x)-g(x)\right| \\
&\quad\le \left|K_n^*(f-g,p,q;x)\right|+\left|(f-g)(x)\right| \\
&\qquad+\left|f(x)-f\left(\frac{1}{[2]_{p,q}[n]_{p,q}}+\frac{1}{q}x\right)\right| \\
&\qquad+\left|K_n^*(g,p,q;x)-g(x)\right| \\
&\quad\le 4\|f-g\|+\omega\left(f,\frac{1}{[2]_{p,q}[n]_{p,q}}+\frac{1-q}{q}x\right) \\
&\qquad+\delta_n(x)\left\|g''\right\|.
\end{aligned} \tag{43}$$

Hence, taking the infimum on the right-hand side over all $g \in C_B^2[0,\infty)$, we have the following result:

$$\begin{aligned}
&\left|K_n(f,p,q;x)-f(x)\right| \\
&\quad\le 4K_2(f,\delta_n(x)) \\
&\qquad+\omega\left(f,\frac{1}{[2]_{p,q}[n]_{p,q}}+\frac{1-q}{q}x\right).
\end{aligned} \tag{44}$$

In view of the property of $K$-functional (32), we get

$$\begin{aligned}
&\left|K_n(f,p,q;x)-f(x)\right| \\
&\quad\le M\omega_2\left(f,\sqrt{\delta_n(x)}\right) \\
&\qquad+\omega\left(f,\frac{1}{[2]_{p,q}[n]_{p,q}}+\frac{1-q}{q}x\right).
\end{aligned} \tag{45}$$

This completes the proof of the theorem. □

## 5. Weighted Approximation

Let $f \in C_2^*[0,\infty) := \{f \in C_2[0,\infty) : \lim_{x\to\infty}(|f(x)|/(1+x^2)) < \infty\}$. Throughout the section, we assume that $(p_n)$ and $(q_n)$ are sequences such that $0 < q_n < p_n \le 1$ and $p_n \to 1$, $q_n \to 1$ as $n \to \infty$.

**Theorem 8.** *For each $f \in C_2^*[0,\infty)$, one has*

$$\lim_{n\to\infty}\left\|K_n(f,p_n,q_n;x)-f(x)\right\|_2 = 0. \tag{46}$$

*Proof.* Using the Korovkin type theorem on weighted approximation in [15], we see that it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty}\left\|K_n\left(t^i,p_n,q_n;x\right)-x^i\right\|_2 = 0, \quad i=0,1,2. \tag{47}$$

Since $K_n(1,p_n,q_n;x)=1$, (47) holds true for $m=0$.

By Lemma 3, we have

$$\begin{aligned}
&\left\|K_n(t,p_n,q_n;x)-x\right\|_2 = \sup_{x\in[0,\infty)}\frac{\left|K_n(t,p_n,q_n;x)-x\right|}{1+x^2} \\
&\quad= \sup_{x\in[0,\infty)}\frac{1}{1+x^2}\left|\frac{1}{q_n}x+\frac{1}{[2]_{p,q}[n]_{p,q}}-x\right| \\
&\quad\le \left(\frac{1}{q_n}-1\right)\sup_{x\in[0,\infty)}\frac{x}{1+x^2} \\
&\qquad+\frac{1}{[2]_{p,q}[n]_{p,q}}\sup_{x\in[0,\infty)}\frac{1}{1+x^2} \\
&\quad\le \frac{1}{q_n}-1+\frac{1}{[2]_{p,q}[n]_{p,q}},
\end{aligned} \tag{48}$$

which implies that the condition in (47) holds for $i=1$ as $n\to\infty$.

Similarly we can write

$$\begin{aligned}
&\left\|K_n\left(t^2,p_n,q_n;x\right)-x^2\right\|_2 \\
&\quad= \sup_{x\in[0,\infty)}\frac{\left|K_n\left(t^2,p_n,q_n;x\right)-x^2\right|}{1+x^2} \\
&\quad\le \left(\frac{p_n}{q_n^3}-1\right)\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2} \\
&\qquad+\left(\frac{2p_n+q_n}{q_n[3]_{p_n,q_n}[n]_{p_n,q_n}}+\frac{1}{q_n^2[n]_{p_n,q_n}}\right)\sup_{x\in[0,\infty)}\frac{x}{1+x^2} \\
&\qquad+\frac{1}{[3]_{p_n,q_n}[n]_{p_n,q_n}^2}\sup_{x\in[0,\infty)}\frac{1}{1+x^2} \le \frac{p_n}{q_n^3}-1 \\
&\qquad+\frac{2p_n+q_n}{q_n[3]_{p_n,q_n}[n]_{p_n,q_n}}+\frac{1}{q_n^2[n]_{p_n,q_n}} \\
&\qquad+\frac{1}{[3]_{p_n,q_n}[n]_{p_n,q_n}^2},
\end{aligned} \tag{49}$$

which implies that

$$\lim_{n\to\infty}\left\|K_n\left(t^2,p_n,q_n;x\right)-x^2\right\|_2 = 0, \tag{50}$$

and equation (47) holds for $i=2$. Thus the proof is completed. □

We give the following theorem to approximate all functions in $C_2^*[0,\infty)$. These types of results are given in [16] for classical Szász operators.

**Theorem 9.** *For each $f \in C_2^*[0,\infty)$ and $\alpha > 0$, one has*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{|K_n(f,p_n,q_n;x) - f(x)|}{(1+x^2)^{1+\alpha}} = 0. \tag{51}$$

*Proof.* Let $x_0 \in [0,\infty)$ be arbitrary but fixed. Then

$$\begin{aligned}
&\sup_{x\in[0,\infty)} \frac{|K_n(f,p_n,q_n;x) - f(x)|}{(1+x^2)^{1+\alpha}} \\
&\quad = \sup_{x\le x_0} \frac{|K_n(f,p_n,q_n;x) - f(x)|}{(1+x^2)^{1+\alpha}} \\
&\qquad + \sup_{x>x_0} \frac{|K_n(f,p_n,q_n;x) - f(x)|}{(1+x^2)^{1+\alpha}} \\
&\quad \le \|K_n(f) - f\|_{C[0,x_0]} \\
&\qquad + \|f\|_2 \sup_{x>x_0} \frac{|K_n(1+t^2,p,q;x)|}{(1+x^2)^{1+\alpha}} \\
&\qquad + \sup_{x>x_0} \frac{|f(x)|}{(1+x^2)^{1+\alpha}}.
\end{aligned} \tag{52}$$

Since $|f(x)| \le \|f\|_2(1+x^2)$, we have $\sup_{x>x_0}(|f(x)|/(1+x^2)^{1+\alpha}) \le \|f\|_2/(1+x_0^2)^\alpha$.

Let $\varepsilon > 0$ be arbitrary. We can choose $x_0$ to be so large that

$$\frac{\|f\|_2}{(1+x_0^2)^\alpha} < \frac{\varepsilon}{3}. \tag{53}$$

In view of Theorem 5, we obtain

$$\begin{aligned}
\|f\|_2 \lim_{n\to\infty} \frac{|K_n(1+t^2,p,q;x)|}{(1+x^2)^{1+\alpha}} &= \frac{1+x^2}{(1+x^2)^{1+\alpha}} \|f\|_2 \\
= \frac{\|f\|_2}{(1+x^2)^\alpha} &\le \frac{\|f\|_2}{(1+x_0^2)^\alpha} < \frac{\varepsilon}{3}.
\end{aligned} \tag{54}$$

Using Theorem 6, we can see that the first term of inequality (52) implies that

$$\|K_n(f) - f\|_{C[0,x_0]} < \frac{\varepsilon}{3}, \quad \text{as } n \longrightarrow \infty. \tag{55}$$

Combining (53)–(55), we get that desired result.

For $f \in C_2^*[0,\infty)$, the weighted modulus of continuity is defined as

$$\Omega_2(f,\delta) = \sup_{x\ge 0, 0<h\le\delta} \frac{|f(x+h) - f(x)|}{1+(x+h)^2}. \tag{56}$$

□

**Lemma 10** (see [17]). *If $f \in C_2^*[0,\infty)$, then*

(i) *$\Omega_2(f,\delta)$ is monotone increasing function of $\delta$,*

(ii) $\lim_{\delta\to 0^+}\Omega_2(f,\delta) = 0$,

(iii) *for any $\lambda \in [0,\infty)$, $\Omega_2(f,\lambda\delta) \le (1+\lambda)\Omega_2(f,\delta)$.*

**Theorem 11.** *If $f \in C_2^*[0,\infty)$, then for sufficiently large $n$ one has*

$$|K_n(f,p,q;x) - f(x)| \le K\left(1+x^{2+\lambda}\right)\Omega_2(f,\delta_n), \quad x \in [0,\infty), \tag{57}$$

*where $\lambda \ge 1$ and $\delta_n = \max\{\alpha_n,\beta_n,\gamma_n\}$, $\alpha_n,\beta_n,\gamma_n$ being*

$$\begin{aligned}
\alpha_n &= \frac{p}{q^3} - \frac{2}{q} + 1, \\
\beta_n &= \frac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}} + \frac{1}{q^2[n]_{p,q}} - \frac{2}{[2]_{p,q}[n]_{p,q}}, \\
\gamma_n &= \frac{1}{[3]_{p,q}[n]_{p,q}^2}.
\end{aligned} \tag{58}$$

*Proof.* From the definition of $\Omega_2(f,\delta)$ and Lemma 10, we may write

$$\begin{aligned}
&|f(t) - f(x)| \\
&\quad \le (1+(x+|t-x|)^m)\left(\frac{|t-x|}{\delta}+1\right)\Omega_2(f,\delta) \\
&\quad \le (1+(2x+t)^m)\left(\frac{|t-x|}{\delta}+1\right)\Omega_2(f,\delta) \\
&\quad := \varphi_x(t)\left(1+\frac{1}{\delta}\psi_x(t)\right)\Omega_2(f,\delta).
\end{aligned} \tag{59}$$

Then we obtain

$$\begin{aligned}
&|K_n(f,p,q;x) - f(x)| \le \Omega_2(f,\delta_n) \\
&\quad \cdot \left(K_n(\varphi_x,p,q;x) + \frac{1}{\delta_n}K_n(\varphi_x\psi_x,p,q;x)\right).
\end{aligned} \tag{60}$$

Applying the Cauchy-Schwartz inequality to the second term on the right-hand side, we get

$$\begin{aligned}
&|K_n(f,p,q;x) - f(x)| \le \Omega_2(f,\delta)\left(K_n(\varphi_x,p,q;x)\right. \\
&\quad \left. + \frac{1}{\delta_n}\sqrt{K_n(\varphi_x^2,p,q;x)}\sqrt{K_n(\psi_x^2,p,q;x)}\right).
\end{aligned} \tag{61}$$

From Lemma 3, we get

$$\begin{aligned}
&\frac{1}{1+x^2}K_n(1+t^2,p,q;x) \\
&\quad = \frac{1}{1+x^2} + \frac{p}{q^3}\frac{x^2}{1+x^2} \\
&\qquad + \left(\frac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}} + \frac{1}{q^2[n]_{p,q}}\right)\frac{x}{1+x^2} \\
&\qquad + \frac{1}{[3]_{p,q}[n]_{p,q}^2}\frac{1}{1+x^2} \le 1 + C_1, \\
&\qquad\qquad \text{for sufficiently large } n,
\end{aligned} \tag{62}$$

where $C_1$ is a positive constant. From (62), there exists a positive constant $K_1$ such that $K_n(\varphi_x,p,q;x) \le K_1(1+x^2)$, for sufficiently large $n$.

Proceeding similarly, $(1/(1+x^4))K_n(1+t^4,p,q;x) \leq 1+C_2$, for sufficiently large $n$, where $C_2$ is a positive constant.

So there exists a positive constant $K_2$ such that $K_n(\varphi_x^2,p,q;x) \leq K_2(1+x^2)$, where $x \in [0,\infty)$ $n$ is large enough. Also we get

$$\begin{aligned} K_n\left(\psi_x^2,p,q;x\right) &= \left(\frac{p}{q^3}-\frac{2}{q}+1\right)x^2 \\ &+ \left(\frac{p+[2]_{p,q}}{q\,[3]_{p,q}\,[n]_{p,q}}+\frac{1}{q^2\,[n]_{p,q}}-\frac{2}{[2]_{p,q}\,[n]_{p,q}}\right)x \\ &+ \frac{1}{[3]_{p,q}\,[n]_{p,q}^2} = \alpha_n x^2+\beta_n x+\gamma_n. \end{aligned} \tag{63}$$

Hence, from (61), we have

$$\begin{aligned} \left|K_n(f,p,q;x)-f(x)\right| &\leq \left(1+x^2\right) \\ &\cdot\left(K_1+\frac{1}{\delta_n}K_2\sqrt{\alpha_n x^2+\beta_n x+\gamma_n}\right)\Omega_2(f,\delta_n). \end{aligned} \tag{64}$$

If we take $\delta_n = \max\{\alpha_n,\beta_n,\gamma_n\}$, then we get

$$\begin{aligned} &\left|K_n(f,p,q;x)-f(x)\right| \\ &\leq \left(1+x^2\right)\left(K_1+K_2\sqrt{x^2+x+1}\right)\Omega_2(f,\delta_n) \\ &\leq K_3\left(1+x^{2+\lambda}\right)\Omega_2(f,\delta_n), \\ &\qquad \text{for sufficiently large } n,\ x\in[0,\infty). \end{aligned} \tag{65}$$

Hence the proof is completed. □

## 6. Conclusion

By using the notion of $(p,q)$-integers, we introduced Kantorovich variant of $(p,q)$-analogue of Szász-Mirakjan operators and established the moments of the operators with the help of a recurrence relation. The local approximation and weighted approximation properties of these new operators in terms of modulus of continuity are studied. These results generalize the approximation results proved for Kantorovich variant of $q$-analogue of Szász-Mirakjan operators which are directly obtained by our results for $p=1$.

## Conflict of Interests

The authors declare that they have no competing interests.

## Authors' Contribution

The authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

## Acknowledgment

The third author (Khursheed J. Ansari) acknowledges the financial support of UGC-BSR Fellowship.

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