1. Introduction

Let $A_n(p)$ denote the class of functions of the form
\[ f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad (p, n \in \mathbb{N} = \{1, 2, 3, \ldots \}), \]
which are analytic in the open unit disk $U = \{z : |z| < 1\}$. A function $f \in A_n(p)$ is said to be starlike of order $\rho$ in $U$ if it satisfies
\[ \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \rho \quad (0 \leq \rho < p; z \in U). \]

For functions $f$ and $g$ analytic in $U$, we say that $f$ is subordinate to $g$ and write $f(z) < g(z) \ (z \in U)$, if there exists an analytic function $w(z)$ in $U$ such that
\[ |w(z)| \leq |z|, \]
\[ f(z) = g(w(z)) \quad (z \in U). \]

Furthermore, if the function $g$ is univalent in $U$, then
\[ f(z) < g(z) \]
\[ (z \in U) \iff f(0) = g(0), \]
\[ f(U) \subset g(U). \]

Let $SL$ be the class of functions defined by
\[ SL = \left\{ f \in A_1(1) : \left| \left( \frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1 \right\}. \]

Thus a function $f \in SL$ if $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$. In terms of subordination, the class $SL$ consists of normalized analytic functions $f$ satisfying $zf'(z)/f(z) \prec \sqrt{1 + z}$. This class $SL$ was introduced by Sokol and Stankiewicz [1]. Several geometric properties had been investigated by Sokol and Stankiewicz [1]. They determined the radius of convexity for functions in the class $SL$. They also obtained structural formula, as well as growth and distortion theorems for these functions. Estimates for the first few coefficients of functions in $SL$ were obtained in [2]. Recently, Sokol [3] determined various radii for functions belonging to the class $SL$; these include the radii of convexity, starlikeness, and strong starlikeness of order $\rho$. The $SL$-radii for certain well-known classes of functions including the Janowski starlike functions were obtained in [4]. General radii problems were also recently considered in [5, 6] wherein certain radii results for the class $SL$ were obtained as special cases.

Motivated essentially by the above and some recent works [7], the main object of the present paper is to investigate some
properties of multivalent analytic functions associated with the lemniscate of Bernoulli.

2. Main Results

The following lemma will be required.

**Lemma 1** (see [8]). Let $q$ be univalent in the unit disk $U$ and let $\theta$ and $\varphi$ be analytic in a domain $D$ containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\varphi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Then $p(z) < q(z)$, and $q$ is the best dominant.

**Theorem 2.** Let $\alpha > 0$. If $f \in A_n(p)$ satisfies

$$
\frac{f(z)}{z^n} \left(1 + \alpha \left(\frac{zf'(z)}{f(z)} - p\right)\right) < \sqrt{1 + z} \quad (z \in U),
$$

then $f(z)/z^n < \sqrt{1 + z}$.

**Proof.** Let $p(z) = f(z)/z^n$. Then $p(z)$ is analytic in $U$ with $p(0) = 1$. Condition (7) becomes that

$$
\frac{f(z)}{z^n} \left(1 + \alpha \left(\frac{zf'(z)}{f(z)} - p\right)\right) = p(z) + \alpha z f'(z) \quad (z \in U).
$$

Define the function $q : U \to \mathbb{C}$ by $q(z) = \sqrt{1 + z}$ with $q(0) = 1$. Since $q(U) = \{w : |w^2 - 1| < 1\}$ is the right-half of the lemniscate of Bernoulli, $q(U)$ is a convex set and hence $q$ is a convex function. This shows that the function $q(z)$ is starlike with respect to the origin. We now show that $p(z) < q(z)$ if the function $p$ satisfies the differential chain

$$
p(z) + \alpha z f'(z) < q(z) + \alpha z q'(z). \quad (9)
$$

For this purpose, let the functions $\theta$ and $\varphi$ be defined by $\theta(w) = w$ and $\varphi(w) = \alpha$. Clearly the functions $\theta$ and $\varphi$ are analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$. Also let $Q$ and $h$ be the functions defined by

$$
Q(z) = zq'(z)\varphi(q(z)) = \alpha z q'(z),
$$

$$
h(z) = \theta(q(z)) + Q(z) = q(z) + \alpha z q'(z)
$$

$$
= \frac{2 + (2 + \alpha)z}{2\sqrt{1 + z}}.
$$

It is obvious that $Q$ is a starlike univalent function in $U$ and

$$
\Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = \frac{1}{\alpha} + \Re \left\{ \frac{zq''(z)}{q'(z)} \right\} > 0
$$

for $z \in U$.

**Theorem 3.** Let $f \in A_n(p)$ and satisfy

$$
\frac{f(z)}{z^n} < \sqrt{1 + z} \quad (z \in U).
$$

Then

$$
\Re \left(1 - \delta \left(\frac{f(z)}{z^n}\right)^2 + \delta \left(\frac{zf'(z)}{f(z)}\right)\right) > \rho
$$

(17)

where $0 \leq \rho < p$, $\rho/p < \delta \leq 1$, and $r_n(\delta, \rho)$ is the smallest root in $(0,1)$ of the equation

$$
2(1 - \delta)r^n - 2(2 - \rho) + \delta(n + 2p - 4) \leq 0.
$$

The result is sharp.

**Proof.** From (16) we can write

$$
\left(\frac{f(z)}{z^n}\right)^2 = 1 + z^n \varphi(z),
$$

where $\varphi$ is analytic and $|\varphi(z)| \leq 1$ in $U$. Differentiating both sides of (19) logarithmically, we arrive at

$$
\frac{zf'(z)}{f(z)} = p + \frac{nz^n \varphi(z)}{2(1 + z^n \varphi(z))} + \frac{z^{n+1} \varphi'(z)}{2(1 + z^n \varphi(z))}.
$$

(20)

$$(z \in U).$$
Put $|z| = r < 1$ and $(f(z)/z^p)^2 = u + iv$ $(u, v \in \mathbb{R})$. Then (19) implies that

$$z^n \varphi(z) = u - 1 + iv,$$

$$1 - r^n \leq u \leq 1 + r^n. \quad (21)$$

With the help of the Carathéodory inequality

$$|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - r^2}, \quad (22)$$

it follows from (21) that

$$\text{Re} \left( (1 - \delta) \left( \frac{f(z)}{z^p} \right)^2 + \delta \left( \frac{z f'(z)}{f(z)} \right) \right) \geq (1 - \delta) u + p\delta + \frac{n\delta}{2} \text{Re} \left( \frac{z^n \varphi(z)}{1 + z^n \varphi(z)} \right) \geq (1 - \delta) u + p\delta + \frac{n\delta}{2} \left( u - 1 \right)^2 + v^2 - r^{2n} + \frac{\delta}{2} \frac{(u - 1)^2 + v^2 - r^{2n}}{r^{n-1} (1 - r^n)^{3/2}} = \mathcal{F}_n(u, v),$$

$$\frac{\partial \mathcal{F}_n(u, v)}{\partial v} = \delta \gamma_n(u, v), \quad (23)$$

where $0 < r < 1$, $0 < \delta \leq 1$, and

$$\gamma_n(u, v) = \frac{nu}{(u^2 + v^2)^2} + \frac{1}{r^{n-1} (1 - r^n)^{1/2}} + \frac{r^{2n} - ((u - 1)^2 + v^2)}{2 r^{n-1} (1 - r^n)^{3/2}} > 0,$$ \quad (25)

because of (21). Since $\mathcal{F}_n(u, v)$ is a dual function of $v$, from (23), (24), and (25), we see that

$$\mathcal{F}_n(u, v) \geq \mathcal{F}_n(u, 0) = (1 - \delta) u + p\delta + \frac{n\delta}{2} \left( \frac{1 - 1}{u} \right) \geq (1 - \delta) u + p\delta + \frac{n\delta}{2} \left( \frac{1 - 1}{u} \right) + \delta \left( u - 1 \right)^2 - r^{2n} = \frac{2(1 - \delta) r^{2n} - \left( 2(2 - \rho) + \delta (n + 2p - 4) \right) r^n + 2(1 - \rho) + 2\delta (p - 1)}{2(1 - r^n)} = \frac{I_n(r)}{2(1 - r^n)},$$

where $I_n(r) = 2(1 - \delta) r^{2n} - \left( 2(2 - \rho) + \delta (n + 2p - 4) \right) r^n + 2(1 - \rho) + 2\delta (p - 1). \quad (32)$

Note that $I_n(0) = 2(1 - \rho) + 2\delta (p - 1) > 0$ and $I_n(1) = -n\delta < 0$. If we let $r_n(\delta, \rho)$ denote the root in $(0, 1)$ of the equation $I_n(r) = 0$, then (31) yields the desired result (17).
To see that the bound \( r_n(\delta, \rho) \) is the best possible, we consider the function
\[
f(z) = z^p (1 - z^n)^{1/2}.
\]
Clearly the function \( f \) defined by (33) satisfies condition (16). Also it is obvious that, for \( z = r \in (r_n(\delta, \rho), 1) \),
\[
(1 - \delta) \left( \frac{f(r)}{r^p} \right)^2 + \delta \left( \frac{rf'(r)}{f(r)} \right) - \rho = \frac{J_n(r)}{2 (1 - r^n)} < 0,
\]
which shows that the bound \( r_n(\delta, \rho) \) can not be increased. The proof of the theorem is completed.

Setting \( \delta = 1 \), Theorem 3 reduces to the following.

**Corollary 4.** Let \( f \) belong to the class \( A_n(p) \) and satisfy condition (16). Then \( f \) is starlike of order \( \rho \) \((0 \leq \rho < p)\) in
\[
|z| < \left( \frac{2(p - \rho)}{2(p - \rho) + n} \right)^{1/n}.
\]
The result is sharp.

**Competing Interests**
The authors declare that there are no competing interests regarding the publication of this paper.

**Acknowledgments**
This research is supported by the National Natural Science Foundation of China (no. 11571299) and the Natural Science Foundation of Jiangsu Province (no. BK20151304). The work is also supported by the Natural Science Foundation of Suqian College (no. 2015KY25).

**References**


