Research Article

Two-Weight Norm Inequality for the One-Sided Hardy-Littlewood Maximal Operators in Variable Lebesgue Spaces

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The authors establish the two-weight norm inequalities for the one-sided Hardy-Littlewood maximal operators in variable Lebesgue spaces. As application, they obtain the two-weight norm inequalities of variable Riemann-Liouville operator and variable Weyl operator in variable Lebesgue spaces on bounded intervals.

1. Introduction and Main Results

The one-sided Hardy-Littlewood maximal operators \( M^+ \) and \( M^- \) are defined by

\[
M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_{x}^{x+h} |f(t)| \, dt,
\]

\[
M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(t)| \, dt.
\]

Sawyer [1] showed that \( M^+ \) is bounded from \( L^p(v) \) to \( L^p(u) \) if the pairs of nonnegative functions \((u, v)\) satisfy Sawyer-type two-weight condition for the one-sided maximal operator. In [2] Martin-Reyes et al. generalized this result to \( M^+_g \), where \( g \) is positive locally integrable function. The similar results are also true for \( M^- \) and \( M^-_g \) (see [1–3]).

Let \( E \) be a measurable set in \( \mathbb{R} \). Given a measurable function \( p(\cdot) : E \rightarrow (0, +\infty) \), we denote

\[
p^-_E = \text{ess inf}_{x \in E} p(x),
\]

\[
p^+_E = \text{ess sup}_{x \in E} p(x).
\]

If \( E = \mathbb{R} \), we will simply note that \( p^- = p^-_{\mathbb{R}} \) and \( p^+ = p^+_{\mathbb{R}} \).

Definition 1. Given \( p(\cdot) : E \rightarrow [1, +\infty) \), \( \sigma \) is a locally integrable function such that \( 0 < \sigma(x) < \infty \) a.e. \( x \in E \). We say that \( p(\cdot) \in P^\sigma(E) \) if there exists a constant \( C > 0 \) such that for every \((x, x+h) \subset E\)

\[
p(x) - \sigma(x, x+h) \leq \frac{C}{\ln \left( \frac{1}{\sigma(x) + |x|} \right)}.
\]

Let \( E \) be a measurable set in \( \mathbb{R} \). Given a measurable function \( p(\cdot) : E \rightarrow (0, +\infty) \), we denote

\[
p^-_E = \text{ess inf}_{x \in E} p(x),
\]

\[
p^+_E = \text{ess sup}_{x \in E} p(x).
\]

If \( E = \mathbb{R} \), we will simply note that \( p^- = p^-_{\mathbb{R}} \) and \( p^+ = p^+_{\mathbb{R}} \).

Definition 2 (see [4]). It is given that \( p(\cdot) : E \rightarrow [1, +\infty) \). We say that \( p(\cdot) \in LH^\sigma(E) \), if there exist constants \( p_\infty \geq 1 \) and \( C_\infty > 0 \) such that for all \( x \in E \)

\[
|p(x) - p_\infty| \leq \frac{C_\infty}{\ln (e + |x|)}.
\]
Let \(1 \leq p_− \leq p_+ < \infty\). The variable Lebesgue space \(L^{p(⋅)}(E)\) is the set of measurable functions \(f\) on \(E\) such that
\[
\rho_{p(⋅),E}(f) = \int_E |f(x)|^{p(⋅)(x)} \, dx < \infty.
\]
(6)
This is a Banach space (see \([4–7]\)) with the norm
\[
\|f\|_{L^{p(⋅)}(E)} = \inf \left\{ \lambda > 0 : \rho_{p(⋅),E}(f^{\lambda}) \leq 1 \right\}.
\]
(7)
If \(E = \mathbb{R}\), we will write \(\|f\|_{L^{p(⋅)}(\mathbb{R})}\) as \(\|f\|_{L^p(⋅)}\). The variable Lebesgue space is the special case of the Musielak-Orlicz space (see \([7, 8]\)). For the detail of \(L^{p(⋅)}\) we refer to \([4–7]\) and so on.

Let \(w\) be a nonnegative locally integrable function on \(E\). The weighted variable Lebesgue space \(L^{p(⋅)}(E;w)\) is the set of measurable functions \(f\) on \(E\) such that \(fw \in L^{p(⋅)}(E)\) and \(\|fw\|_{L^{p(⋅)}(E)} = \|f\|_{L^{p(⋅)}\cdot(E;w)}\). When \(p(⋅) = p\) is a constant, \(L^{p(⋅)}(E;w)\) coincides with the classical weighted Lebesgue space \(L^p(E;w)\).

Edmunds et al. \([9]\) investigated the boundedness of \(M^+\) and \(M^-\) in the variable Lebesgue spaces \(L^{p(⋅)}\). The two-weight weak type modular inequalities of \(M^+\) and \(M^-\) in \(L^{p(⋅)}\) were discussed in \([10]\). In \([11]\), Kolkashvili et al. acquired the sufficient condition such that \(M^+\) and \(M^-\) are bounded from \(L^{p(⋅)}(\mathbb{R}^-)\) to \(L^{p(⋅)}(\mathbb{R}^+),\) where \(p(⋅)\) is constant on some interval \((a, +\infty)\) \((a > 0)\) and \(T_{vw}\) is bounded in \(L^{p(⋅)}(\mathbb{R}^+)\) with
\[
T_{vw}f(x) = v(x) \int_x^{+\infty} \frac{f(y)}{yw(y)} \, dy, \quad x \in \mathbb{R}^+.
\]
(8)
Throughout this paper, \(u\) and \(v\) are nonnegative locally integrable functions and \(C\) is a positive constant whose value may change from one occurrence to the next. For exponent function \(p(⋅)\) with \(p(\cdot) \geq 1\), its conjugate exponent will be denoted by \(p'(⋅)\) with \(1/p(x) + 1/p'(x) = 1\). For a Lebesgue measurable set \(E \subset \mathbb{R}, \chi_E\) will be its characteristic function.

Definition 3. It is given that \(p(⋅) : E \to [1, +\infty)\) such that \(1 \leq p_− \leq p_+ < \infty.\) We say that \((u, v) \in S^{p(⋅)}(E)\) if there exists a constant \(C > 0\) such that for every interval \(I = (a, b) < E,\)
\[
\int_I M^+(\chi_I \sigma)(x) u(x)^{p(x)} \, dx \leq C \int_I \sigma(x) \, dx
\]
< +\infty,
where \(v(x) > 0\) a.e. \(x \in E\) and \(\sigma(x) = v(x)^{-p'(x)}\).

Definition 4. It is given that \(p(⋅) : E \to [1, +\infty)\) such that \(1 \leq p_− \leq p_+ < \infty.\) We say that \((u, v) \in S^{p(⋅)}(E)\) if there exist constants \(m \geq 1/p_−\) and \(C_1, C_2 > 0\) such that for every interval \(I = (a, b) < E,\)
\[
\int_I M^+(\chi_I \sigma)(x) u(x)^{p(x)} \, dx \geq 0,
\]
(10)
\[
\int_I \frac{u(x)^{p(x)}}{(e + |x|)^{mp}} \, dx \leq C \int_I \frac{\sigma(x)}{(e + |x|)^{mp}} \, dx \leq C_2,
\]
where \(v(x) > 0\) a.e. \(x \in E\) and \(\sigma(x) = v(x)^{-p'(x)}\).

\(S^p(E)\) can be considered to be the generalization of Sawyer-type two-weight condition (see \([1]\)) for the one-sided maximal operator in variable exponents case. If \(E = \mathbb{R}\) we will simply write \(S^p(E)\) as \(S^p_\mathbb{R}\) and \(S^{p^+}(\mathbb{R})\) as \(S^{p^+}_\mathbb{R}\). We can define \(p(⋅) \in \mathcal{P}_σ(E), (u, v) \in S^{p^+}(E),\) and \((u, v) \in S^{p^+}(E)\) in similar ways.

Our main results are the following theorems.

Theorem 5. Let \(D = (−\infty, d) < [1, +\infty)\) such that \(1 < p_D < p^+ < +\infty.\) Then \((u, v) \in S^{p(⋅)}(D)\) and \(p(⋅) \in (\mathcal{P}_σ(\mathbb{R}) \cap LH_{co}(\mathbb{R}))\) with \(\sigma(x) = v(x)^{-p'(x)}\), then
\[
\|M^+(f)\|_{L^{p(⋅)}(D)} \leq C \|f\|_{L^{p(⋅)}(D)}.
\]
(11)

Theorem 6. Let \(D = (d, +\infty) < (−\infty, +\infty)\) such that \(1 < p^- \leq p < +\infty.\) Then \((u, v) \in S^{p^-}(D)\) and \(p(⋅) \in (\mathcal{P}_σ(\mathbb{R}) \cap LH_{co}(\mathbb{R}))\) with \(\sigma(x) = v(x)^{-p'(x)}\), then
\[
\|M^+(f)\|_{L^{p(⋅)}(D)} \leq C \|f\|_{L^{p(⋅)}(D)}.
\]
(12)

Theorem 7. Let \(D\) be a bounded interval and \(p(⋅) : D \to [1, +\infty)\) such that \(1 < p^+ \leq p^- < +\infty.\) Then \((u, v) \in S^{p^+}(D)\) and \(p(⋅) \in (\mathcal{P}_σ(\mathbb{R}) \cap LH_{co}(\mathbb{R}))\) with \(\sigma(x) = v(x)^{-p'(x)}\), then
\[
\|M^+(f)\|_{L^{p(⋅)}(D)} \leq C \|f\|_{L^{p(⋅)}(D)}.
\]
(13)

Corollary 8. It is given that \(p(⋅) : \mathbb{R} \to [1, +\infty)\) such that \(1 < p < p^+ < +\infty.\) Then \((u, v) \in S^{p^+}(\mathbb{R})\) and \(p(⋅) \in (\mathcal{P}_σ(\mathbb{R}) \cap LH_{co}(\mathbb{R} \} [-c, c])\) with \(\sigma(x) = v(x)^{-p'(x)}\), then
\[
\|M^+(f)\|_{L^{p(⋅)}(\mathbb{R})} \leq C \|f\|_{L^{p(⋅)}(\mathbb{R})}.
\]
(14)

Corollary 9. It is given that \(p(⋅) : \mathbb{R} \to [1, +\infty)\) such that \(1 < p \leq p^+ < +\infty.\) Then \((u, v) \in S^{p^+}(\mathbb{R})\) and \(p(⋅) \in (\mathcal{P}_σ(\mathbb{R}) \cap LH_{co}(\mathbb{R} \} [-c, c])\) with \(\sigma(x) = v(x)^{-p'(x)}\), then
\[
\int \frac{\sigma(x)}{(e + |x|)^{mp}} \, dx \leq C,
\]
(15)

where \(\sigma(x) = v(x)^{-p'(x)}\).
\[
\| (M^+ f) u \|_{P(D)} \leq C \| f \|_{P(D)}. \tag{16}
\]

Remark 10. In Theorems 5 and 6, the set \( S^p_{P(D)} \cap S^{p'}_{P(D)} \) is not empty. In fact, if \( v(x) \geq 1 \) and \( 0 \leq u(x) \leq v(x)^{1/(1-p(x))} \), when \( x \in D \), then \( (u, v) \in (S^p_{P(D)} \cap S^{p'}_{P(D)}(D)) \).

Remark 11. If we take \( u(x) = 1 \) and \( v(x) = 1 \) for a.e. \( x \in D \) whenever \( D \) is an open interval, then \( \mathcal{B}^p_\alpha(D) = \mathcal{B}^p_\alpha(D) = \mathcal{B}^{p'}_\alpha(D) \) (see Definition 18 below) and the results of this paper coincide with those of [9].

Remark 12. If we use the Sawyer-type condition was used earlier in [12] to characterize the two-weight boundedness of the classical Hardy-Littlewood maximal operator \( M \).

Corresponding results for variable Lebesgue spaces can be found in [13–15].

2. Proof of the Main Results

In order to establish our main results, we will need following lemmas.

Lemma 14 (see [4]). It is given that \( E \) and \( p(\cdot) : E \rightarrow [1, +\infty) \) such that \( P_E^p < \infty \).

(a) If \( \| f \|_{L^{p(\cdot)}(E)} \leq 1 \), then \( p_E^{p(\cdot)}(E) f^{1/p_E^p} \leq \| f \|_{L^{p(\cdot)}(E)} \leq p_E^{p(\cdot)}(E) f^{1/p_E^p} \).

(b) If \( \| f \|_{L^{p(\cdot)}(E)} > 1 \), then \( p_E^{p(\cdot)}(E) f^{1/p_E^p} \leq \| f \|_{L^{p(\cdot)}(E)} \leq p_E^{p(\cdot)}(E) f^{1/p_E^p} \).

In particular, if \( \| f \|_{L^{p(\cdot)}(E)} \leq 1 \), then \( p_E^{p(\cdot)}(E) f \leq 1 \).

Lemma 15 (see [4]). If \( P_E^p < \infty \), the set of bounded functions with compact support is dense in \( L^{p(\cdot)}(E) \).

Lemma 16 (see [1]). Suppose \( g \geq 0 \) is integrable function with compact support on \( \mathbb{R} \). If \( I = (a, b) \) is a component interval of the open set \( \{ x \in \mathbb{R} : M^+ g(x) > \lambda \} \) with \( \lambda > 0 \), then

\[
\frac{1}{b-x} \int_x^b g(t) \, dt \geq \lambda \quad \text{for} \quad a \leq x < b. \tag{17}
\]

Lemma 17 (see [16]). It is given that a set \( G \subseteq \mathbb{R} \) and two exponents \( s(\cdot) \) and \( r(\cdot) \) such that

\[
|s(y) - r(y)| \leq \frac{C_0}{\ln(e + |y|)}. \tag{18}
\]

Then for every \( t \geq 1 \) there exists a constant \( C = C(t, C_0) \) such that for all functions \( f \) with \( |f(y)| \leq 1 \),

\[
\int_G [f(y)]^{s(y)} \, d\mu(y) \leq C \left( \int_G [f(y)]^{r(y)} \, d\mu(y) \right)^{1/r},
\]

where \( \mu \) is a given nonnegative measure.

Proof of Theorem 5. By the homogeneity, Lemma 15, the Fatou lemma for variable Lebesgue spaces [4] and Lemma 14, we only need to prove

\[
\int_{-\infty}^d M^+ f(x)^{p(x)} u(x)^{p(x)} \, dx \leq C \tag{20}
\]

for the nonnegative bounded function \( f \) with compact support and \( \| f \|_{L^{p(\cdot)}(E)} = 1 \). Let \( N > |d| \) be a positive integer and for \( k \in \mathbb{Z} \), define

\[
\Omega_{k,N} = \{ x \in \mathbb{R} : M^+ f(x) > 2^k \} \cap (-N, d). \tag{21}
\]

Obviously each \( \Omega_{k,N} \) is a bounded open set. Let \( I_{jk} = (a_{jk}, b_{jk}) \) be the component intervals of \( \Omega_{k,N} \), where \( j \) is an integer. Applying Lemma 16 to a fixed \( I_{jk} \), we have

\[
\frac{1}{b_{jk} - a_{jk}} \int_{a_{jk}}^{b_{jk}} f(t) \, dt \geq 2^k \quad \text{for} \quad x \in [a_{jk}, b_{jk}). \tag{22}
\]

Let \( E_{jk} = I_{jk} \setminus \Omega_{k+1,N} \), then the sets \( \{ E_{jk} \} \) are pairwise disjoint and for every \( k \)

\[
\bigcup_j E_{jk} = \{ x \in \mathbb{R} : 2^k < M^+ f(x) \leq 2^{k+1} \} \cap (-N, d). \tag{23}
\]

Therefore

\[
\int_{-N}^d M^+ f(x)^{p(x)} u(x)^{p(x)} \, dx
\]

\[
= \sum_{k,j} \int_{E_{jk}} M^+ f(x)^{p(x)} u(x)^{p(x)} \, dx
\]

\[
\leq 2^{p_E^p} \sum_{k,j} \int_{E_{jk}} \left( \frac{1}{b_{jk} - a_{jk}} \int_{a_{jk}}^{b_{jk}} f(t) \, dt \right)^{p(x)} u(x)^{p(x)} \, dx
\]

\[
= 2^{p_E^p} \sum_{k,j} \int_{-N}^d (A_{jk} (f^{-1})(x))^{p(x)} \, dx,
\]
where
\[ A_{jk}g(x) = \left( \frac{1}{\int_{b^j_k}^{b^j_k} g(t) \sigma(t) \, dt} \right) \chi_{E_{jk}}(x), \]
\[ \varphi_{jk}(x) = \left( \frac{1}{b^j_k - x} \right) \left( \frac{1}{\int_{b^j_k}^{b^j_k} g(t) \sigma(t) \, dt} \right) \chi_{E_{jk}}(x). \]

We show that for every \( j \) and \( k \) the inequality
\[ A_{jk}g(x)^{p(x)} \leq C \left( A_{jk} \left( g^{p(\cdot)/p_D}(x) \right)^{p_D} + A_{jk} R(x)^{p_D} + R(x)^{p_D} \right) \]
(26)
is valid, where \( g = f\sigma^{-1}, R(x) = C/(e + |x|)^m \) (\( m \geq 1/p_D \) is the same as that in (10)), and \( C \) is independent of \( j \) and \( k \).

Let \( q(x) = p(x)/p_D \), then \( q(x) \in \mathcal{P}_D(\sigma) \). By \( \|f\|_{L^{p(x)}(D)} = 1 \) and Lemma 14 we get
\[ \int_D g(x)^{p(x)} \sigma(x) \, dx = \int_D f(x)^{p(x)} \nu(x)^{p(x)} \, dx \leq 1. \]
(27)
Let \( g(x) = g_1(x) + g_2(x) \), where
\[ g_1(x) = g(x) \mathbb{1}_{\{x \in D, g(x) > 1\}}(x), \]
\[ g_2(x) = g(x) \mathbb{1}_{\{x \in D, g(x) < 1\}}(x), \]
then
\[ A_{jk}g(x)^{p(x)} \leq 2^{p(x)-1} \left( A_{jk}g_1(x)^{p(x)} + A_{jk}g_2(x)^{p(x)} \right). \]
(29)
Since \( g_1(t) \geq 1 \), by (27) we can get
\[ \int_{b^j_k}^{b^j_k} g_1(t) \sigma(t) \, dt \leq \int_D g_1(t)^{p(t)} \sigma(t) \, dt \leq 1 \]
for \( x \in E_{jk} \). Then
\[ A_{jk}g_1(x)^{q(x)} \leq \left( \int_{b^j_k}^{b^j_k} \sigma(t) \, dt \right)^{-q(x)} \]
\[ \cdot \left( \int_{b^j_k}^{b^j_k} g_1(t) \sigma(t) \, dt \right)^{q(x)} \]
\[ = \left( \int_{b^j_k}^{b^j_k} \sigma(t) \, dt \right)^{-q(x)-q(x)} \]
\[ \cdot \left( \int_{b^j_k}^{b^j_k} g_1(t) \sigma(t) \, dt \right)^{q(x)} \]
\[ \leq \left( \int_{b^j_k}^{b^j_k} \sigma(t) \, dt \right)^{-q(x)-q(x)} \]
\[ \cdot \left( \int_{b^j_k}^{b^j_k} g_1(t) \sigma(t) \, dt \right)^{q(x)} \]
\[ \leq \left( \int_{b^j_k}^{b^j_k} \sigma(t) \, dt \right)^{-q(x)-q(x)}/C. \]
(31)
If \( \int_{b^j_k}^{b^j_k} \sigma(t) \, dt \leq 1/2 \), by (4), we have
\[ \left( \int_{b^j_k}^{b^j_k} \sigma(t) \, dt \right)^{-q(x)-q(x)} \]
\[ \leq \left( \int_{b^j_k}^{b^j_k} \sigma(t) \, dt \right)^{-q(x)-q(x)} \]
\[ \leq C. \]
(32)
If \( \int_{b^j_k}^{b^j_k} \sigma(t) \, dt > 1/2 \), then
\[ \left( \int_{b^j_k}^{b^j_k} \sigma(t) \, dt \right)^{-q(x)-q(x)} \leq 2^{-q(x)-q(x)} \leq 2^{q(x)} \leq C. \]
(33)
Therefore
\[ A_{jk}g_1(x)^{p(x)} \]
\[ \leq C \left( \int_{b^j_k}^{b^j_k} \sigma(t) \, dt \int_{b^j_k}^{b^j_k} g_1(t) \sigma(t) \, dt \right)^{q(x)}/C. \]
(34)
Thus, by the Hölder inequality and \( g_1(t) \geq 1 \) we can get
\[ A_{jk}g_2(x)^{p(x)} \]
\[ \leq C \left( \int_{b^j_k}^{b^j_k} \sigma(t) \, dt \int_{b^j_k}^{b^j_k} g_2(t) \sigma(t) \, dt \right)^{p(x)}/C. \]
(35)
Next, we will estimate \( A_{jk}g_2(x) \). If \( A_{jk}g_2(x) \leq R(x) \), by \( R(x) \leq 1 \), we have
\[ A_{jk}g_2(x)^{q(x)} \leq R(x)^{q(x)} \leq R(x). \]
(36)
On the other hand, if \( A_{jk}g_2(x) > R(x) > 0 \), noticing that \( A_{jk}g_2(x) \leq 1 \) and
\[ q(x) - q \geq -|q(x) - q| \quad \text{for } x \in \mathbb{R}, \]
(37)
thus
\[ 0 < A_{jk}g_2(x)^{q(x)} \leq A_{jk}g_2(x)^{-|q(x)|} \leq R(x)^{-|q(x)|}. \]
(38)
Applying (38), \( q(\cdot) \in LH_\infty(D) \) and the Hölder inequality,
\[
A_{jk} g_2(x)^{q(x)} \leq A_{jk} g_2(x)^{-q(x)\cdot q_\infty} A_{jk} g_2(x)^{q_\infty} \\
\leq R(x)^{-q(x)\cdot q_\infty} \\
\frac{1}{b_x} \int_{b_x}^{b_x} g_2(t) \sigma(t) \, dt \\
\leq C \left( \int_{b_x}^{b_x} \sigma(t) \, dt \right)^{-1} \int_{b_x}^{b_x} g_2(t)^{q_\infty} \sigma(t) \, dt.
\]
By \( 0 \leq g_2(t) \leq 1 \) and Lemma 17, we get
\[
A_{jk} g_2(x)^{q(x)} \leq C \left( \int_{b_x}^{b_x} \sigma(t) \, dt \right)^{-1} \\
\left( \int_{b_x}^{b_x} g_2(y)^{q(y)} \sigma(y) \, dy \right) \\
+ \int_{b_x}^{b_x} \sigma(y) \left( e + |y|^m \right) \, dy \\
\leq C \left( A_{jk} \left( g_2^{q(\cdot)} \right)(x) \right) \\
+ A_{jk} R(x).
\]
Hence, combining (36) and (40)
\[
A_{jk} g_2(x)^{p(x)} \leq C \left( A_{jk} \left( g_2^{q(\cdot)} \right)(x) + A_{jk} R(x) \right) \\
+ R(x)^{p_D} \leq C \left( \left( A_{jk} \left( g_2^{q(\cdot)} \right)(x) \right)^{p_D} \right) \\
+ A_{jk} R(x)^{p_D} + R(x)^{p_D}.
\]
This completes the proof of (26) by (29), (35), and (41). Applying (26) to (24), we obtain
\[
\int_{-N}^{d} M^\ast f(x)^{p(x)} u(x)^{p(x)} \, dx \\
\leq C \sum_{k,j} \int_{-N}^{d} \left( A_{jk} \left( \left( f \sigma^{-1} \right)^{p(\cdot)} \right)(x) \right)^{p_D} \varphi_{jk}(x) \, dx \\
+ C \sum_{k,j} \int_{-N}^{d} A_{jk} R(x)^{p_D} \varphi_{jk}(x) \, dx \\
+ C \sum_{k,j} \int_{-N}^{d} R(x)^{p_D} \varphi_{jk}(x) \, dx = I_1 + I_2 + I_3.
\]
Let \( T \) be the following linear operator:
\[
Th(x) = \sum_{k,j} A_{jk} h(x)
\]
and \( \varphi(x) = \sum_{k,j} \varphi_{jk}(x) \). Since \( \|f\|_{L^p(D)} = 1 \), if we show that the operator \( T \) is bounded from \( L^{p(D)}((-N,d),\sigma dx) \) to \( L^{p(\lambda)}((-N,d),\sigma dx) \), we obtain
\[
I_1 = C \int_{-N}^{d} \left( \left( f \sigma^{-\lambda} \right)^{p(\lambda)} \right)(x) \, dx \\
\leq C \int_{-N}^{d} \left( f(x) \sigma(x)^{-\lambda} \right)^{p(\lambda)} \sigma(x) \, dx \\
\leq C \int_{-N}^{d} f(x)^{p(x)} \varphi(x) \, dx \leq C.
\]

It is easy to see that \( T \) is bounded on \( L^{p(\lambda)}(D) \). Therefore, we only need to show that \( T \) is bounded from \( L^1((-N,d),\sigma dx) \) to \( L^{1,\infty}((-N,d),\sigma dx) \) to complete our proof by the Marcinkiewicz interpolation theorem. For arbitrary \( \lambda > 0 \) and each pair \( (j,k) \), define
\[
F_{jk} (\lambda) = E_{jk} \cap \{ x \in (-N,d) : Th(x) > \lambda \}.
\]
Obviously, \( F_{jk} (\lambda) \) are pairwise disjoint. Let \( c_{jk}(\lambda) = \inf F_{jk} (\lambda) \) and \( J_{jk} = [c_{jk}(\lambda),b_{jk}] \). It is clear that any two intervals of \( \{J_{jk}\} \) are disjoint or one is contained in the other. By the definition of \( J_{jk} \), we also have
\[
\int_{J_{jk}} h(t) \sigma(t) \, dt \geq \lambda \int_{J_{jk}} \sigma(t) \, dt.
\]
Let \( \{I_j\} \) be the maximal elements of the family \( \{\{J_{jk}\}\} \). These maximal elements exist since the intervals \( J_{jk} \) have uniformly bounded length. The intervals \( I_j \) also satisfy (46). Then, by (9) and (46), we obtain
\[
\int_{\{x \in (-N,d) : Th(x) > \lambda\}} \varphi(x) \, dx \\
= \sum_{i \in \{k,j \}} \int_{\{x \in (-N,d) \}} \left( \frac{1}{b_{jk} - x} \int_{x}^{b_{jk}} \chi_{I_i}(t) \sigma(t) \, dt \right)^{p(x)} u(x)^{p(x)} \, dx \\
\cdot u(x)^{p(x)} \, dx \leq \sum_{i} M^\ast \left( \chi_{I_i} \sigma \right)^{p(x)} \, dx \leq C \sum_{i} \frac{1}{\lambda} \\
\cdot \int_{I_i} h(x) \sigma(x) \, dx \leq C \int_{-N}^{d} h(x) \sigma(x) \, dx.
\]
This has proved the weak (1,1) inequality for \( T \). Hence the estimate for \( I_1 \) is completed.
Since linear operator \( T \) is bounded from \( L^{p(\lambda)}((-N,d),\sigma dx) \) to \( L^{p(\lambda)}((-N,d),\sigma dx) \), by the second inequality of (10),
\[
I_2 = C \int_{-N}^{d} TR(x)^{p(\lambda)} \varphi(x) \, dx \\
\leq C \int_{-N}^{d} R(x)^{p(\lambda)} \sigma(x) \, dx \leq C \int_{D} \frac{\sigma(x)}{(e+|x|)^{mp_D}} \, dx \leq C.
\]
Similarly, by (10), we have
\[
I_3 = C \sum_{k,j} \int_{-N}^{d} \left( \frac{1}{b_{jk} - x} \int_{x}^{b_k} \sigma(t) \, dt \right)^{p(x)} \cdot u(x)^{p(x)} \chi_{E_{jk}}(x) \, dx \leq C \int_{-N}^{d} \left( M^+ (\sigma \chi(-N,d)) \right)^{p(x)} \cdot u(x)^{p(x)} \, dx \tag{49}
\]
\[
\cdot \left( \frac{u(x)^{p(x)}}{(e + |x|)^{mp_D}} \right) \, dx \leq C \int_{-N}^{d} R(x)^{p_D} \sigma(x) \, dx \leq C.
\]

Therefore, by (42), (44), (48), and (49), we have
\[
\int_{-N}^{d} M^+ f(x)^{p(x)} u(x)^{p(x)} \, dx \leq C, \tag{50}
\]
where \( C \) is independent of \( N \). Let \( N \) tend to infinity and the proof of Theorem 5 is finished. \( \square \)

Theorem 6 can be proved similarly.

Proof of Theorem 7. We can assume \( D \) to be a bounded open interval and \( f \) to be nonnegative with \( \|f\|_{\text{Lip}^1(D)} = 1 \). It is sufficient to prove that
\[
\int_D M^+ f(x)^{p(x)} u(x)^{p(x)} \, dx \leq C. \tag{51}
\]
For \( k \in \mathbb{Z} \), define
\[
\Omega_k = \left\{ x \in D : M^+ f(x) > 2^k \right\}. \tag{52}
\]
Let \( I_{jk} = (a_{jk}, b_{jk}) \) be the component intervals of \( \Omega_k \) and \( E_{jk} = I_{jk} \setminus \Omega_{k+1} \), where \( I \) is an integer. Using the same procedure as (24), we obtain
\[
\int_D M^+ f(x)^{p(x)} u(x)^{p(x)} \, dx \leq 2^k \sum_{k,j} \int_{A_{jk}} \left( f \sigma^{-1}(x) \right)^{p(x)} \varphi_{jk}(x) \, dx, \tag{53}
\]
where
\[
A_{jk}g(x) = \left( \frac{1}{\int_{x}^{b_k} \sigma(t) \, dt} \int_{x}^{b_k} g(t) \sigma(t) \, dt \right) \chi_{E_{jk}}(x), \tag{54}
\]
\[
\varphi_{jk}(x) = \left( \frac{1}{b_{jk} - x} \int_{x}^{b_k} \sigma(t) \, dt \right)^{p(x)} u(x)^{p(x)} \chi_{E_{jk}}(x). \tag{55}
\]
Let \( g(x) = f(x) \sigma^{-1}(x) \), \( g_1(x) = g(x) \chi_{E_{jk}}(x) \) and \( g_2(x) = g(x) \chi_{E_{jk}}(x) \chi_{\Omega_{k+1}}(x) \). The estimate for \( A_{jk}g_1(x) \) is the same as (35). Since \( A_{jk}g_2(x) \leq 1 \), we have
\[
A_{jk}g(x)^{p(x)} \leq C \left( A_{jk}g_1(x)^{p(x)} + A_{jk}g_2(x)^{p(x)} \right) \leq C \left( A_{jk} \left( g^{p(x)/p_D} \right)^{p_D} + 1 \right), \tag{56}
\]
Combining (53) and (55), we obtain
\[
\int_D M^+ f(x)^{p(x)} u(x)^{p(x)} \, dx \leq C \sum_{k,j} \int_D \left( A_{jk} \left( f \sigma^{-1}(x)^{p(x)/p_D} \right)^{p_D} \right) \varphi_{jk}(x) \, dx \tag{57}
\]
\[
\cdot \chi_{E_{jk}}(x) \, dx \leq C \int_D M^+ (\sigma \chi(D)) (x) \varphi_{jk}(x) \, dx \tag{58}
\]
\[
\cdot \left( \frac{u(x)^{p(x)}}{(e + |x|)^{mp_D}} \right) \, dx \leq C \int_D \sigma(x) \, dx \leq C.
\]
\( \square \)

The Corollary 8 can be obtained by the results of Theorems 5, 6, and 7 directly.

Proof of Corollary 9. Without loss of generality, we can assume that \( f \) is nonnegative and bounded with a compact support and \( \|f\|_{\text{Lip}} = 1 \). Let \( H = \mathbb{R} \setminus [-c,c] \). Due to Theorem 7 we only need to prove
\[
\int_H M^+ f(x)^{p(x)} u(x)^{p(x)} \, dx \leq C. \tag{59}
\]
Let \( N > c \) be a positive integer and \( D = (-N,c) \cup (c,N) \). For \( k \in \mathbb{Z} \), define
\[
\Omega_k = \left\{ x \in H : M^+ f(x) > 2^k \right\} \cap D. \tag{60}
\]
Let \( I_{jk} = (a_{jk}, b_{jk}) \) be the component intervals of \( \Omega_k \) and \( E_{jk} = I_{jk} \setminus \Omega_{k+1} \), where \( j \) is an integer. Using the same procedure as (24), we obtain
\[
\int_D M^+ f(x)^{p(x)} u(x)^{p(x)} \, dx \leq 2^k \sum_{k,j} \int_{A_{jk}} \left( f \sigma^{-1}(x) \right)^{p(x)} \varphi_{jk}(x) \, dx, \tag{61}
\]
where \( A_{jk} \) and \( \varphi_{jk} \) have the similar definitions as those in the proof of Theorem 5. Let \( g(x) = f(x) \sigma^{-1}(x) \), \( g_1(x) = g(x) \chi_{E_{jk}}(x) \) and \( g_2(x) = g(x) \chi_{E_{jk}}(x) \chi_{\Omega_{k+1}}(x) \). The estimate for \( A_{jk}g_1(x) \) is also the same as (35). Let \( q(x) = p(x)/p_D \), then \( q(x) \in \text{Lip}_{p_D}(H) \) and \( q(x) \geq q_{\infty} = p_{co}/p \) for
\( x \in H \). Since \( A_{jk}g_2(x) \leq 1 \) for every \( j \) and \( k \), by the Hölder inequality and Lemma 17,
\[
A_{jk}g_2(x)\eta(x) \leq A_{jk}g_2(x) \eta^{\infty} \leq C \left( \int_x^{b_{jk}} \sigma(t) \, dt \right)^{-1}
\]
\[
\cdot \int_x^{b_{jk}} g_2(t) \eta^{\infty} \sigma(t) \, dt \leq C \left( \int_x^{b_{jk}} \sigma(t) \, dt \right)^{-1}
\]
\[
\cdot \left( \int_x^{b_{jk}} g_2(t) \eta(t) \sigma(t) \, dt + \int_x^{b_{jk}} \frac{\sigma(t)}{(e + |t|)^{mp\alpha_{jk}}} \, dt \right)
\]
\[
\leq C \left( A_{jk} \left( g_2^{\eta(t)} \right)(x) + A_{jk} ((e + |x|)^{-m}) (x) \right).
\]
By (35) and (61), we get
\[
A_{jk}g_2(x)\eta(x) \leq C \left( A_{jk}g_1(x)\eta(x) + A_{jk}g_2(x)\eta(x) \right)^{\rho^-}
\]
\[
\leq C \left( A_{jk} \left( g^{\eta(t)\rho^-} \right)(x) \right)^{\rho^-}
\]
\[
+ \left( A_{jk} ((e + |x|)^{-m}) (x) \right)^{\rho^-}.
\]
Therefore we have
\[
\int_D M^+f(x)u(x)\eta(x) \, dx
\]
\[
\leq C \sum_{k,j} \int_D \left( A_{jk} \left( (f\sigma^{-1})^{\eta(t)\rho^-} \right)(x) \right)^{\rho^-} \varphi_{jk}(x) \, dx
\]
\[
+ C \sum_{k,j} \int_D \left( A_{jk} ((e + |x|)^{-m}) (x) \right)^{\rho^-} \varphi_{jk}(x) \, dx
\]
\[
= I_6 + I_7.
\]
By the similar estimates of \( I_1 \) and \( I_2 \), we get \( I_6 \leq C \) and
\[
I_7 \leq C \int_D \frac{\sigma(x)}{(e + |x|)^{mp}} \, dx
\]
\[
\leq C \left( \int_0^\infty \frac{\sigma(x)}{(e + |x|)^{mp}} \, dx + \int_{-N}^N \frac{\sigma(x)}{(e + |x|)^{mp}} \, dx \right)
\]
\[
\leq C.
\]
Let \( N \) tend to infinity and the proof is complete. \( \square \)

3. Applications

In this section, we assume that \( I = (0, b) \) with \( 0 < b < +\infty \). Let
\[
\mathcal{R}_{\alpha}f(x) = \int_0^x f(t) (x - t)^{\alpha(x) - 1} \, dt, \quad x \in (0, b),
\]
\[
\mathcal{W}_{\alpha}f(x) = \int_x^b f(t) (x - t)^{\alpha(x) - 1} \, dt, \quad x \in (0, b),
\]
where \( 0 < \alpha(x) < 1 \). If \( \alpha(x) \) is a constant function, \( \mathcal{R}_{\alpha} \) is the classical Riemann-Liouville operator and \( \mathcal{W}_{\alpha} \) is the classical Weyl operator. In [17], Andersen and Sawyer obtained the two weighted norm inequalities of \( \mathcal{R}_{\alpha} \) and \( \mathcal{W}_{\alpha} \) from \( L^p(R, \eta(x)) \) to \( L^q(R, \varphi(x)) \). Other results about \( \mathcal{R}_{\alpha} \) and \( \mathcal{W}_{\alpha} \) on \( L^p \) can be seen in [18–21] and so forth.

Edmunds et al. [9] studied the boundedness of \( \mathcal{R}_{\alpha} \), \( \mathcal{W}_{\alpha} \), \( \mathcal{R}_{\alpha} \), and \( \mathcal{W}_{\alpha} \) in variable Lebesgue spaces. Kokilashvili et al. [11] discussed the two weighted norm inequalities of \( \mathcal{R}_{\alpha} \) and \( \mathcal{W}_{\alpha} \) in \( L_{\eta}^p(R) \) and \( L_{\eta}^q(R_\star) \). In this section, we will discuss the two-weight inequalities of \( \mathcal{R}_{\alpha} \) and \( \mathcal{W}_{\alpha} \) in \( L_{\eta}^p(I) \).

Definition 18 (see [9]). Given \( p(\cdot) : E \to [1, +\infty) \), we say that \( p(\cdot) \in \mathcal{P}_\alpha(E) \) if there exists a constant \( C > 0 \) such that for all \( x, y \in E \), with \( 0 < x - y < 1/2 \),
\[
p(x) - p(y) \leq \frac{C}{-\ln(x - y)}.
\]
We also say that \( p(\cdot) \in \mathcal{P}_\alpha(E) \) if there exists a constant \( C > 0 \) such that for all \( x, y \in E \) with \( 0 < y - x < 1/2 \),
\[
p(x) - p(y) \leq \frac{C}{-\ln(y - x)}.
\]

Our results in this section are the following theorems:

Theorem 19. It is given that \( p(\cdot) : I \to [1, +\infty) \) such that \( 1 < p_1^\star < +\infty \) and \( 0 < \alpha^\star \leq \alpha(x) < 1/p_1^\star \). If \( (q^{(\eta(t))\rho^-}, p(\cdot), \eta(\cdot)) \in S_{p_1^\star}^+(I) \) and \( p(\cdot) \in (\mathcal{P}_{\alpha}^+(I) \cap \mathcal{P}_{\alpha}^-(I)) \), where \( \sigma(x) = \nu(x)^{-\rho^-} \) and \( q(x) = p(x)/(1 - \alpha(x)p(x)) \), then
\[
\| (\mathcal{R}_{\alpha}f) \|_{L_{\nu(x)}^p(I)} \leq C \| f \|_{L_{\nu(x)}^p(I)}.
\]

Theorem 20. Given \( p(\cdot) : I \to [1, +\infty) \) such that \( 1 < p_1^\star < +\infty \) and \( 0 < \alpha^\star \leq \alpha(x) < 1/p_1^\star \). If \( (u^{(\eta(t))\rho^-}, p(\cdot), \eta(\cdot)) \in S_{p_1^\star}^+(I) \) and \( p(\cdot) \in (\mathcal{P}_{\alpha}^+(I) \cap \mathcal{P}_{\alpha}^-(I)) \), where \( \sigma(x) = \nu(x)^{-\rho^-} \) and \( q(x) = p(x)/(1 - \alpha(x)p(x)) \), then
\[
\| (\mathcal{W}_{\alpha}f) \|_{L_{\nu(x)}^p(I)} \leq C \| f \|_{L_{\nu(x)}^p(I)}.
\]

Remark 21. The sets \( (\mathcal{P}_{\alpha}^+(I) \cap \mathcal{P}_{\alpha}^+(I)) \) and \( (\mathcal{P}_{\alpha}^-(I) \cap \mathcal{P}_{\alpha}^-(I)) \) are not empty. Let \( p(\cdot) \in (\mathcal{P}_{\alpha}^+(I) \cap \mathcal{P}_{\alpha}^-(I)) \). If \( \sigma(x) \) is lower Ahlfors \( Q \)-regular (\( Q > 0 \)), which means there exists a constant \( C = C(Q) > 0 \) such that \( \int_0^1 \sigma(t) \, dt \geq C(b - a)^2 \) for every bounded interval \( (a, b) \subset I \), then \( p(\cdot) \in (\mathcal{P}_{\alpha}^+(I)) \) (resp., \( p(\cdot) \in (\mathcal{P}_{\alpha}^-(I)) \)).

In order to prove Theorem 19, we need the following lemma.

Lemma 22 (see [9]). It is given that \( p(\cdot) \in \mathcal{P}_\alpha(I), 0 < \alpha \leq \alpha(x) < 1/p_1^\star \), and \( g(x) = p(x)/(1 - \alpha(x)p(x)) \). If \( \| f \|_{L_{\nu(x)}^p(I)} \leq 1 \), then there exists a positive constant \( C \) depending only on \( p(\cdot) \) and \( \alpha(x) \) such that
\[
\| \mathcal{W}_{\alpha}f \|_{L_{\nu(x)}^p(I)} \leq C \| f \|_{L_{\nu(x)}^p(I)}^{\rho^-/\rho+x} \chi(I).
\]
Proof of Theorem 19. By the homogeneity of norm, we can assume that \( \|f\|_{L^p(I)} \leq 1 \) and \( \|f\|_{L^p(I)} \leq 1 \). It is sufficient to prove
\[
\int_I (W_{a(x)} f)(x) y(x) u(x) y(x) \, dx \leq C. \tag{71}
\]
Applying (70) and Theorem 7 we have
\[
\int_I (W_{a(x)} f)(x) y(x) u(x) y(x) \, dx \\
\leq C \int_I (M^+ f)(x) p(x) \left( u(x)^p(x)/p(x) \right)^{p(x)} \, dx \leq C. \tag{72}
\]

Using the similar proving method as that of Lemma 3.1 in [9], we can prove the following lemma.

Lemma 23. It is given that \( p(\cdot) \in \mathcal{P}(I), 0 < \alpha_{\ell} \leq \alpha(x) < 1/\ell^p, \) and \( q(x) = p(x)/(1 - \alpha(x) p(x)). \) If \( \|f\|_{L^p(I)} \leq 1 \), then there exists a positive constant \( C \) depending only on \( p(\cdot) \) and \( \alpha(\cdot) \) such that
\[
\mathcal{R}_{a(\cdot)}(|f|)(x) \leq C [M^+ f(x)]^{p(x)/q(x)}, \quad x \in I. \tag{73}
\]

By this lemma and the two weighted results of \( M^+ \), we can get the result of Theorem 20 directly.

Competing Interests
The authors declare that there are no competing interests regarding the publication of this paper.

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References