Research Article

Trace Operators on Wiener Amalgam Spaces

Jayson Cunanan and Yohei Tsutsui

Department of Mathematical Sciences, Faculty of Science, Shinshu University, Asahi 3-1-1, Matsumoto, Nagano 390-8621, Japan

Correspondence should be addressed to Jayson Cunanan; jcunanan@shinshu-u.ac.jp

Received 24 February 2016; Accepted 27 March 2016

Academic Editor: Huo Xiong Wu

Copyright © 2016 J. Cunanan and Y. Tsutsui. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The paper deals with trace operators of Wiener amalgam spaces using frequency uniform decomposition operators and maximal inequalities, obtaining sharp results. Additionally, we provide the embedding between standard and anisotropic Wiener amalgam spaces.

1. Introduction

The aim of this paper is to study the trace problem: what can be said about the trace operator $\mathbb{T}$,

$$
\mathbb{T}: f(x) \mapsto f(\mathbb{X},0), \quad \mathbb{X} = (x_1, x_2, \ldots, x_{n-1}),
$$

as a mapping from $W^p_{s}(\mathbb{R}^n)$ to $W^p_{s}(\mathbb{R}^{n-1})$. We note that, for a tempered distribution $f$ defined on $\mathbb{R}^n$, $f(x,0)$ has no straightforward meaning and the question is how to define the trace for a class of tempered distributions. One can resort to the Schwartz function $\phi$, which has a pointwise trace $\phi(\mathbb{X},0)$. It can be extended to (quasi-)Banach function spaces which contain the Schwartz space $\mathcal{S}'$ as a dense subspace.

Our setting is on Wiener amalgam spaces. These spaces, together with modulation spaces, were introduced by Feichtinger [1–3] in the 80s and are now widely used function spaces for various problems in PDE and harmonic analysis [4–10]. They resemble Triebel-Lizorkin spaces in the sense that we are taking $L^p(\ell^1)$ norms but differ with the decomposition operator being used. Instead of the dyadic decomposition operators $\Delta_k \sim \mathcal{F}^{-1}X_{k-1/2} \mathcal{F}$ used for Triebel-Lizorkin spaces, Wiener amalgam spaces use frequency uniform decomposition operators $\square_k \sim \mathcal{F}^{-1}X_k \mathcal{F}$, where $X_k$ denotes a unit cube with center $k$ and $\cup_{k \in \mathbb{Z}^n} X_k = \mathbb{R}^n$.

The concept of trace operator plays an important role in studying the existence and uniqueness of solutions to boundary value problems, that is, to partial differential equations with prescribed boundary conditions [11, 12]. The trace operator makes it possible to extend the notion of restriction of a function to the boundary of its domain to "generalized" functions in various function spaces with regularity. Now, we give a formal definition for the trace operators.

**Definition 1.** Let $X$ and $Y$ be quasi-Banach function spaces defined on $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively. Assume that the Schwartz class $\mathcal{S}'$ is dense in $X$. Denote

$$
\mathbb{T}: f(x) \mapsto f(\mathbb{X},0), \quad f \in \mathcal{S}'.
$$

Assuming that there exists a constant $C > 0$ such that

$$
\| \mathbb{T}f \|_Y \leq C \| f \|_X, \quad \forall f \in \mathcal{S}',
$$

one can extend $\mathbb{T}: X \to Y$ by the density of $\mathcal{S}'$ in $X$ and we write $f(\mathbb{X},0) = \mathbb{T}f$, which is said to be the trace of $f \in X$. Moreover, if there exists a continuous linear operator $\mathbb{T}^{-1}: Y \to X$ such that $\mathbb{T}\mathbb{T}^{-1}$ is the identity operator on $Y$, then $\mathbb{T}$ is said to be a trace-retraction from $X$ onto $Y$.

For $(\alpha \cdot)$-modulation spaces, Besov spaces, and Triebel-Lizorkin spaces, trace theorems have been extensively studied [12–14]. Feichtinger et al. [13] considered the trace theorems on anisotropic modulation spaces $M^{s,p,q,r}_\alpha$ with $0 < p,q,r < \infty$, $s \in \mathbb{R}$ and they obtained $\mathbb{T}M^{s,p,q,r}_\alpha(\mathbb{R}^n) = M^{s,p}_\alpha(\mathbb{R}^{n-1})$. In [15, 16], we find that, for $0 < p,q \leq \infty$, and $s - 1/p > (n - 1)(1/p - 1)$, we have $\mathbb{T}B^{p,q}_s(\mathbb{R}^n) = B^{p,q}_{s-1/p}(\mathbb{R}^{n-1})$ and $\mathbb{T}F^p_{s,p}(\mathbb{R}^n) = F^p_{s-1/p}(\mathbb{R}^{n-1})$ (the case $F^\infty_{s,p}$ is omitted). The use
of atoms as a framework in studying trace problems can be found in [16] and the references within.

Our main results are the following.

**Theorem 2.** Let \( n \geq 2 \), \( 0 < p, q < \infty \), \( s \in \mathbb{R} \). Then
\[
\mathbb{T} : f(x) \mapsto f(\overline{x},0), \quad \overline{x} = (x_1, x_2, \ldots, x_{n-1})
\]
is a trace-retraction from \( W^{p,q}_{s+1/(1\wedge p-1/q)e} (\mathbb{R}^n) \) to \( W^{p,q}_{s} (\mathbb{R}^{n-1}) \).

In view of the embedding in Theorem 6(ii), we immediately have the following corollary.

**Corollary 3.** Let \( n \geq 2 \), \( 0 < p, q < \infty \), \( s \geq 0 \). Then for any \( \epsilon > 0 \)
\[
\mathbb{T} : W^{p,q}_{s+1/(1\wedge p-1/q)e} (\mathbb{R}^n) \rightarrow W^{p,q}_{s} (\mathbb{R}^{n-1}) .
\]

We remark that Corollary 3 is an improvement of an older trace theorem found in [14] and that our result is sharp at least for \( 1 < p, q < \infty \). Moreover, our result shows independence of \( p \). This is due to the pointwise estimates we were able to prove in Section 3. An interesting observation is that the trace theorem of Triebel-Lizorkin spaces stated above shows independence in \( q \). This difference might be due to the decomposition operators used in the norm of each of the function spaces.

The paper is organised as follows. In Section 2, the embedding between standard and anisotropic Wiener amalgam spaces is given. We also define notations, function spaces, and some lemmas to be used throughout this paper. In Section 3, we prove our main result, Theorem 2, and the sharpness of Corollary 3.

**2. Preliminaries**

**Notations.** The Schwartz class of test functions on \( \mathbb{R}^n \) will be denoted by \( \mathcal{S}(\mathbb{R}^n) \) and its dual and the space of tempered distributions will be denoted by \( \mathcal{S}'(\mathbb{R}^n) \). \( L^p(\mathbb{R}^n) \) norm is given by \( \|f\|_{L^p} = \left( \int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} \) whenever \( 1 \leq p < \infty \) and \( \|f\|_{L^\infty} = \text{ess sup}_{x \in \mathbb{R}^n} |f(x)| \). The Fourier transform of a function \( f \in \mathcal{S}(\mathbb{R}^n) \) is given by
\[
\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx,
\]
which is an isomorphism of the Schwartz space \( \mathcal{S}(\mathbb{R}^n) \) onto itself that extends to the tempered distributions \( \mathcal{S}'(\mathbb{R}^n) \) by duality. The inverse Fourier transform is given by \( \mathcal{F}^{-1} f(x) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \hat{f}(\xi) d\xi \). Given \( 1 \leq p \leq \infty \), we denote by \( p' \) the conjugate exponent of \( p \) (i.e., \( 1/p + 1/p' = 1 \)). We use the notation \( u \leq v \) to denote \( u \leq cv \) for a positive constant \( c \) independent of \( u \) and \( v \). We write \( a \land b = \min(a, b) \) and \( a \lor b = \max(a, b) \). We now define the function spaces in this paper.

Let \( \eta : \mathbb{R} \rightarrow [0, 1] \) be a smooth bump function satisfying
\[
\eta(\xi) = \begin{cases} 1, & |\xi| \leq 1 \\ \text{smooth}, & 1 < |\xi| \leq 2 \\ 0, & |\xi| \geq 2. \end{cases}
\]

We write, for \( k = (k_1, \ldots, k_n) \) and \( \xi = (\xi_1, \ldots, \xi_n) \),
\[
\phi_k = \eta(2(\xi - k)).
\]

Put
\[
\psi_k = \phi_{k_1} (\xi_1) \cdots \phi_{k_n} (\xi_n) / \sum_{k \in \mathbb{Z}^n} \phi_{k_1} (\xi_1) \cdots \phi_{k_n} (\xi_n), \quad k \in \mathbb{Z}^n.
\]

**Definition 4 (Wiener amalgam spaces).** For \( 0 < p, q \leq \infty \), and \( s \in \mathbb{R} \), the Wiener amalgam space \( \mathcal{W}^{p,q}_{s,\mathbb{R}^n} \) consists of all tempered distributions \( f \in \mathcal{S}' \) for which the following is finite:
\[
\|f\|_{\mathcal{W}^{p,q}_{s,\mathbb{R}^n}} = \left\| \int \langle k \rangle^s \Box_k f \right\|_{L^p},
\]
with \( \Box_k f = \mathcal{F}^{-1} (\phi_k \hat{f}) \).

We note that (10) is a quasi-norm if \( 0 < p, q \leq \infty \) and norm if \( 1 \leq p, q \leq \infty \). Moreover, (10) is independent of the choice of \( \varphi = \{\varphi_k\}_{k \in \mathbb{Z}^n} \). We refer the reader to [1, 2, 17] for equivalent definitions (continuous versions).

We write \( \overline{x} = (x_1, x_2, \ldots, x_{n-1}) \) and define the anisotropic Wiener amalgam spaces \( \mathcal{W}^{p,q}_{s,\mathbb{R}^{n-1}} \) by the following norm:
\[
\|f\|_{\mathcal{W}^{p,q}_{s,\mathbb{R}^{n-1}}} = \left\| \int \langle k \rangle^s \Box_k f \right\|_{L^p(\mathbb{R}^{n-1})}.
\]

Similarly, for \( \overline{x} = (x_1, x_2, \ldots, x_{n-2}) \), we define
\[
\|f\|_{\mathcal{W}^{p,q}_{s,\mathbb{R}^{n-2}}} = \left\| \int \langle k \rangle^s \Box_k f \right\|_{L^p(\mathbb{R}^{n-2})}.
\]

Comparing amalgam spaces \( \mathcal{W}^{p,q}_{s,\mathbb{R}^n} \) with anisotropic amalgam spaces \( \mathcal{W}^{p,q}_{s,\mathbb{R}^{n-1}} \), we see that \( \mathcal{W}^{p,q}_{s,\mathbb{R}^n} \) is rotational invariant but \( \mathcal{W}^{p,q}_{s,\mathbb{R}^{n-1}} \) is not. Using the almost orthogonality of \( \varphi \) we see that \( \mathcal{W}^{p,q}_{s,\mathbb{R}^n} \) is independent of \( \varphi \). Moreover, recalling that \( \| f \|_{\mathcal{L}_p} \) is the function sequence \( \{\Box_k f\}_{k \in \mathbb{Z}^n} \) equipped with the \( \mathcal{L}_p \) norm, it is easy to see that \( \mathcal{W}^{p,q}_{s,\mathbb{R}^n} \) is a quasi-Banach space for any \( s \in \mathbb{R} \), \( p, q, r \in (0, \infty] \) and a Banach space for any \( s \in \mathbb{R} \), \( 1 \leq p, q, r \leq \infty \). Moreover, the Schwartz space is dense in \( \mathcal{W}^{p,q}_{s,\mathbb{R}^n} \) if \( p, q, r < \infty \). The proofs are similar to those of amalgam spaces in [1, 2, 17].

We collect properties of Wiener amalgam spaces in the following lemma.

**Lemma 5.** Let \( p, q, p_i, q_i \in [1, \infty) \) for \( i = 1, 2 \) and \( s_j, t_j \in \mathbb{R} \) for \( j = 1, 2 \). Then one has the following:
\[
\begin{align*}
(1) \mathcal{S}'(\mathbb{R}^n) & \hookrightarrow \mathcal{W}^{p,q}_{s,\mathbb{R}^n} \hookrightarrow \mathcal{S}'(\mathbb{R}^n); \\
(2) \mathcal{S} & \text{ is dense in } \mathcal{W}^{p,q}_{s,\mathbb{R}^n} \text{ if } p \text{ and } q < \infty; \\
(3) \text{if } q_1 \leq q_2 \text{ and } p_1 \leq p_2, \text{ then } \mathcal{W}^{p_1,q_1}_{s,\mathbb{R}^n} \hookrightarrow \mathcal{W}^{p_2,q_2}_{s,\mathbb{R}^n};
\end{align*}
\]
(4) if \( s_1 \geq s_2 \), then \( W_{s_1}^{p,q} \hookrightarrow W_{s_2}^{p,q} \);

(5) (complex interpolation) let \( 0 < \theta < 1 \), \( 1/p = \theta/p_1 + (1 - \theta)/p_2 \), \( 1/q = \theta/q_1 + (1 - \theta)/q_2 \), and \( s = \theta s_1 + (1 - \theta)s_2 \). Then
\[
[W_{s_1}^{p_1,q_1}, W_{s_2}^{p_2,q_2} |_{[\theta]} = W_{s}^{p,q}.
\]

The proofs of these statements can be found in [1, 3, 17, 18].

**Theorem 6** (embedding: \( W_{s}^{p,q} \hookrightarrow W_{s'}^{p',q'} \)). Let \( p, q, r \in (0, \infty) \) and \( s \geq 0 \).

(I) The case \( r = q \).

(I-i) The case \( r = q = \infty \):
\[
W_{s}^{p,\infty} \hookrightarrow W_{s}^{p,\infty}.
\]

(I-ii) The case \( r = q < \infty \):
\[
W_{s}^{p,q} \hookrightarrow W_{s}^{p,q}.
\]

(II) The case \( r < q \).

(II-i) The case \( q = \infty \). If \( s > 1/r \), then
\[
W_{s}^{p,\infty} \hookrightarrow W_{s}^{p,\infty}.
\]

(II-ii) The case \( q < \infty \). If \( s > (1/r - 1/q) \), then
\[
W_{s}^{p,q} \hookrightarrow W_{s}^{p,q}.
\]

(III) The case \( q < r \).

(III-i) The case \( r = \infty \):
\[
W_{s}^{p,\infty} \hookrightarrow W_{s}^{p,\infty}.
\]

(III-ii) The case \( r < \infty \):
\[
W_{s}^{p,q} \hookrightarrow W_{s}^{p,q}.
\]

Proof. For part (I), it suffices to show the following estimates.

(I-i) Consider
\[
sup_{k} \sup_{k} \sum_{k} \langle k \rangle^{s} |\mathfrak{D}_k f|^q \leq \sum_{k} \langle k \rangle^{s} |\mathfrak{D}_k f|^q.
\]

(I-ii) Consider
\[
\left( \sum_{k} \sum_{k} \langle k \rangle^{s} |\mathfrak{D}_k f|^q \right)^{1/q} \leq \left( \sum_{k} \langle k \rangle^{s} |\mathfrak{D}_k f|^q \right)^{1/q}.
\]

(II-i) Let \( s' = s - 1/r - \varepsilon \), \( (\varepsilon > 0) \). We may assume that
\[
\left( \sum_{k} \sum_{k} \langle k \rangle^{s} |\mathfrak{D}_k f|^q \right)^{1/r} \leq \left( \sum_{k} \langle k \rangle^{s} |\mathfrak{D}_k f|^q \right)^{1/r}.
\]

(II-ii) Let \( s' = s - 1/r - 1/q - \varepsilon \), \( (\varepsilon > 0) \). It suffices to show the embedding in the case \( s' \geq 0 \). Remark that \( q/r \in (1, \infty) \) and \((q/r)' = 1/(r(1/r - 1/q)) \). Let \( \alpha = 1 - r/q + \varepsilon \).

\[
\left( \sum_{k} \sum_{k} \langle k \rangle^{s} |\mathfrak{D}_k f|^q \right)^{1/r} \leq \left( \sum_{k} \sum_{k} \langle k \rangle^{s} |\mathfrak{D}_k f|^q \right)^{1/r}.
\]

Here, we have used \( \alpha(q/r)' = 1 + \varepsilon/(1/r - 1/q) > 1 \). Because \( \alpha/r = 1/r - 1/q + \varepsilon = s - s' \), \( s - s' \geq 0 \), and \( s' \geq 0 \),
\[
\langle k \rangle^{s} |\mathfrak{D}_k f|^q \leq \left( \frac{\langle k \rangle}{\langle k \rangle} \right)^{s-s'} \langle k \rangle^{s'} \leq 1.
\]
(III-i) Consider
\[ \sup_{k} \left( \sum_{k} \langle k \rangle^{sq} |\Box_k f|^q \right)^{1/q} \leq \left( \sum_{k} \langle k \rangle^{sq} |\Box_k f|^q \right)^{1/q}. \] (26)

Here, we have used \( s \geq 0. \)

(III-ii) Using the embedding \( \ell^1 \hookrightarrow \ell^r, \)
\[ \left\{ \sum_{k} \left( \sum_{k} \langle k \rangle^{sq} |\Box_k f|^q \right)^{r/q} \right\}^{1/r} \leq \left( \sum_{k} \langle k \rangle^{sq} |\Box_k f|^q \right)^{1/q}. \] (27)

In the last inequality, we need \( s \geq 0. \) \( \square \)

Lemma 7 (Triebel [12]). Let \( 0 < p < \infty \) and \( 0 < q \leq \infty. \) Let \( \Omega = \{ \Omega_k \}_{k \in \mathbb{Z}^n} \) be a sequence of compact subsets of \( \mathbb{R}^n. \) Let \( d_k \) be the diameter of \( \Omega_k. \) If \( 0 < r < \min(p, q), \) then there exists a constant \( c \) such that
\[ \left\| \sup_{z \in \mathbb{R}^n} |f_k| \right\|_{L^p(\ell^r)} \leq c \left\| f_k \right\|_{L^p(\ell^r)} \] (28)
holds for all \( f \in L^p(\ell^r), \) where \( f = \{f_k\}_{k \in \mathbb{Z}^n} \in \ell^r, \) supp\( \mathcal{F} f_k \)
\[ \subset \Omega_k, \left\| f_k \right\|_{L^p(\ell^r)} < \infty \}. \] (29)

Definition 8 (maximal functions). Let \( b > 0 \) and \( f \in \ell^r. \) Then
\[ \Box_k f(x) := \frac{|\Box_k f(x-y)|}{1 + |y|^b} x \in \mathbb{R}^n, k \in \mathbb{Z}^n. \] (30)

Proposition 9. Let \( 0 < p < \infty \) and \( 0 < q \leq \infty, b > n/\min(p, q). \) Then
\[ \left\| \left( \sum_{k \in \mathcal{Z}^n} \langle k \rangle^{sq} |\Box_k f|^q \right)^{1/q} \right\|_{L^p(\ell^r)} \]
\[ \left\| \left( \sum_{k \in \mathcal{Z}^{n-1}} \left( \sum_{k \in \mathcal{Z}^{n-1}} \langle k \rangle^{sq} |\Box_k f|^q \right)^{r/q} \right)^{1/r} \right\|_{L^p(\ell^r)} \]
are equivalent norms in \( W^{p,q}_b(\mathbb{R}^n) \) and \( W^{p,q}_b(\mathbb{R}^n) \), respectively.

The proof is a direct consequence of Lemma 7, taking \( f_k = \Box_k f. \) See also [14, Proposition].

3. Proof of the Main Results

First, we narrate the idea of the proof. We give an equivalent formulation for \( \mathcal{F} \mathcal{F}(\mathcal{T} f)(x) \), a function in \( \mathbb{R}^{n-1} \), via some \( \mathcal{D} \mathcal{D}_j f(x, 0) \), a function in \( \mathbb{R}^n. \) Then we compute for pointwise estimates between the corresponding \( \ell^1 \) norms and \( \ell^q \) norms for cases \( 0 < q < 1 \) and \( 1 \leq q < \infty. \) Finally, taking \( L^p(\mathbb{R}^{n-1}) \) norms and using our equivalent norms in Proposition 9, we arrive to our conclusion.

We denote by \( \mathcal{F} \mathcal{F}(\mathcal{T} f) \) the partial (inverse) Fourier transform on \( \mathbb{R}^n \). Write \( \{ \eta_j \}_{j \in \mathbb{Z}^{n-1}} \) as versions of (9) in \( \mathbb{R}^{n-1}. \) By the support property of \( \eta_j, \) we observe
\[ \mathcal{D} \mathcal{D}_j (\mathcal{T} f)(x) = \left( \mathcal{F}^{-1}_{\mathbb{R}^n} \eta_j \mathcal{F}_{\mathbb{R}^n} \right)(\mathcal{T} f)(x) \]
\[ = \sum_{l \in \mathbb{Z}^{n-1}} \left( \mathcal{F}^{-1}_{\mathbb{R}^n} \eta_j \mathcal{F}_{\mathbb{R}^n} \left( (\mathcal{T} f)(y, 0) \right) \right)(x), \] (33)
where \( \psi_j = \mathcal{F}^{-1}_{\mathbb{R}^n} \eta_j \), \( l = (l_j), \) and \( \mathcal{D} \mathcal{D}_j f := \mathcal{F}^{-1}_{\mathbb{R}^n} \psi_j \mathcal{F}_{\mathbb{R}^n} f. \) Note that the left-hand side is a function in \( \mathbb{R}^{n-1} \) while the right-hand side is a function in \( \mathbb{R}^n. \)

Recall our maximal function (30) and take \( y_1 = y_2 = \cdots = y_{n-1} = 0, y_n = x_n; \) we have, for \( |x_n| \leq 1, \)
\[ \left\| \Box_{e_j} f(x, 0) \right\|_p \leq \left\| e_j f(x) \right\|_p, \] (34)

Proof of Theorem 2. We start by taking the \( \ell^1 \)-norm of (33).

We write
\[ \left( \sum_{k \in \mathbb{Z}^{n-1}} \langle k \rangle^{sq} |\Box_k f(x, 0)|^q \right)^{1/q} \]
\[ = \left( \sum_{k \in \mathbb{Z}^{n-1}} \langle k \rangle^{sq} \left( \sum_{l \in \mathbb{Z}^{n-1}} \chi_{(l_j-1) \leq l_j} |\Box_k f(x, 0)|^q \right)^{1/q} \right)^{1/q} \] (35)

For \( 0 < q < 1, \) we estimate (35) by
\[ \left( \sum_{k \in \mathbb{Z}^{n-1}} \langle k \rangle^{sq} |\Box_k f(x, 0)|^q \right)^{1/q} \]
\[ \leq \left( \sum_{l \in \mathbb{Z}^{n-1}} \chi_{(l_j-1) \leq l_j} |\Box_{e_j} f(x, 0)|^q \right)^{1/q} \] (36)
\[ = \left( \sum_{l \in \mathbb{Z}^{n-1}} \chi_{(l_j-1) \leq l_j} |\Box_{e_j} f(x, 0)|^q \right)^{1/q} \]

Note that \( \sum_{k \in \mathbb{Z}^{n-1}} \chi_{(l_j-1) \leq l_j} |\Box_{e_j} f(x, 0)|^q = \sum_{j=1}^{n-1} \left| \Box_{e_j} f(x, 0) \right|^q, \) where \( e_j \) is the \( j \)-th column of the identity matrix. In the
sequel, it suffices to consider only the case \( j = 1 \). Moreover, we write \( \Box \psi_j f := \Box_{\tau_{\psi_j} f} \) for some \( \psi_j \) satisfying (9). Using (34) we have

\[
\left( \sum_{l_i \in Z} \sum_{l \in Z} \langle \tilde{t} \rangle^{s_q} |\Box_{\tau_{\psi_j} f} (x_0) |^q \right)^{1/q} \leq \left( \sum_{l_i \in Z} \sum_{l \in Z} \langle \tilde{t} \rangle^{s_q} |\Box^s f (x, x_n) |^q \right)^{1/q}.
\]

(37)

Combining (36) and (37), then taking the \( L^p(\mathbb{R}^{n-1}) \)-norm, and raising to \( p \)th power give

\[
\| f (x, 0) \|_{W^s_p(\mathbb{R}^{n-1})}^p \leq \left( \sum_{l_i \in Z} \sum_{l \in Z} \langle \tilde{t} \rangle^{s_q} |\Box^s f (x, x_n) |^q \right)^{1/q} \left( \sum_{l_i \in Z} \sum_{l \in Z} \langle \tilde{t} \rangle^{s_q} |\Box_{\tau_{\psi_j} f} (x_0) |^q \right)^{1/q}.
\]

(38)

Integrating over \( x_n \in [0, 1] \),

\[
\| f (x, 0) \|_{W^s_p(\mathbb{R}^{n-1})} \leq \left( \sum_{l \in Z} \langle \tilde{t} \rangle^{s_q} \| \Box^s f (x, x_n) \|_{L^p(\mathbb{R}^{n-1})} \right)^{1/q} \left( \sum_{l \in Z} \langle \tilde{t} \rangle^{s_q} \| \Box_{\tau_{\psi_j} f} (x_0) \|_{L^p(\mathbb{R}^{n-1})} \right)^{1/q}.
\]

(39)

Note that the last inequality follows from Proposition 9.

For \( 1 \leq q \leq \infty \), we use Minkowski's inequality to give an upper bound of (35) as follows:

\[
\left( \sum_{k \in Z} \langle \tilde{k} \rangle^{s_q} |\Box_k (T f) (x) |^q \right)^{1/q} \leq \left( \sum_{l \in Z} \langle \tilde{t} \rangle^{s_q} \left( \sum_{k \in Z} \chi_{[l_i \in Z]} |\Box_k f (x_0) | \right)^q \right)^{1/q}.
\]

(42)

Thus, \( T^{-1} : W^{s_q}_p(\mathbb{R}^{n-1}) \to W^{s_q-p,1}_p(\mathbb{R}^n) \).
As the end of this paper, we discuss the optimality of Corollary 3. We recall the counterexample given in [13]. For $1 < p, q < \infty$, there exists a function which shows

$$ T : M^{p,q}_{1/q} (\mathbb{R}^n) \rightarrow M_{0}^{p,q} (\mathbb{R}^{n-1}) \quad (43) $$

Since $M^{p,q} = W^{p,q}_{1/q}(\mathbb{R}^n)$, we also have $T : W^{p,q}_{1/q} (\mathbb{R}^n) \rightarrow W_{0}^{p,q} (\mathbb{R}^{n-1})$. Hence, Corollary 3 is sharp for $p = q$, $1 < p, q < \infty$ (refer to Figure 1). We now claim that it is also sharp for all $1 < p, q < \infty$. Contrary to our claim, suppose $s = 1/q'$ implies $TW_{p,q} (\mathbb{R}^n) = W^{p,q}_{q'} (\mathbb{R}^{n-1})$. Then, by interpolation with the estimate for a point $Q(p_1, q_1)$ with $s = 1/q_1'$, one would obtain an improvement for the segment connecting $P(p, q)$ and $Q(p_1, q_1)$ (refer to Figure 2), which is not possible.

**Competing Interests**

The authors declare that they have no competing interests.

**Acknowledgments**

This work was supported by JSPS, through “Program to Disseminate Tenure Tracking System.” Yohei Tsutsui was also partially supported by JSPS, through Grant-in-Aid for Young Scientists (B) (no. 15K20919).

**References**


Submit your manuscripts at
http://www.hindawi.com