Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2016, Article ID 1648281, 8 pages
http://dx.doi.org/10.1155/2016/1648281

Research Article

Two-Weight Norm Inequality for the One-Sided Hardy-Littlewood Maximal Operators in Variable Lebesgue Spaces

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Received 24 June 2016; Revised 16 October 2016; Accepted 27 October 2016

Academic Editor: Stanislav Hencl

The authors establish the two-weight norm inequalities for the one-sided Hardy-Littlewood maximal operators in variable Lebesgue spaces. As application, they obtain the two-weight norm inequalities of variable Riemann-Liouville operator and variable Weyl operator in variable Lebesgue spaces on bounded intervals.

1. Introduction and Main Results

The one-sided Hardy-Littlewood maximal operators M^+ and M^- are defined by

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(t)| \, dt,$$
$$M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(t)| \, dt. \tag{1}$$

Sawyer [1] showed that M^+ is bounded from $L^p(v)$ to $L^p(u)$ if the pairs of nonnegative functions (u, v) satisfy Sawyer-type two-weight condition for the one-sided maximal operator. In [2] Martín-Reyes et al. generalized this result to M_g^+, where

$$M_g^+ f(x) = \sup_{h>0} \frac{1}{\int_x^{x+h} g(t) \, dt} \int_x^{x+h} |f(t)| \, g(t) \, dt \tag{2}$$

with g is positive locally integrable function. The similar results are also true for M^- and M_g^- (see [1–3]).

Let E be a measurable set in $\mathbb{R}$. Given a measurable function $p(\cdot) : E \to (0, +\infty)$, we denote

$$p_E^- = \operatorname*{ess\,inf}_{x \in E} p(x),$$
$$p_E^+ = \operatorname*{ess\,sup}_{x \in E} p(x). \tag{3}$$

If $E = \mathbb{R}$, we will simply note that $p^- = p_{\mathbb{R}}^-$ and $p^+ = p_{\mathbb{R}}^+$.

Definition 1. Given $p(\cdot) : E \to [1, +\infty)$, σ is a locally integrable function such that $0 < \sigma(x) < \infty$ a.e. $x \in E$. We say that $p(\cdot) \in \mathscr{P}_\sigma^+(E)$ if there exists a constant $C > 0$ such that for every $(x, x+h) \subset E$, with $\int_x^{x+h} \sigma(t) dt \leqslant 1/2$,

$$p(x) - p_{(x,x+h)}^- \leqslant \frac{C}{-\ln\left(\int_x^{x+h} \sigma(t) \, dt\right)}. \tag{4}$$

Definition 2 (see [4]). It is given that $p(\cdot) : E \to [1, +\infty)$. We say that $p(\cdot) \in LH_\infty(E)$, if there exist constants $p_\infty \geqslant 1$ and $C_\infty > 0$ such that for all $x \in E$

$$|p(x) - p_\infty| \leqslant \frac{C_\infty}{\ln(e + |x|)}. \tag{5}$$

Let $1 \leqslant p_E^- \leqslant p_E^+ < \infty$. The variable Lebesgue space $L^{p(\cdot)}(E)$ is the set of measurable functions f on E such that

$$\rho_{p(\cdot),E}(f) := \int_E |f(x)|^{p(x)} \, dx < \infty. \tag{6}$$

This is a Banach space (see [4–7]) with the norm

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\left\{\lambda > 0 : \rho_{p(\cdot),E}\left(\frac{f}{\lambda}\right) \leqslant 1\right\}. \tag{7}$$

If $E = \mathbb{R}$, we will write $\|f\|_{L^{p(\cdot)}(E)}$ as $\|f\|_{p(\cdot)}$. The variable Lebesgue space is the special case of the Musielak-Orlicz space (see [7, 8]). For the detail of $L^{p(\cdot)}$ we refer to [4–7] and so on.

Let w be a nonnegative locally integrable function on E. The weighted variable Lebesgue space $L_w^{p(\cdot)}(E)$ is the set of measurable functions f on E such that $fw \in L^{p(\cdot)}(E)$ and $\|f\|_{L_w^{p(\cdot)}(E)} = \|fw\|_{L^{p(\cdot)}(E)}$. When $p(\cdot) = p$ is a constant, $L_w^{p(\cdot)}(E)$ coincides with the classical weighted Lebesgue space $L^p(w^p)(E)$.

Edmunds et al. [9] investigated the boundedness of M^+ and M^- in the variable Lebesgue spaces $L^{p(\cdot)}$. The two-weight weak type modular inequalities of M^+ and M^- in $L^{p(\cdot)}$ were discussed in [10]. In [11], Kokilashvili et al. acquired the sufficient condition such that M^+ and M^- are bounded from $L_w^{p(\cdot)}(\mathbb{R}_+)$ to $L_v^{p(\cdot)}(\mathbb{R}_+)$, where $p(\cdot)$ is constant on some interval $(a, +\infty)$ $(a > 0)$ and $T_{v,w}$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$ with

$$T_{v,w} f(x) = v(x) \int_x^{+\infty} \frac{f(y)}{yw(y)} dy, \quad x \in \mathbb{R}_+. \tag{8}$$

Throughout this paper, u and v are nonnegative locally integrable functions and C is a positive constant whose value may change from one occurrence to the next. For exponent function $p(\cdot)$ with $p(x) \geqslant 1$, its conjugate exponent will be denoted by $p'(\cdot)$ with $1/p(x) + 1/p'(x) = 1$. For a Lebesgue measurable set $E \subset \mathbb{R}$, χ_E will be its characteristic function.

Definition 3. It is given that $p(\cdot) : E \to [1, +\infty)$ such that $1 \leqslant p_E^- \leqslant p_E^+ < \infty$. We say that $(u, v) \in S_{p(\cdot)}^+(E)$ if there exists a constant $C > 0$ such that for every interval $I = (a, b) \subset E$, with $\int_{-\infty}^a u(x)^{p(x)} dx > 0$,

$$\int_I M^+(\chi_I \sigma)(x)^{p(x)} u(x)^{p(x)} \, dx \leqslant C \int_I \sigma(x) \, dx < +\infty, \tag{9}$$

where $v(x) > 0$ a.e. $x \in E$ and $\sigma(x) = v(x)^{-p'(x)}$.

Definition 4. It is given that $p(\cdot) : E \to [1, +\infty)$ such that $1 \leqslant p_E^- \leqslant p_E^+ < \infty$. We say that $(u, v) \in S_{p(\cdot)}'^+(E)$ if there exist constants $m \geqslant 1/p_E^-$ and $C_1, C_2 > 0$ such that for every interval $I = (a, b) \subset E$, with $\int_{-\infty}^a u(x)^{p(x)} dx > 0$,

$$\int_I M^+(\chi_I \sigma)(x)^{p(x)} \frac{u(x)^{p(x)}}{(e + |x|)^{mp_E^-}} dx \leqslant C_1 \int_I \frac{\sigma(x)}{(e + |x|)^{mp_E^-}} dx \leqslant C_2, \tag{10}$$

where $v(x) > 0$ a.e. $x \in E$ and $\sigma(x) = v(x)^{-p'(x)}$.

$S_{p(\cdot)}^+(E)$ and $S_{p(\cdot)}'^+(E)$ can be considered to be the generalization of Sawyer-type two-weight condition (see [1]) for the one-sided maximal operator in variable exponents case. If $E = \mathbb{R}$ we will simply write $S_{p(\cdot)}^+(\mathbb{R})$ as $S_{p(\cdot)}^+$ and $S_{p(\cdot)}'^+(\mathbb{R})$ as $S_{p(\cdot)}'^+$. We can define $p(\cdot) \in \mathscr{P}_\sigma^-(E)$, $(u, v) \in S_{p(\cdot)}^-(E)$, and $(u, v) \in S_{p(\cdot)}'^-(E)$ in similar ways.

Our main results are the following theorems.

Theorem 5. *Let $D = (-\infty, d)$, $d < +\infty$, and $p(\cdot) : D \to [1, +\infty)$ such that $1 < p_D^- \leqslant p_D^+ < +\infty$. If $(u, v) \in (S_{p(\cdot)}^+(D) \cap S_{p(\cdot)}'^+(D))$ and $p(\cdot) \in (\mathscr{P}_\sigma^+(D) \cap LH_\infty(D))$ with $\sigma(x) = v(x)^{-p'(x)}$, then*

$$\|(M^+ f) u\|_{L^{p(\cdot)}(D)} \leqslant C \|f v\|_{L^{p(\cdot)}(D)}. \tag{11}$$

Theorem 6. *Let $D = (d, +\infty)$, $d > -\infty$, and $p(\cdot) : D \to [1, +\infty)$ such that $1 < p_D^- \leqslant p_D^+ < +\infty$. If $(u, v) \in (S_{p(\cdot)}^+(D) \cap S_{p(\cdot)}'^+(D))$ and $p(\cdot) \in (\mathscr{P}_\sigma^+(D) \cap LH_\infty(D))$ with $\sigma(x) = v(x)^{-p'(x)}$, then*

$$\|(M^+ f) u\|_{L^{p(\cdot)}(D)} \leqslant C \|f v\|_{L^{p(\cdot)}(D)}. \tag{12}$$

Theorem 7. *Let D be a bounded interval and let $p(\cdot) : D \to [1, +\infty)$ such that $1 < p_D^- \leqslant p_D^+ < +\infty$. If $(u, v) \in S_{p(\cdot)}^+(D)$ and $p(\cdot) \in \mathscr{P}_\sigma^+(D)$ with $\sigma(x) = v(x)^{-p'(x)}$, then*

$$\|(M^+ f) u\|_{L^{p(\cdot)}(D)} \leqslant C \|f v\|_{L^{p(\cdot)}(D)}. \tag{13}$$

Corollary 8. *It is given that $p(\cdot) : \mathbb{R} \to [1, +\infty)$ such that $1 < p^- \leqslant p^+ < +\infty$. If there exists a constant $c > 0$ such that $(u, v) \in (S_{p(\cdot)}^+ \cap S_{p(\cdot)}'^+(\mathbb{R} \setminus [-c, c]))$ and $p(\cdot) \in (\mathscr{P}_\sigma^+(\mathbb{R}) \cap LH_\infty(\mathbb{R} \setminus [-c, c]))$ with $\sigma(x) = v(x)^{-p'(x)}$, then*

$$\|(M^+ f) u\|_{p(\cdot)} \leqslant C \|f v\|_{p(\cdot)}. \tag{14}$$

Corollary 9. *It is given that $p(\cdot) : \mathbb{R} \to [1, +\infty)$ such that $1 < p^- \leqslant p^+ < +\infty$, $(u, v) \in S_{p(\cdot)}^+$, and $p(\cdot) \in \mathscr{P}_\sigma^+(\mathbb{R})$ with $\sigma(x) = v(x)^{-p'(x)}$. If*

(a) *there exists a constant $c > 0$ such that $p(\cdot) \in LH_\infty(\mathbb{R} \setminus [-c, c])$ and $p(x) \geqslant p_\infty$ for $x \in (\mathbb{R} \setminus [-c, c])$,*

(b) *there exist constants $m \geqslant 1/p^-$ and $C > 0$ such that for every interval $I = (a, b) \subset (\mathbb{R} \setminus [-c, c])$, with $\int_{-\infty}^a u(x)^{p(x)} dx > 0$,*

$$\int_I \frac{\sigma(x)}{(e + |x|)^{mp^-}} dx \leqslant C. \tag{15}$$

Then

$$\left\|\left(M^{+}f\right)u\right\|_{p(\cdot)} \leqslant C\left\|fv\right\|_{p(\cdot)}. \tag{16}$$

Remark 10. In Theorems 5 and 6, the set $S^{+}_{p(\cdot)}(D) \cap S'^{+}_{p(\cdot)}(D)$ is not empty. In fact, if $v(x) \geqslant 1$ and $0 \leqslant u(x) \leqslant v(x)^{1/(1-p(x))}$, when $x \in D$, then $(u, v) \in (S^{+}_{p(\cdot)}(D) \cap S'^{+}_{p(\cdot)}(D))$.

Remark 11. If we change the conditions $S^{+}_{p(\cdot)}$, $S'^{+}_{p(\cdot)}$, $\mathscr{P}^{+}_{\sigma}$ to $S^{-}_{p(\cdot)}$, $S'^{-}_{p(\cdot)}$, and $\mathscr{P}^{-}_{\sigma}$, respectively, in above theorems, we will obtain similar results of M^{-}.

Remark 12. If we take $u(x) = 1$ and $v(x) = 1$ for a.e. $x \in D$ whenever D is an open interval, then $\mathscr{P}^{+}_{\sigma}(D) = \mathscr{P}^{+}_{1}(D) = \mathscr{P}^{+}(D)$ (see Definition 18 below) and the results of this paper coincide with those of [9].

Remark 13. The Sawyer-type condition was used earlier in [12] to characterize the two-weight boundedness of the classical Hardy-Littlewood maximal operator M. Corresponding results for variable Lebesgue spaces can be found in [13–15].

2. Proof of the Main Results

In order to establish our main results, we will need following lemmas.

Lemma 14 (see [4]). *It is given that E and $p(\cdot) : E \to [1, +\infty)$ such that $p^{+}_{E} < \infty$.*

(a) *If $\|f\|_{L^{p(\cdot)}(E)} \leqslant 1$, then $\rho_{p(\cdot),E}(f)^{1/p^{-}_{E}} \leqslant \|f\|_{L^{p(\cdot)}(E)} \leqslant \rho_{p(\cdot),E}(f)^{1/p^{+}_{E}}$.*

(b) *If $\|f\|_{L^{p(\cdot)}(E)} > 1$, then $\rho_{p(\cdot),E}(f)^{1/p^{+}_{E}} \leqslant \|f\|_{L^{p(\cdot)}(E)} \leqslant \rho_{p(\cdot),E}(f)^{1/p^{-}_{E}}$.*

In particular, if $\|f\|_{L^{p(\cdot)}(E)} \leqslant 1$, then $\rho_{p(\cdot),E}(f) \leqslant 1$.

Lemma 15 (see [4]). *If $p^{+}_{E} < \infty$, the set of bounded functions with compact support is dense in $L^{p(\cdot)}(E)$.*

Lemma 16 (see [1]). *Suppose $g \geqslant 0$ is integrable function with compact support on $\mathbb{R}$. If $I = (a, b)$ is a component interval of the open set $\{x \in \mathbb{R} : M^{+}g(x) > \lambda\}$ with $\lambda > 0$, then*

$$\frac{1}{b-x}\int_{x}^{b} g(t)\,\mathrm{d}t \geqslant \lambda \quad \text{for } a \leqslant x < b. \tag{17}$$

Lemma 17 (see [16]). *It is given that a set $G \subseteq \mathbb{R}$ and two exponents $s(\cdot)$ and $r(\cdot)$ such that*

$$\left|s(y) - r(y)\right| \leqslant \frac{C_0}{\ln\left(e + |y|\right)}. \tag{18}$$

Then for every $t \geqslant 1$ there exists a constant $C = C(t, C_0)$ such that for all functions f with $|f(y)| \leqslant 1$,

$$\begin{aligned}\int_{G}\left|f(y)\right|^{s(y)}\mathrm{d}\mu(y) &\leqslant C\int_{G}\left|f(y)\right|^{r(y)}\mathrm{d}\mu(y)\\ &\quad+\int_{G}\frac{1}{\left(e+|y|\right)^{tr^{-}_{G}}}\mathrm{d}\mu(y),\end{aligned} \tag{19}$$

where μ is a given nonnegative measure.

Proof of Theorem 5. By the homogeneity, Lemma 15, the Fatou lemma for variable Lebesgue spaces [4] and Lemma 14, we only need to prove

$$\int_{-\infty}^{d} M^{+}f(x)^{p(x)}\,u(x)^{p(x)}\,\mathrm{d}x \leqslant C \tag{20}$$

for the nonnegative bounded function f with compact support and $\|fv\|_{L^{p(\cdot)}(D)} = 1$. Let $N > |d|$ be a positive integer and for $k \in \mathbb{Z}$, define

$$\Omega_{k,N} = \left\{x \in \mathbb{R} : M^{+}f(x) > 2^{k}\right\} \cap (-N, d). \tag{21}$$

Obviously each $\Omega_{k,N}$ is a bounded open set. Let $I_{jk} = (a_{jk}, b_{jk})$ be the component intervals of $\Omega_{k,N}$, where j is an integer. Applying Lemma 16 to a fixed I_{jk}, we have

$$\frac{1}{b_{jk}-x}\int_{x}^{b_{jk}} f(t)\,\mathrm{d}t \geqslant 2^{k} \quad \text{for } x \in \left[a_{jk}, b_{jk}\right). \tag{22}$$

Let $E_{jk} = I_{jk} \setminus \Omega_{k+1,N}$, then the sets $\{E_{jk}\}$ are pairwise disjoint and for every k

$$\bigcup_{j} E_{jk} = \left\{x \in \mathbb{R} : 2^{k} < M^{+}f(x) \leqslant 2^{k+1}\right\} \cap (-N, d). \tag{23}$$

Therefore

$$\begin{aligned}&\int_{-N}^{d} M^{+}f(x)^{p(x)}\,u(x)^{p(x)}\,\mathrm{d}x\\ &= \sum_{k,j}\int_{E_{jk}} M^{+}f(x)^{p(x)}\,u(x)^{p(x)}\,\mathrm{d}x\\ &\leqslant \sum_{k,j}\int_{E_{jk}}\left(2^{k+1}\right)^{p(x)}u(x)^{p(x)}\,\mathrm{d}x\\ &\leqslant 2^{p^{+}_{D}}\sum_{k,j}\int_{E_{jk}}\left(\frac{1}{b_{jk}-x}\int_{x}^{b_{jk}} f(t)\,\mathrm{d}t\right)^{p(x)}u(x)^{p(x)}\,\mathrm{d}x\\ &= 2^{p^{+}_{D}}\sum_{k,j}\int_{-N}^{d}\left(A_{jk}\left(f\sigma^{-1}\right)(x)\right)^{p(x)}\varphi_{jk}(x)\,\mathrm{d}x,\end{aligned} \tag{24}$$

where

$$\begin{aligned} A_{jk}g(x) &= \left(\frac{1}{\int_x^{b_{jk}}\sigma(t)\,\mathrm{d}t}\int_x^{b_{jk}} g(t)\sigma(t)\,\mathrm{d}t\right)\chi_{E_{jk}}(x), \\ \varphi_{jk}(x) &= \left(\frac{1}{b_{jk}-x}\int_x^{b_{jk}}\sigma(t)\,\mathrm{d}t\right)^{p(x)} u(x)^{p(x)}\chi_{E_{jk}}(x). \end{aligned} \tag{25}$$

We show that for every j and k the inequality

$$\begin{aligned} A_{jk}g(x)^{p(x)} \leqslant C\Bigg(&\left(A_{jk}\left(g^{p(\cdot)/p_D^-}\right)(x)\right)^{p_D^-} \\ &+ A_{jk}R(x)^{p_D^-} + R(x)^{p_D^-}\Bigg) \end{aligned} \tag{26}$$

is valid, where $g = f\sigma^{-1}$, $R(x) = C/(e+|x|)^m$ ($m \geqslant 1/p_D^-$ is the same as that in (10)), and C is independent of j and k.

Let $q(x) = p(x)/p_D^-$, then $q(\cdot) \in (\mathscr{P}_\sigma^+(D) \cap LH_\infty(D))$. By $\|fv\|_{L^{p(\cdot)}(D)} = 1$ and Lemma 14 we get

$$\int_D g(x)^{p(x)}\sigma(x)\,\mathrm{d}x = \int_D f(x)^{p(x)} v(x)^{p(x)}\,\mathrm{d}x \leqslant 1. \tag{27}$$

Let $g(x) = g_1(x) + g_2(x)$, where

$$\begin{aligned} g_1(x) &= g(x)\chi_{\{x\in D: g(x)>1\}}(x), \\ g_2(x) &= g(x)\chi_{\{x\in D: g(x)\leqslant 1\}}(x), \end{aligned} \tag{28}$$

then

$$\begin{aligned} A_{jk}g(x)^{p(x)} &\leqslant 2^{p_D^+-1}\left(A_{jk}g_1(x)^{p(x)} + A_{jk}g_2(x)^{p(x)}\right). \end{aligned} \tag{29}$$

Since $g_1(t) \geqslant 1$, by (27) we can get

$$\int_x^{b_{jk}} g_1(t)\sigma(t)\,\mathrm{d}t \leqslant \int_D g_1(t)^{p(t)}\sigma(t)\,\mathrm{d}t \leqslant 1 \tag{30}$$

for $x \in E_{jk}$. Then

$$\begin{aligned} A_{jk}g_1(x)^{q(x)} &\leqslant \left(\int_x^{b_{jk}}\sigma(t)\,\mathrm{d}t\right)^{-q(x)} \\ &\quad\cdot\left(\int_x^{b_{jk}} g_1(t)\sigma(t)\,\mathrm{d}t\right)^{q_{(x,b_{jk})}^-} \\ &= \left(\int_x^{b_{jk}}\sigma(t)\,\mathrm{d}t\right)^{-(q(x)-q_{(x,b_{jk})}^-)} \\ &\quad\cdot\left(\frac{1}{\int_x^{b_{jk}}\sigma(t)\,\mathrm{d}t}\int_x^{b_{jk}} g_1(t)\sigma(t)\,\mathrm{d}t\right)^{q_{(x,b_{jk})}^-}. \end{aligned} \tag{31}$$

If $\int_x^{b_{jk}}\sigma(t)\mathrm{d}t \leqslant 1/2$, by (4), we have

$$\begin{aligned} &\left(\int_x^{b_{jk}}\sigma(t)\,\mathrm{d}t\right)^{-(q(x)-q_{(x,b_{jk})}^-)} \\ &\quad\leqslant \left(\int_x^{b_{jk}}\sigma(t)\,\mathrm{d}t\right)^{C/\ln(\int_x^{b_{jk}}\sigma(t)\mathrm{d}t)} \leqslant C. \end{aligned} \tag{32}$$

If $\int_x^{b_{jk}}\sigma(t)\mathrm{d}t > 1/2$, then

$$\left(\int_x^{b_{jk}}\sigma(t)\,\mathrm{d}t\right)^{-(q(x)-q_{(x,b_{jk})}^-)} \leqslant 2^{q(x)-q_{(x,b_{jk})}^-} \leqslant 2^{q_D^+} \leqslant C. \tag{33}$$

Therefore

$$A_{jk}g_1(x)^{p(x)} \leqslant C\left(\frac{1}{\int_x^{b_{jk}}\sigma(t)\,\mathrm{d}t}\int_x^{b_{jk}} g_1(t)\sigma(t)\,\mathrm{d}t\right)^{q_{(x,b_{jk})}^- p_D^-}. \tag{34}$$

Thus, by the Hölder inequality and $g_1(t) \geqslant 1$ we can get

$$\begin{aligned} A_{jk}g_1(x)^{p(x)} &\leqslant C\left(\frac{1}{\int_x^{b_{jk}}\sigma(t)\,\mathrm{d}t}\int_x^{b_{jk}} g_1(t)^{q_{(x,b_{jk})}^-}\sigma(t)\,\mathrm{d}t\right)^{p_D^-} \\ &\leqslant C\left(\frac{1}{\int_x^{b_{jk}}\sigma(t)\,\mathrm{d}t}\int_x^{b_{jk}} g_1(t)^{q(t)}\sigma(t)\,\mathrm{d}t\right)^{p_D^-} \\ &\leqslant C\left(A_{jk}\left(g^{p(\cdot)/p_D^-}\right)(x)\right)^{p_D^-}. \end{aligned} \tag{35}$$

Next, we will estimate $A_{jk}g_2(x)$. If $A_{jk}g_2(x) \leqslant R(x)$, by $R(x) \leqslant 1$, we have

$$A_{jk}g_2(x)^{q(x)} \leqslant R(x)^{q(x)} \leqslant R(x). \tag{36}$$

On the other hand, if $A_{jk}g_2(x) > R(x) > 0$, noticing that $A_{jk}g_2(x) \leqslant 1$ and

$$q(x) - q_\infty \geqslant -|q(x) - q_\infty| \quad \text{for } x \in \mathbb{R}, \tag{37}$$

thus

$$\begin{aligned} 0 < A_{jk}g_2(x)^{q(x)-q_\infty} &\leqslant A_{jk}g_2(x)^{-|q(x)-q_\infty|} \\ &\leqslant R(x)^{-|q(x)-q_\infty|}. \end{aligned} \tag{38}$$

Applying (38), $q(\cdot) \in LH_\infty(D)$ and the Hölder inequality,

$$\begin{aligned} A_{jk} g_2(x)^{q(x)} &\leqslant A_{jk} g_2(x)^{-|q(x)-q_\infty|} A_{jk} g_2(x)^{q_\infty} \\ &\leqslant R(x)^{-|q(x)-q_\infty|} \\ &\cdot \left(\frac{1}{\int_x^{b_{jk}} \sigma(t)\,\mathrm{d}t} \int_x^{b_{jk}} g_2(t)\,\sigma(t)\,\mathrm{d}t \right)^{q_\infty} \\ &\leqslant C \left(\int_x^{b_{jk}} \sigma(t)\,\mathrm{d}t \right)^{-1} \int_x^{b_{jk}} g_2(t)^{q_\infty} \sigma(t)\,\mathrm{d}t. \end{aligned} \tag{39}$$

By $0 \leqslant g_2(t) \leqslant 1$ and Lemma 17, we get

$$\begin{aligned} A_{jk} g_2(x)^{q(x)} &\leqslant C \left(\int_x^{b_{jk}} \sigma(t)\,\mathrm{d}t \right)^{-1} \\ &\cdot \left(\int_x^{b_{jk}} g_2(y)^{q(y)} \sigma(y)\,\mathrm{d}y \right. \\ &\left. + \int_x^{b_{jk}} \frac{\sigma(y)}{(e+|y|)^{mq^-_{(x,b_{jk})}}} \mathrm{d}y \right) \leqslant C \left(A_{jk} \left(g_2^{q(\cdot)} \right)(x) \right. \\ &\left. + A_{jk} R(x) \right). \end{aligned} \tag{40}$$

Hence, combining (36) and (40)

$$\begin{aligned} A_{jk} g_2(x)^{p(x)} &\leqslant C \left(A_{jk} \left(g_2^{q(\cdot)} \right)(x) + A_{jk} R(x) \right. \\ &\left. + R(x) \right)^{p_D^-} \leqslant C \left(\left(A_{jk} \left(g^{p(\cdot)/p_D^-} \right)(x) \right)^{p_D^-} \right. \\ &\left. + A_{jk} R(x)^{p_D^-} + R(x)^{p_D^-} \right). \end{aligned} \tag{41}$$

This completes the proof of (26) by (29), (35), and (41). Applying (26) to (24), we obtain

$$\begin{aligned} &\int_{-N}^{d} M^+ f(x)^{p(x)} u(x)^{p(x)}\,\mathrm{d}x \\ &\leqslant C \sum_{k,j} \int_{-N}^{d} \left(A_{jk} \left(\left(f\sigma^{-1} \right)^{p(\cdot)/p_D^-} \right)(x) \right)^{p_D^-} \varphi_{jk}(x)\,\mathrm{d}x \\ &+ C \sum_{k,j} \int_{-N}^{d} A_{jk} R(x)^{p_D^-} \varphi_{jk}(x)\,\mathrm{d}x \\ &+ C \sum_{k,j} \int_{-N}^{d} R(x)^{p_D^-} \varphi_{jk}(x)\,\mathrm{d}x =: \mathbf{I_1} + \mathbf{I_2} + \mathbf{I_3}. \end{aligned} \tag{42}$$

Let T be the following linear operator:

$$Th(x) = \sum_{k,j} A_{jk} h(x) \tag{43}$$

and $\varphi(x) = \sum_{k,j} \varphi_{jk}(x)$. Since $\|fv\|_{L^{p(\cdot)}(D)} = 1$, if we show that the operator T is bounded from $L^{p_D^-}((-N,d),\sigma\mathrm{d}x)$ to $L^{p_D^-}((-N,d),\varphi\mathrm{d}x)$, we obtain

$$\begin{aligned} \mathbf{I_1} &= C \int_{-N}^{d} \left(T \left(\left(f\sigma^{-1} \right)^{p(\cdot)/p_D^-} \right)(x) \right)^{p_D^-} \varphi(x)\,\mathrm{d}x \\ &\leqslant C \int_{-N}^{d} \left(f(x)\,\sigma(x)^{-1} \right)^{p(x)} \sigma(x)\,\mathrm{d}x \\ &\leqslant C \int_D f(x)^{p(x)} v(x)^{p(x)}\,\mathrm{d}x \leqslant C. \end{aligned} \tag{44}$$

It is easy to see that T is bounded on $L^\infty(D)$. Therefore, we only need to show that T is bounded from $L^1((-N,d),\sigma\mathrm{d}x)$ to $L^{1,\infty}((-N,d),\varphi\mathrm{d}x)$ to complete our proof by the Marcinkiewicz interpolation theorem. For arbitrary $\lambda > 0$ and each pair (j,k), define

$$F_{jk}(\lambda) := E_{jk} \cap \{x \in (-N,d) : Th(x) > \lambda\}. \tag{45}$$

Obviously, $F_{jk}(\lambda)$ are pairwise disjoint. Let $c_{jk}(\lambda) := \inf F_{jk}(\lambda)$ and $J_{jk} = J_{jk}(\lambda) := [c_{jk}(\lambda), b_{jk})$. It is clear that any two intervals of $\{J_{jk}\}$ are disjoint or one is contained in the other. By the definition of J_{jk}, we also have

$$\int_{J_{jk}} h(t)\,\sigma(t)\,\mathrm{d}t \geqslant \lambda \int_{J_{jk}} \sigma(t)\,\mathrm{d}t. \tag{46}$$

Let $\{J_i\}$ be the maximal elements of the family $\{J_{jk}\}$. These maximal elements exist since the intervals J_{jk} have uniformly bounded length. The intervals J_i also satisfy (46). Then, by (9) and (46), we obtain

$$\begin{aligned} &\int_{\{x\in(-N,d):Th(x)>\lambda\}} \varphi(x)\,\mathrm{d}x \\ &= \sum_i \sum_{\{k,j:J_i\supset J_{jk}\}} \int_{F_{jk}(\lambda)} \left(\frac{1}{b_{jk}-x} \int_x^{b_{jk}} \chi_{J_i}(t)\,\sigma(t)\,\mathrm{d}t \right)^{p(x)} \\ &\cdot u(x)^{p(x)}\,\mathrm{d}x \leqslant \sum_i \int_{J_i} M^+ \left(\chi_{J_i}\sigma \right)(x)^{p(x)} \\ &\cdot u(x)^{p(x)}\,\mathrm{d}x \leqslant C \sum_i \int_{J_i} \sigma(x)\,\mathrm{d}x \leqslant C \sum_i \frac{1}{\lambda} \\ &\cdot \int_{J_i} h(x)\,\sigma(x)\,\mathrm{d}x \leqslant \frac{C}{\lambda} \int_{-N}^{d} h(x)\,\sigma(x)\,\mathrm{d}x. \end{aligned} \tag{47}$$

This has proved the weak $(1,1)$ inequality for T. Hence the estimate for $\mathbf{I_1}$ is completed.

Since linear operator T is bounded from $L^{p_D^-}((-N,d),\sigma\mathrm{d}x)$ to $L^{p_D^-}((-N,d),\varphi\mathrm{d}x)$, by the second inequality of (10),

$$\begin{aligned} \mathbf{I_2} &= C \int_{-N}^{d} TR(x)^{p_D^-} \varphi(x)\,\mathrm{d}x \\ &\leqslant C \int_{-N}^{d} R(x)^{p_D^-} \sigma(x)\,\mathrm{d}x \leqslant C \int_D \frac{\sigma(x)}{(e+|x|)^{mp_D^-}} \mathrm{d}x \\ &\leqslant C. \end{aligned} \tag{48}$$

Similarly, by (10), we have

$$\begin{aligned}\mathbf{I_3} &= C\sum_{k,j}\int_{-N}^{d}\frac{1}{(e+|x|)^{mp_D^-}}\left(\frac{1}{b_{jk}-x}\int_x^{b_{jk}}\sigma(t)\,\mathrm{d}t\right)^{p(x)}\\ &\quad\cdot u(x)^{p(x)}\chi_{E_{jk}}(x)\,\mathrm{d}x \leqslant C\int_{-N}^{d}\left(M^+\left(\sigma\chi_{(-N,d)}\right)\right.\\ &\quad\left.\cdot(x)\right)^{p(x)}\frac{u(x)^{p(x)}}{(e+|x|)^{mp_D^-}}\mathrm{d}x \leqslant C\int_{-N}^{d}R(x)^{p_D^-}\sigma(x)\,\mathrm{d}x\\ &\quad\leqslant C.\end{aligned}\tag{49}$$

Therefore, by (42), (44), (48), and (49), we have

$$\int_{-N}^{d}M^+f(x)^{p(x)}u(x)^{p(x)}\,\mathrm{d}x\leqslant C,\tag{50}$$

where C is independent of N. Let N tend to infinity and the proof of Theorem 5 is finished. □

Theorem 6 can be proved similarly.

Proof of Theorem 7. We can assume D to be a bounded open interval and f to be nonnegative with $\|fv\|_{L^{p(\cdot)}(D)}=1$. It is sufficient to prove that

$$\int_D M^+f(x)^{p(x)}u(x)^{p(x)}\,\mathrm{d}x\leqslant C.\tag{51}$$

For $k\in\mathbb{Z}$, define

$$\Omega_k=\left\{x\in D: M^+f(x)>2^k\right\}.\tag{52}$$

Let $I_{jk}=(a_{jk},b_{jk})$ be the component intervals of Ω_k and $E_{jk}=I_{jk}\setminus\Omega_{k+1}$, where j is an integer. Using the same procedure as (24), we obtain

$$\begin{aligned}&\int_D M^+f(x)^{p(x)}u(x)^{p(x)}\,\mathrm{d}x\\ &\quad\leqslant 2^{p^+}\sum_{k,j}\int_D A_{jk}\left(f\sigma^{-1}\right)(x)^{p(x)}\varphi_{jk}(x)\,\mathrm{d}x,\end{aligned}\tag{53}$$

where

$$\begin{aligned}&A_{jk}g(x)\\ &\quad=\left(\frac{1}{\int_x^{b_{jk}}\sigma(t)\,\mathrm{d}t}\int_x^{b_{jk}}g(t)\sigma(t)\,\mathrm{d}t\right)\chi_{E_{jk}}(x),\\ &\varphi_{jk}(x)\\ &\quad=\left(\frac{1}{b_{jk}-x}\int_x^{b_{jk}}\sigma(t)\,\mathrm{d}t\right)^{p(x)}u(x)^{p(x)}\chi_{E_{jk}}(x).\end{aligned}\tag{54}$$

Let $g(x)=f(x)\sigma^{-1}(x)$, $g_1(x)=g(x)\chi_{\{x\in D:g(x)>1\}}(x)$ and $g_2(x)=g(x)\chi_{\{x\in D:g(x)\leqslant 1\}}(x)$. The estimate for $A_{jk}g_1(x)$ is the same as (35). Since $A_{jk}g_2(x)\leqslant 1$, we have

$$\begin{aligned}A_{jk}g(x)^{p(x)}&\leqslant C\left(A_{jk}g_1(x)^{p(x)}+A_{jk}g_2(x)^{p(x)}\right)\\ &\leqslant C\left(\left(A_{jk}\left(g^{p(\cdot)/p_D^-}\right)(x)\right)^{p_D^-}+1\right).\end{aligned}\tag{55}$$

Combining (53) and (55), we obtain

$$\begin{aligned}&\int_D M^+f(x)^{p(x)}u(x)^{p(x)}\,\mathrm{d}x\\ &\quad\leqslant C\sum_{k,j}\int_D\left(A_{jk}\left(\left(f\sigma^{-1}\right)^{p(\cdot)/p_D^-}\right)(x)\right)^{p_D^-}\varphi_{jk}(x)\,\mathrm{d}x\\ &\quad+C\sum_{k,j}\int_D\varphi_{jk}(x)\,\mathrm{d}x=:\mathbf{I_4}+\mathbf{I_5}.\end{aligned}\tag{56}$$

Similarly as $\mathbf{I_1}$ we can estimate $\mathbf{I_4}$ to get $\mathbf{I_4}\leqslant C$. Applying (9) to $\mathbf{I_5}$, we have

$$\begin{aligned}\mathbf{I_5}&=C\sum_{k,j}\int_D\left(\frac{1}{b_{jk}-x}\int_x^{b_{jk}}\sigma(t)\,\mathrm{d}t\right)^{p(x)}u(x)^{p(x)}\\ &\quad\cdot\chi_{E_{jk}}(x)\,\mathrm{d}x\leqslant C\int_D M^+\left(\sigma\chi_D\right)(x)^{p(x)}\\ &\quad\cdot u(x)^{p(x)}\,\mathrm{d}x\leqslant C\int_D\sigma(x)\,\mathrm{d}x\leqslant C.\end{aligned}\tag{57}$$

□

The Corollary 8 can be obtained by the results of Theorems 5, 6, and 7 directly.

Proof of Corollary 9. Without loss of generality, we can assume that f is nonnegative and bounded with a compact support and $\|fv\|_{p(\cdot)}=1$. Let $H=\mathbb{R}\setminus[-c,c]$. Due to Theorem 7 we only need to prove

$$\int_H M^+f(x)^{p(x)}u(x)^{p(x)}\,\mathrm{d}x\leqslant C.\tag{58}$$

Let $N>c$ be a positive integer and $D=(-N,c)\cup(c,N)$. For $k\in\mathbb{Z}$, define

$$\Omega_k=\left\{x\in H: M^+f(x)>2^k\right\}\cap D.\tag{59}$$

Let $I_{jk}=(a_{jk},b_{jk})$ be the component intervals of Ω_k and $E_{jk}=I_{jk}\setminus\Omega_{k+1}$, where j is an integer. Using the same procedure as (24), we obtain

$$\begin{aligned}&\int_D M^+f(x)^{p(x)}u(x)^{p(x)}\,\mathrm{d}x\\ &\quad\leqslant 2^{p^+}\sum_{k,j}\int_D A_{jk}\left(f\sigma^{-1}\right)(x)^{p(x)}\varphi_{jk}(x)\,\mathrm{d}x,\end{aligned}\tag{60}$$

where A_{jk} and φ_{jk} have the similar definitions as those in the proof of Theorem 5. Let $g(x)=f(x)\sigma^{-1}(x)$, $g_1(x)=g(x)\chi_{\{x\in D:g(x)>1\}}(x)$ and $g_2(x)=g(x)\chi_{\{x\in D:g(x)\leqslant 1\}}(x)$. The estimate for $A_{jk}g_1(x)$ is also the same as (35). Let $q(x)=p(x)/p^-$, then $q(x)\in LH_\infty(H)$ and $q(x)\geqslant q_\infty=p_\infty/p^-$ for

$x \in H$. Since $A_{jk}g_2(x) \leqslant 1$ for every j and k, by the Hölder inequality and Lemma 17,

$$\begin{aligned} A_{jk}g_2(x)^{q(x)} &\leqslant A_{jk}g_2(x)^{q_\infty} \leqslant C\left(\int_x^{b_{jk}} \sigma(t)\,\mathrm{d}t\right)^{-1} \\ &\cdot \int_x^{b_{jk}} g_2(t)^{q_\infty}\sigma(t)\,\mathrm{d}t \leqslant C\left(\int_x^{b_{jk}} \sigma(t)\,\mathrm{d}t\right)^{-1} \\ &\cdot \left(\int_x^{b_{jk}} g_2(t)^{q(t)}\sigma(t)\,\mathrm{d}t + \int_x^{b_{jk}} \frac{\sigma(t)}{(e+|t|)^{mq^-_{(x,b_{jk})}}}\mathrm{d}t\right) \\ &\leqslant C\left(A_{jk}\left(g_2^{q(\cdot)}\right)(x) + A_{jk}\left((e+|x|)^{-m}\right)(x)\right). \end{aligned} \tag{61}$$

By (35) and (61), we get

$$\begin{aligned} A_{jk}g(x)^{p(x)} &\leqslant C\left(A_{jk}g_1(x)^{q(x)} + A_{jk}g_2(x)^{q(x)}\right)^{p^-} \\ &\leqslant C\left(\left(A_{jk}\left(g^{p(\cdot)/p^-}\right)(x)\right)^{p^-}\right. \\ &\quad \left. + \left(A_{jk}\left((e+|x|)^{-m}\right)(x)\right)^{p^-}\right). \end{aligned} \tag{62}$$

Therefore we have

$$\begin{aligned} &\int_D M^+f(x)^{p(x)}u(x)^{p(x)}\,\mathrm{d}x \\ &\leqslant C\sum_{k,j}\int_D \left(A_{jk}\left(\left(f\sigma^{-1}\right)^{p(\cdot)/p^-}\right)(x)\right)^{p^-}\varphi_{jk}(x)\,\mathrm{d}x \\ &\quad + C\sum_{k,j}\int_D \left(A_{jk}\left((e+|x|)^{-m}\right)(x)\right)^{p^-}\varphi_{jk}(x)\,\mathrm{d}x \\ &=: \mathbf{I_6} + \mathbf{I_7}. \end{aligned} \tag{63}$$

By the similar estimates of $\mathbf{I_1}$ and $\mathbf{I_2}$, we get $\mathbf{I_6} \leqslant C$ and

$$\begin{aligned} \mathbf{I_7} &\leqslant C\int_D \frac{\sigma(x)}{(e+|x|)^{mp^-}}\mathrm{d}x \\ &\leqslant C\left(\int_{-N}^{-c} \frac{\sigma(x)}{(e+|x|)^{mp^-}}\mathrm{d}x + \int_c^N \frac{\sigma(x)}{(e+|x|)^{mp^-}}\mathrm{d}x\right) \\ &\leqslant C. \end{aligned} \tag{64}$$

Let N tend to infinity and the proof is complete. □

3. Applications

In this section, we assume that $I = (0,b)$ with $0 < b < +\infty$. Let

$$\begin{aligned} \mathscr{R}_{\alpha(\cdot)}f(x) &= \int_0^x f(t)(x-t)^{\alpha(x)-1}\,\mathrm{d}t, \quad x \in (0,b), \\ \mathscr{W}_{\alpha(\cdot)}f(x) &= \int_x^b f(t)(x-t)^{\alpha(x)-1}\,\mathrm{d}t, \quad x \in (0,b), \end{aligned} \tag{65}$$

where $0 < \alpha(x) < 1$. If $\alpha(x)$ is a constant function, $\mathscr{R}_\alpha$ is the classical Riemann-Liouville operator and $\mathscr{W}_\alpha$ is the classical Weyl operator. In [17], Andersen and Sawyer obtained the two weighted norm inequalities of $\mathscr{R}_\alpha$ and $\mathscr{W}_\alpha$ from $L^p(\mathbb{R}, u\mathrm{d}x)$ to $L^q(\mathbb{R}, v\mathrm{d}x)$. Other results about $\mathscr{R}_\alpha$ and $\mathscr{W}_\alpha$ on L^p can be seen in [18–21] and so forth.

Edmunds et al. [9] studied the boundedness of $\mathscr{R}_\alpha$, $\mathscr{W}_\alpha$, $\mathscr{R}_{\alpha(\cdot)}$, and $\mathscr{W}_{\alpha(\cdot)}$ in variable Lebesgue spaces. Kokilashvili et al. [11] discussed the two weighted norm inequalities of $\mathscr{R}_\alpha$ and $\mathscr{W}_\alpha$ from $L_w^{p(\cdot)}(\mathbb{R}_+)$ to $L_v^{q(\cdot)}(\mathbb{R}_+)$. In this section, we will discuss the two-weight inequalities of $\mathscr{R}_{\alpha(\cdot)}$ and $\mathscr{W}_{\alpha(\cdot)}$ in $L^{p(\cdot)}(I)$.

Definition 18 (see [9]). Given $p(\cdot): E \to [1,+\infty)$, we say that $p(\cdot) \in \mathscr{P}^-(E)$ if there exists a constant $C > 0$ such that for all $x, y \in E$, with $0 < x - y \leqslant 1/2$,

$$p(x) - p(y) \leqslant \frac{C}{-\ln(x-y)}. \tag{66}$$

We also say that $p(\cdot) \in \mathscr{P}^+(E)$ if there exists a constant $C > 0$ such that for all $x, y \in E$ with $0 < y - x \leqslant 1/2$,

$$p(x) - p(y) \leqslant \frac{C}{-\ln(y-x)}. \tag{67}$$

Our results in this section are the following theorems:

Theorem 19. *It is given that $p(\cdot): I \to [1,+\infty)$ such that $1 < p_I^- \leqslant p_I^+ < +\infty$ and $0 < \alpha_I^- \leqslant \alpha(x) < 1/p_I^+$. If $(u^{q(\cdot)/p(\cdot)}, v) \in S_{p(\cdot)}^+(I)$ and $p(\cdot) \in (\mathscr{P}_\sigma^+(I) \cap \mathscr{P}^+(I))$, where $\sigma(x) = v(x)^{-p'(x)}$ and $q(x) = p(x)/(1-\alpha(x)p(x))$, then*

$$\left\|\left(\mathscr{W}_{\alpha(\cdot)}f\right)u\right\|_{L^{q(\cdot)}(I)} \leqslant C\left\|fv\right\|_{L^{p(\cdot)}(I)}. \tag{68}$$

Theorem 20. *Given $p(\cdot): I \to [1,+\infty)$ such that $1 < p_I^- \leqslant p_I^+ < +\infty$ and $0 < \alpha_I^- \leqslant \alpha(x) < 1/p_I^+$. If $(u^{q(\cdot)/p(\cdot)}, v) \in S_{p(\cdot)}^-(I)$ and $p(\cdot) \in (\mathscr{P}_\sigma^-(I) \cap \mathscr{P}^-(I))$, where $\sigma(x) = v(x)^{-p'(x)}$ and $q(x) = p(x)/(1-\alpha(x)p(x))$, then*

$$\left\|\left(\mathscr{R}_{\alpha(\cdot)}f\right)u\right\|_{L^{q(\cdot)}(I)} \leqslant C\left\|fv\right\|_{L^{p(\cdot)}(I)}. \tag{69}$$

Remark 21. The sets $\mathscr{P}_\sigma^+(I) \cap \mathscr{P}^+(I)$ and $\mathscr{P}_\sigma^-(I) \cap \mathscr{P}^-(I)$ are not empty. Let $p(\cdot) \in \mathscr{P}^+(I)$ (resp., $p(\cdot) \in \mathscr{P}^-(I)$). If σ is lower Ahlfors Q-regular ($Q > 0$), which means there exists a constant $C = C(Q) > 0$ such that $\int_a^b \sigma(t)\mathrm{d}t \geqslant C(b-a)^Q$ for every bounded interval $(a,b) \subset I$, then $p(\cdot) \in \mathscr{P}_\sigma^+(I)$ (resp., $p(\cdot) \in \mathscr{P}_\sigma^-(I)$).

In order to prove Theorem 19, we need the following lemma.

Lemma 22 (see [9]). *It is given that $p(\cdot) \in \mathscr{P}^+(I)$, $0 < \alpha_I^- \leqslant \alpha(x) < 1/p_I^+$, and $q(x) = p(x)/(1-\alpha(x)p(x))$. If $\|f\|_{L^{p(\cdot)}(I)} \leqslant 1$, then there exists a positive constant C depending only on $p(\cdot)$ and $\alpha(\cdot)$ such that*

$$\mathscr{W}_{\alpha(\cdot)}(|f|)(x) \leqslant C\left[M^+f(x)\right]^{p(x)/q(x)}, \quad x \in I. \tag{70}$$

Proof of Theorem 19. By the homogeneity of norm, we can assume that $\|fv\|_{L^{p(\cdot)}(I)} \leqslant 1$ and $\|f\|_{L^{p(\cdot)}(I)} \leqslant 1$. It is sufficient to prove

$$\int_I \left(\mathscr{W}_{\alpha(x)} f\right)(x)^{q(x)} u(x)^{q(x)} \, dx \leqslant C. \tag{71}$$

Applying (70) and Theorem 7 we have

$$\begin{aligned} &\int_I \left(\mathscr{W}_{\alpha(x)} f\right)(x)^{q(x)} u(x)^{q(x)} \, dx \\ &\quad \leqslant C \int_I \left(M^+ f\right)(x)^{p(x)} \left(u(x)^{q(x)/p(x)}\right)^{p(x)} dx \leqslant C. \end{aligned} \tag{72}$$

□

Using the similar proving method as that of Lemma 3.1 in [9], we can prove the following lemma.

Lemma 23. *It is given that $p(\cdot) \in \mathscr{P}^-(I)$, $0 < \alpha_I^- \leqslant \alpha(x) < 1/p_I^+$, and $q(x) = p(x)/(1-\alpha(x)p(x))$. If $\|f\|_{L^{p(\cdot)}(I)} \leqslant 1$, then there exists a positive constant C depending only on $p(\cdot)$ and $\alpha(\cdot)$ such that*

$$\mathscr{R}_{\alpha(\cdot)}\left(|f|\right)(x) \leqslant C\left[M^- f(x)\right]^{p(x)/q(x)}, \quad x \in I. \tag{73}$$

By this lemma and the two weighted results of M^-, we can get the result of Theorem 20 directly.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

Acknowledgments

This research is supported by NNSF-China (Grants nos. 11171345 and 51234005).

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