Research Article

On the Superstability of Lobačevskiĭ's Functional Equations with Involution

Jaeyoung Chung,¹ Bogeun Lee,² and Misuk Ha²

¹Department of Mathematics, Kunsan National University, Kunsan 573-701, Republic of Korea

²Department of Mathematics, Jeonbuk National University, Jeonju 561-756, Republic of Korea

Correspondence should be addressed to Bogeun Lee; akiles@jbnu.ac.kr

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Let G be a uniquely 2-divisible commutative group and let $f, g : G \rightarrow \mathbb{C}$ and $\sigma : G \rightarrow G$ be an involution. In this paper, generalizing the superstability of Lobačevskiĭ's functional equation, we consider $|f((x + \sigma y)/2) - g(x)f(y)| \leq \psi(x)$ or $\psi(y)$ for all $x, y \in G$, where $\psi : G \rightarrow \mathbb{R}^+$. As a direct consequence, we find a weaker condition for the functions f satisfying the Lobačevskiĭ functional inequality to be unbounded, which refines the result of Găvrută and shows the behaviors of bounded functions satisfying the inequality. We also give various examples with explicit involutions on Euclidean space.

1. Main Results

Throughout this paper, we denote by G a uniquely 2-divisible commutative group (i.e., for each $x \in G$, there exists a unique $y \in G$ such that $y + y = x$) and by $\mathbb{R}, \mathbb{R}^+, \mathbb{C}$, and $\mathbb{R}^n$ the set of real numbers, nonnegative real numbers, complex numbers, and the n -dimensional Euclidean space, respectively. A function $m : G \rightarrow \mathbb{C}$ is called an *exponential function* provided that $m(x + y) = m(x)m(y)$ for all $x, y \in G$ and $\sigma : G \rightarrow G$ is called an *involution* provided that $\sigma(x + y) = \sigma(x) + \sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in G$. We denote $\sigma(x)$ by σx . An exponential function $m : G \rightarrow \mathbb{C}$ is called *σ -exponential function* if m satisfies $m(\sigma x) = m(x)$ for all $x \in G$ and is denoted by m_σ . In [1] (see also [2, Theorem 6.5]), Găvrută investigated the superstability of Lobačevskiĭ's functional equation:

$$\left| f\left(\frac{x+y}{2}\right) - f(x)f(y) \right| \leq \delta \quad (1)$$

for all $x, y \in G$ and for some $\delta > 0$ (see Albert and Baker [3] and Baker [4] for the superstability of the exponential functional equation). As a result he proved the following.

Theorem 1. Assume that $f : G \rightarrow \mathbb{C}$ satisfies (1). Then, every bounded function f satisfies

$$|f(x)| \leq \frac{|f(0)| + \sqrt{|f(0)|^2 + 4\delta}}{2} := K_{f,\delta} \quad (2)$$

for all $x \in G$, and if there exists $y_0 \in G$ such that $|f(y_0)| > K_{f,\delta}$, then f satisfies

$$f\left(\frac{x+y}{2}\right)^2 = f(x)f(y) \quad (3)$$

for all $x, y \in G$.

A careful observation shows that the upper bound $K_{f,\delta}$ in (2) is not optimal and may be replaced by a better one; namely, $K_{f,\delta}^* = (1/2)(m_f + \sqrt{m_f^2 + 4\delta})$ with $m_f = \inf_{x \in G} |f(x)|$. For example, let $f(x) = \sqrt{\delta} \cos x$. Then, $K_{f,\delta} = (1/2)(1 + \sqrt{5})\sqrt{\delta}$ and $K_{f,\delta}^* = \sqrt{\delta}$. Using the better upper bound, we investigate the behavior of bounded functions satisfying (1) (see Remark 3).

As main results in this paper, we generalize Găvrută's result and consider the inequalities:

$$\left| f\left(\frac{x+\sigma y}{2}\right)^2 - g(x)f(y) \right| \leq \psi(x), \quad (4)$$

$$\left| f\left(\frac{x+\sigma y}{2}\right)^2 - g(x)f(y) \right| \leq \psi(y) \quad (5)$$

for all $x, y \in G$, where $\psi : G \rightarrow \mathbb{R}^+$ is an unbounded function.

Note that the cases $|f((x+\sigma y)/2)^2 - f(x)g(y)| \leq \psi(y)$ and $|f((x+\sigma y)/2)^2 - f(x)g(y)| \leq \psi(x)$ can be reduced to the special case of form (4) or (5) when $\sigma = I$, the identity involution on G (replacing $g(y)$ by $g(\sigma y)$ and $\psi(y)$ by $\psi(\sigma y)$).

As a direct consequence of our results, we obtain the following.

Theorem 2. Assume that $f : G \rightarrow \mathbb{C}$ satisfies

$$\left| f\left(\frac{x+\sigma y}{2}\right)^2 - f(x)f(y) \right| \leq \delta \quad (6)$$

for all $x, y \in G$. Then, every bounded function f satisfies

$$f(x) \leq \frac{m_f + \sqrt{m_f^2 + 4\delta}}{2} := K_{f,\delta}^* \quad (7)$$

for all $x \in G$, and if there exists $y_0 \in G$ such that $|f(y_0)| > K_{f,\delta}^*$, then there exist σ -exponential function m_σ and $\lambda \in \mathbb{C}$ such that

$$f(x) = \lambda m_\sigma(x) \quad (8)$$

for all $x \in G$.

Remark 3. Let $\sigma = I$ be the identity involution in Theorem 2; we obtain a refined version of Theorem 1. Now, from (7), we have

$$\begin{aligned} M_f - m_f &\leq \frac{-m_f + \sqrt{m_f^2 + 4\delta}}{2} \leq \frac{2\delta}{m_f + \sqrt{m_f^2 + 4\delta}} \\ &\leq \frac{2\delta}{\sqrt{4\delta}} = \sqrt{\delta}. \end{aligned} \quad (9)$$

Thus, if $\{f_n\}$, $n = 1, 2, 3, \dots$, is a sequence of functions satisfying

$$\left| f_n\left(\frac{x+y}{2}\right)^2 - f_n(x)f_n(y) \right| \leq \frac{1}{n} \quad (10)$$

for all $x, y \in G$, then there exists a sequence $r_n \in \mathbb{R}$ such that $|f_n(x) - r_n|$ converges uniformly to 0 as $n \rightarrow \infty$ (i.e., $|f_n|$ tends to straight lines).

We refer the reader to [5–7] for related functional equations and their stabilities. We also refer the reader to [8, 9] for some important recent developments on the issues of stability and superstability for functional equations.

2. Stability of (4)

In this section, we prove the superstability of (4). For the proof, we need the following.

Lemma 4. Assume that $m_1, m_2 : G \rightarrow \mathbb{C}$ are unbounded exponential functions satisfying

$$|m_1(x) - \lambda m_2(x)| \leq \delta \quad (11)$$

for all $x \in G$ and for some $\lambda \in \mathbb{C}$. Then, one has $\lambda = 1$ and $m_1 = m_2$.

Proof. Since m_2 is unbounded exponential function, it is easy to see that $m_2(x) \neq 0$ for all $x \in G$. Replacing x by kx , $k = 1, 2, 3, \dots$, in (11) and dividing the result by $|m_2(kx)|$ we have

$$\left| \left(\frac{m_1(x)}{m_2(x)} \right)^k - \lambda \right| \leq \frac{\delta}{|m_2(x)|^k}. \quad (12)$$

If $|m_2(x)| > 1$, we have

$$\lim_{k \rightarrow \infty} \left(\frac{m_1(x)}{m_2(x)} \right)^k = \lambda. \quad (13)$$

Since $\lambda \neq 0$, from (13), we have

$$m_1(x) = m_2(x) \quad (14)$$

for all $x \in G$ such that $|m_2(x)| > 1$ and hence $\lambda = 1$. Let $|m_2(x)| \leq 1$. Choose $y \in G$ such that $|m_2(y)| > |m_2(x)|^{-1}$. Then, we have $|m_2(y)| > 1$ and $|m_2(x+y)| = |m_2(x)m_2(y)| > 1$. Thus, it follows from (14) that $m_1(y) = m_2(y)$ and $m_1(x+y) = m_2(x+y)$ and hence

$$m_1(x) = \frac{m_1(x+y)}{m_1(y)} = \frac{m_2(x+y)}{m_2(y)} = m_2(x) \quad (15)$$

for all $x \in G$ such that $|m_2(x)| \leq 1$. This completes the proof. $\square$

From now on, we denote $M_f = \sup_{x \in G} |f(x)|$ and $m_f = \inf_{x \in G} |f(x)|$.

Theorem 5. Assume that $f, g : G \rightarrow \mathbb{C}$ satisfy (4). Then, every bounded function f satisfies

$$M_f \leq \inf_{x \in G} \frac{|g(x)| + \sqrt{|g(x)|^2 + 4\psi(x)}}{2}. \quad (16)$$

If f is unbounded, then there exists an exponential function m such that

$$g(x) = g(0)m(x), \quad (17)$$

$$|f(x)| \leq \sqrt{|f(0)g(0)m(2x)| + \psi(2x)}$$

for all $x \in G$, and, in particular, if

$$\inf_{x \in G} \left(\frac{\psi(x)}{|g(x)|} + \frac{1}{|g(x)|} \right) = 0, \quad (18)$$

then there exist σ -exponential function m_σ and $\lambda \in \mathbb{C}$ such that

$$f(x) = g(x) = \lambda m_\sigma(x) \quad (19)$$

for all $x \in G$.

Proof. Assume that f is bounded. Replacing y by $-\sigma x + 2\sigma y$ in (4) and using the triangle inequality with the result, we have

$$\begin{aligned} |f(y)|^2 &\leq |g(x)| |f(-\sigma x + 2\sigma y)| + \psi(x) \\ &\leq |g(x)| M_f + \psi(x) \end{aligned} \quad (20)$$

for all $x, y \in G$. Taking the supremum of the left hand side of (20), we have

$$M_f^2 \leq |g(x)| M_f + \psi(x) \quad (21)$$

for all $x \in G$. Fixing x and solving the quadratic inequality (21), we obtain

$$M_f \leq \frac{|g(x)| + \sqrt{|g(x)|^2 + 4\psi(x)}}{2} \quad (22)$$

for all $x \in G$. Taking the infimum of the right hand side of (22), we get (16). Now, assume that f is unbounded. Putting $x = 0$ in (4), we have

$$\left| f\left(\frac{\sigma y}{2}\right)^2 - g(0) f(y) \right| \leq \psi(0) \quad (23)$$

for all $y \in G$. Choosing a sequence $y_n \in G$, $n = 1, 2, 3, \dots$, such that $|f(y_n)| \rightarrow \infty$ as $n \rightarrow \infty$, putting $y = y_n$, $n = 1, 2, 3, \dots$, in (4), dividing the result by $|f(y_n)|$, letting $n \rightarrow \infty$, and using (23), we have

$$\begin{aligned} g(x) &= \lim_{n \rightarrow \infty} \frac{f((x + \sigma y_n)/2)^2}{f(y_n)} \\ &= \lim_{n \rightarrow \infty} \frac{g(0) f(\sigma x + y_n)}{f(y_n)} \end{aligned} \quad (24)$$

for all $x \in G$. Multiplying both sides of (24) by $g(y)$ and using (4) and (24), we have

$$\begin{aligned} g(y) g(x) &= \lim_{n \rightarrow \infty} \frac{g(0) g(y) f(\sigma x + y_n)}{f(y_n)} \\ &= \lim_{n \rightarrow \infty} \frac{g(0) f((y + x + \sigma y_n)/2)^2}{f(y_n)} \\ &= g(0) g(y + x) \end{aligned} \quad (25)$$

for all $x, y \in G$. Thus, $g_0(x) = g(x)/g(0)$ is an exponential function and let $g_0 = m$; then, we get

$$g(x) = g(0) m(x) \quad (26)$$

for all $x \in G$. Putting $y = 0$ in (4), replacing x by $2x$, and using (26) and the triangle inequality, we have $|f(x)| \leq$

$\sqrt{|f(0)g(0)m(x)^2| + \psi(2x)}$ for all $x \in G$, which gives (17). Assume that (18) holds. Then, we can choose a sequence x_n , $n = 1, 2, 3, \dots$, such that

$$\lim_{n \rightarrow \infty} \frac{\psi(x_n)}{|g(x_n)|} = \lim_{n \rightarrow \infty} \frac{1}{|g(x_n)|} = 0. \quad (27)$$

Putting $x = x_n$, $n = 1, 2, 3, \dots$, in (4), dividing the result by $|g(x_n)|$, letting $n \rightarrow \infty$, and using (23), we have

$$\begin{aligned} f(y) &= \lim_{n \rightarrow \infty} \frac{f((x_n + \sigma y)/2)^2}{g(x_n)} \\ &= \lim_{n \rightarrow \infty} \frac{g(0) f(\sigma x_n + y)}{g(x_n)} \end{aligned} \quad (28)$$

for all $y \in G$. Multiplying both sides of (28) by $g(x)$ and using (4) and (28), we have

$$\begin{aligned} g(x) f(y) &= \lim_{n \rightarrow \infty} \frac{g(0) g(x) f(\sigma x_n + y)}{g(x_n)} \\ &= \lim_{n \rightarrow \infty} \frac{g(0) f((x + \sigma y + x_n)/2)^2}{g(x_n)} \\ &= g(0) f(\sigma x + y) \end{aligned} \quad (29)$$

for all $x, y \in G$. Putting $y = 0$ in (29), replacing x by σx , using (26), and dividing the result by $g(0)$, we have

$$f(x) = f(0) m(\sigma x) \quad (30)$$

for all $x \in G$. Putting (30) in (23), we have

$$|f(0)|^2 |m(y) - g(0) f(0)^{-1} m(\sigma y)| \leq \psi(0) \quad (31)$$

for all $y \in G$. Applying Lemma 4 in (31), we have $m(y) = m(\sigma y)$ for all $y \in G$ and $g(0) = f(0)$. Letting $g(0) = f(0) := \lambda$, we get (19). This completes the proof. $\square$

In particular, if $g = f$ in Theorem 5, we obtain the following.

Corollary 6. Assume that $f : G \rightarrow \mathbb{C}$ satisfies

$$\left| f\left(\frac{x + \sigma y}{2}\right)^2 - f(x) f(y) \right| \leq \psi(x) \quad (32)$$

for all $x, y \in G$. Then, every bounded function f satisfies

$$M_f \leq \inf_{x \in G} \frac{|f(x)| + \sqrt{|f(x)|^2 + 4\psi(x)}}{2}, \quad (33)$$

and if f is unbounded, then there exists σ -exponential function m_σ such that

$$f(x) = f(0) m_\sigma(x) \quad (34)$$

for all $x \in G$.

Proof. By Theorem 5, we get (33) and every unbounded function f has the form $f(x) = f(0)m(x)$ for some exponential function m . Putting $x = 0$ and $f(y) = f(0)m(y)$ in (32), we have

$$|f(0)|^2 |m(\sigma y) - m(y)| \leq \psi(0) \quad (35)$$

for all $y \in G$. Using Lemma 4, we have $m(y) = m(\sigma y)$ for all $y \in G$. Thus, we get (34). This completes the proof. $\square$

Remark 7. In particular, if $\psi(x) = \delta$ in Corollary 6, we obtain Theorem 2.

3. Stability of (5)

In this section, we prove the stability of (5). We exclude the trivial case when $f = 0$ or $g = 0$.

Theorem 8. Assume that $f, g : G \rightarrow \mathbb{C}$ satisfy (5). Then, every bounded function f satisfies

$$\begin{aligned} \sup_{y \in G} (M_g |f(y)| - \psi(y)) &\leq M_f^2 \\ &\leq \inf_{y \in G} (M_g |f(y)| + \psi(y)). \end{aligned} \quad (36)$$

If f is unbounded, then there exists an exponential function m such that

$$f(x) = f(0)m(x) \quad (37)$$

for all $x \in G$. In particular, if

$$\inf_{x \in G} \left(\frac{\psi(x)}{|f(x)|} + \frac{1}{|f(x)|} \right) = 0, \quad (38)$$

then there exist σ -exponential function m_σ and $\lambda \in \mathbb{C}$ such that

$$f(x) = g(x) = \lambda m_\sigma(x) \quad (39)$$

for all $x \in G$.

Proof. Choosing $y_0 \in G$ such that $f(y_0) \neq 0$ and putting $y = y_0$ in (5), we get

$$\left| f\left(\frac{x + \sigma y_0}{2}\right)^2 - g(x)f(y_0) \right| \leq \psi(y_0) \quad (40)$$

for all $x \in G$, and it follows that f is bounded if and only if g is bounded. Assume that f and g are bounded. Replacing x by $2x - \sigma y$ in (5) and using the triangle inequality with the result, we have

$$\begin{aligned} |f(x)|^2 &\leq |g(2x - \sigma y)| |f(y)| + \psi(y) \\ &\leq M_g |f(y)| + \psi(y) \end{aligned} \quad (41)$$

for all $x, y \in G$. Taking the supremum of the left hand side of (41), we have

$$M_f^2 \leq M_g |f(y)| + \psi(y) \quad (42)$$

for all $y \in G$. Taking the infimum of the right hand side of (42), we have

$$M_f^2 \leq \inf_{y \in G} (M_g |f(y)| + \psi(y)). \quad (43)$$

Now, using the triangle inequality with (5) again, we have

$$|g(x)f(y)| \leq \left| f\left(\frac{x + \sigma y}{2}\right)^2 \right| + \psi(y) \leq M_f^2 + \psi(y) \quad (44)$$

for all $x, y \in G$. Taking the supremum of $g(x)$ in the left hand side of (44) and subtracting $\psi(y)$ from both sides of the result, we have

$$M_g |f(y)| - \psi(y) \leq M_f^2 \quad (45)$$

for all $y \in G$. Taking the supremum of the left hand side of (45), we have

$$\sup_{y \in G} (M_g |f(y)| - \psi(y)) \leq M_f^2. \quad (46)$$

From (43) and (46), we get (36). Now, we assume that f and g are unbounded. Putting $y = 0$ in (5), we have

$$\left| f\left(\frac{x}{2}\right)^2 - g(x)f(0) \right| \leq \psi(0) \quad (47)$$

for all $x \in G$. Choosing a sequence $x_n \in G$, $n = 1, 2, 3, \dots$, such that $|g(x_n)| \rightarrow \infty$ as $n \rightarrow \infty$, putting $x = x_n$, $n = 1, 2, 3, \dots$, in (5), dividing the result by $|g(x_n)|$, letting $n \rightarrow \infty$, and using (47), we have

$$\begin{aligned} f(y) &= \lim_{n \rightarrow \infty} \frac{f((x_n + \sigma y)/2)^2}{g(x_n)} \\ &= \lim_{n \rightarrow \infty} \frac{f(0)g(x_n + \sigma y)}{g(x_n)} \end{aligned} \quad (48)$$

for all $y \in G$. Multiplying both sides of (48) by $f(z)$ and using (5) and (48), we have

$$\begin{aligned} f(y)f(z) &= \lim_{n \rightarrow \infty} \frac{f(0)g(x_n + \sigma y)f(z)}{g(x_n)} \\ &= \lim_{n \rightarrow \infty} \frac{f(0)f((x_n + \sigma y + \sigma z)/2)^2}{g(x_n)} \\ &= f(0)f(y + z) \end{aligned} \quad (49)$$

for all $y, z \in G$. Thus, $f_0(x) = f(x)/f(0)$ is an exponential function, say $f_0 = m$; then, we get

$$f(x) = f(0)m(x) \quad (50)$$

for all $x \in G$. Assume that (38) holds. Then, we can choose a sequence y_n , $n = 1, 2, 3, \dots$, such that

$$\lim_{n \rightarrow \infty} \frac{\psi(y_n)}{|f(y_n)|} = \lim_{n \rightarrow \infty} \frac{1}{|f(y_n)|} = 0. \quad (51)$$

Putting $y = y_n$, $n = 1, 2, 3, \dots$, in (5), dividing the result by $|f(y_n)|$, letting $n \rightarrow \infty$, and using (47), we have

$$\begin{aligned} g(x) &= \lim_{n \rightarrow \infty} \frac{f((x + \sigma y_n)/2)^2}{f(y_n)} \\ &= \lim_{n \rightarrow \infty} \frac{f(0) g(x + \sigma y_n)}{f(y_n)} \end{aligned} \quad (52)$$

for all $y \in G$. Multiplying both sides of (52) by $f(y)$ and using (5) and (52), we have

$$\begin{aligned} g(x) f(y) &= \lim_{n \rightarrow \infty} \frac{f(0) g(x + \sigma y_n) f(y)}{f(y_n)} \\ &= \lim_{n \rightarrow \infty} \frac{f(0) f((x + \sigma y_n + \sigma y)/2)^2}{f(y_n)} \\ &= f(0) g(x + \sigma y) \end{aligned} \quad (53)$$

for all $x, y \in G$. Putting $x = 0$ in (53), replacing y by σy , and using (50), we have

$$g(x) = g(0) m(\sigma x) \quad (54)$$

for all $x \in G$. Putting (50) and (54) in (47), we have

$$|f(0)|^2 |m(x) - g(0) f(0)^{-1} m(\sigma x)| \leq \psi(0) \quad (55)$$

for all $x \in G$. Applying Lemma 4, we have $m(x) = m(\sigma x)$ for all $x \in G$ and $g(0) = f(0)$. Letting $g(0) = f(0) := \lambda$, we get (39). This completes the proof. $\square$

In particular, if $g = f$ in Theorem 8, we obtain the following.

Corollary 9. Assume that $f : G \rightarrow \mathbb{C}$ satisfies

$$\left| f\left(\frac{x + \sigma y}{2}\right)^2 - f(x) f(y) \right| \leq \psi(y) \quad (56)$$

for all $x, y \in G$. Then, every bounded function f satisfies

$$M_f \leq \inf_{x \in G} \frac{|f(x)| + \sqrt{|f(x)|^2 + 4\psi(x)}}{2}, \quad (57)$$

and if f is unbounded, then there exists σ -exponential function m_σ such that

$$f(x) = f(0) m_\sigma(x) \quad (58)$$

for all $x \in G$.

Proof. By Theorem 8, every bounded function f satisfies

$$M_f^2 \leq M_f |f(y)| + \psi(y) \quad (59)$$

for all $y \in G$. Solving (59), we have

$$M_f \leq \frac{|f(y)| + \sqrt{|f(y)|^2 + 4\psi(y)}}{2} \quad (60)$$

for all $y \in G$, which gives (57). If f is unbounded, then by Theorem 8 we have

$$f(x) = f(0) m(x) \quad (61)$$

for all $x \in G$. Putting (61) in (56), we have

$$|f(0)^2 m(x)| |m(\sigma y) - m(y)| \leq \psi(y) \quad (62)$$

for all $x, y \in G$. Since m is unbounded, we have $m(y) = m(\sigma y)$ for all $y \in G$, which gives (58). This completes the proof. $\square$

In particular, if $\psi(y) = \delta$ for all $y \in G$ in Theorems 5 and 8, we obtain the following.

Corollary 10. Assume that $f, g : G \rightarrow \mathbb{C}$ satisfy

$$\left| f\left(\frac{x + \sigma y}{2}\right)^2 - g(x) f(y) \right| \leq \delta \quad (63)$$

for all $x, y \in G$. If f is bounded, then f satisfies

$$M_g M_f - \delta \leq M_f^2 \leq \min \{M_g m_f, M_f m_g\} + \delta. \quad (64)$$

If f is unbounded, then there exist σ -exponential function m_σ and $\lambda \in \mathbb{C}$ such that

$$f(x) = g(x) = \lambda m_\sigma(x) \quad (65)$$

for all $x \in G$.

Proof. By Theorem 5, every bounded function f satisfies

$$M_f^2 \leq M_f m_g + \delta \quad (66)$$

and, by Theorem 8, f satisfies

$$M_g M_f - \delta \leq M_f^2 \leq M_g m_f + \delta. \quad (67)$$

From (66) and (67), we get (64). If f is unbounded, then condition (38) holds. Thus, by Theorem 8, we get (65). This completes the proof. $\square$

4. Examples

In this section, employing some involutions on Euclidean space, we give various examples satisfying inequalities (4) and (5). We denote by $c \cdot x$ the inner product of $c = (c_1, c_2, \dots, c_n) \in \mathbb{C}^n$ and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ defined as $c \cdot x = \sum_{j=1}^n c_j x_j$,

$|x| = \sqrt{x_1^2 + \dots + x_n^2}$, and $\Re c = (\Re c_1, \dots, \Re c_n)$, where $\Re c_j$ are the real parts of c_j , $j = 1, 2, \dots, n$. It is easy to see that if G is uniquely 2-divisible, then m_σ is σ -exponential if and only if

$$m_\sigma(x) = m\left(\frac{x + \sigma x}{2}\right), \quad \forall x \in G \quad (68)$$

for some exponential function $m : G \rightarrow \mathbb{C}$.

Example 1. Let $G = \langle \mathbb{R}^2, + \rangle$ in Theorem 5 and let

$$\sigma = \begin{bmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{bmatrix} \quad (69)$$

be a 2×2 matrix, where $a, b \in \mathbb{R}$ with $b \neq 0$. Then, σ defines an involution on $\mathbb{R}^2$. Let $f, g : \mathbb{R}^2 \rightarrow \mathbb{C}$ satisfy

$$\left| f\left(\frac{t+au+bv}{2}, \frac{s+b^{-1}(1-a^2)u-av}{2}\right) - g(t, s) f(u, v) \right| \leq \epsilon (|t|^p + |s|^p) \quad (70)$$

for all $t, s, u, v \in \mathbb{R}$ and for some $\epsilon > 0$, $p > 0$. Then, by Theorem 5, every bounded function f satisfies

$$|f(u, v)| \leq \inf_{(t,s) \in \mathbb{R}^2} \frac{|g(t, s)| + \sqrt{|g(t, s)|^2 + 4\epsilon(|t|^p + |s|^p)}}{2} \quad (71)$$

for all $u, v \in \mathbb{R}$, and if f is unbounded, then

$$g(t, s) = g(0, 0) m(t, s) \quad (72)$$

for all $(t, s) \in \mathbb{R}^2$, where $m : \mathbb{R}^2 \rightarrow \mathbb{C}$ is an exponential function. In particular, if g is unbounded, then we can choose $(t_0, s_0) \in \mathbb{R}^2$ such that $|m(t_0, s_0)| > 1$, and hence

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{\psi(nt_0, ns_0)}{|g(nt_0, ns_0)|} \\ & \leq \lim_{n \rightarrow \infty} \frac{\epsilon(|t_0|^p + |s_0|^p)}{|g(0, 0)|} \lim_{n \rightarrow \infty} \frac{n^p}{m(t_0, s_0)^n} = 0, \end{aligned} \quad (73)$$

which implies that condition (18) holds. Thus, by Theorem 5, g, f have the form

$$g(t, s) = f(t, s) = \lambda m(t, s) \quad (74)$$

for all $(t, s) \in \mathbb{R}^2$, where $\lambda \in \mathbb{C}$ and $m : \mathbb{R}^2 \rightarrow \mathbb{C}$ is an exponential function satisfying

$$m(t, s) = m(at + bs, b^{-1}(1-a^2)t - as) \quad (75)$$

for all $(t, s) \in \mathbb{R}^2$. In particular, if g is Lebesgue measurable, m is Lebesgue measurable and has the form $m(t, s) = e^{c_1 t + c_2 s}$ for some $c_1, c_2 \in \mathbb{C}$. From condition (75), we have $c_1 = (1/b)(1+a)c_2$. Thus, g, f has the form

$$g(t, s) = f(t, s) = \lambda e^{d((1+a)t + bs)} \quad (76)$$

for all $(t, s) \in \mathbb{R}^2$ and for some $d, \lambda \in \mathbb{C}$ with $\Re d \neq 0$.

Example 2. Let σ be an $n \times n$ matrix such that $\sigma^2 = E$, the unit matrix. Suppose that $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$ satisfy

$$\left| f\left(\frac{x + \sigma y}{2}\right) - g(x) f(y) \right| \leq e^{|x|^q} \quad (77)$$

for all $x, y \in \mathbb{R}^n$ and for some $0 \leq q < 1$. Then, by Theorem 5, every bounded function f satisfies

$$|f(y)| \leq \inf_{x \in \mathbb{R}^n} \frac{|g(x)| + \sqrt{|g(x)|^2 + 4e^{|x|^q}}}{2} \quad (78)$$

for all $y \in \mathbb{R}^n$, and if f is unbounded, then

$$g(x) = g(0) m(x) \quad (79)$$

for all $x \in \mathbb{R}^n$. If g is unbounded, it is easy to see that condition (18) holds. Thus, by Theorem 5, there exist σ -exponential function $m_\sigma : \mathbb{R}^n \rightarrow \mathbb{C}$ and $\lambda \in \mathbb{C}$ such that

$$g(x) = f(x) = \lambda m_\sigma(x) \quad (80)$$

for all $x \in \mathbb{R}^n$. In particular, if g is continuous, then by (68) there exist $c \in \mathbb{C}^n$, $\Re(c + c\sigma) \neq 0$ and $\lambda \in \mathbb{C}$ such that

$$g(x) = f(x) = \lambda e^{(1/2)(c + c\sigma) \cdot x} \quad (81)$$

for all $x \in \mathbb{R}^n$.

Example 3. Suppose that $f, g : \mathbb{C} \rightarrow \mathbb{C}$ are unbounded function satisfying

$$\left| f\left(\frac{z + \bar{w}}{2}\right) - g(z) f(w) \right| \leq \epsilon |w|^p \quad (82)$$

for all $z, w \in \mathbb{C}$ and for some $p > 0$. Then, by Theorem 8, there exists an exponential function $m : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$f(z) = g(z) = \lambda m(z) \quad (83)$$

for all $z \in \mathbb{C}$. Choose $z_0 \in \mathbb{C}$ such that $|m(z_0)| > 1$. Then, we have

$$\lim_{n \rightarrow \infty} \frac{|nz_0|^p}{|f(nz_0)|} = \frac{|z_0|^p}{|f(0)|} \lim_{n \rightarrow \infty} \frac{n^p}{|m(z_0)|^n} = 0, \quad (84)$$

which implies that condition (38) holds. Thus, again, by Theorem 8, there exist an exponential function m satisfying $m(z) = m(\bar{z})$ and $\lambda \in \mathbb{C}$ such that $f(z) = g(z) = \lambda m(z)$ for all $z \in \mathbb{C}$. Now, from $m(z) = m(\bar{z})$, m is given by $m(z) = m(\Re z)$. Thus, we have

$$f(z) = g(z) = \lambda m(\Re z) \quad (85)$$

for all $z \in \mathbb{C}$.

Remark 11. Let $a, b \in \mathbb{R}^n$ be two nonzero vectors that are not parallel; that is, $b \neq ra$ for all $r \in \mathbb{R}$. Then, the hyperplane $b \cdot x = 0$ is not parallel to $(b-a) \cdot x = 0$ and hence there exists $x_0 \in \mathbb{R}^n$ such that $b \cdot x_0 > 0$ and $(b-a) \cdot x_0 > 0$. If $b = ta$ for some $t \in \mathbb{R}$, then there exists $x_0 \in \mathbb{R}^n$ such that $b \cdot x_0 > 0$ and $(b-a) \cdot x_0 > 0$ if and only if $t > 1$. Thus, if $b \neq ta$ for all $t \leq 1$, then there exists $x_0 \in \mathbb{R}^n$ such that $b \cdot x_0 > 0$ and $(b-a) \cdot x_0 > 0$.

Example 4. Let $a \in \mathbb{R}^n$ be fixed. Suppose that $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$ are unbounded continuous functions satisfying

$$\left| f\left(\frac{x + \sigma y}{2}\right)^2 - g(x)f(y) \right| \leq e^{a \cdot y} \quad (86)$$

for all $x, y \in \mathbb{R}^n$. By Theorem 8, we have

$$f(x) = f(0)e^{c \cdot x} \quad (87)$$

for all $x \in \mathbb{R}^n$ and for some $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$. If $\Re c \neq ta$ for all $t \leq 1$, then, by Remark 11, there exists $x_0 \in \mathbb{R}^n$ such that

$$\begin{aligned} \Re c \cdot x_0 &> 0, \\ \Re c \cdot x_0 &> a \cdot x_0. \end{aligned} \quad (88)$$

From (87) and (88), we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1 + e^{a \cdot kx_0}}{|f(kx_0)|} &= \frac{1}{|f(0)|} \lim_{k \rightarrow \infty} \frac{1 + e^{a \cdot kx_0}}{e^{\Re c \cdot kx_0}} \\ &= \frac{1}{|f(0)|} \lim_{k \rightarrow \infty} \frac{1 + e^{ka \cdot x_0}}{e^{k\Re c \cdot x_0}} = 0. \end{aligned} \quad (89)$$

Thus, condition (38) holds. By Theorem 8, we get

$$g(x) = f(x) = \lambda e^{(1/2)(c + c\sigma) \cdot x} \quad (90)$$

for all $x \in \mathbb{R}^n$.

Competing Interests

The authors declare that they have no competing interests.

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