Research Article

Stability of a Quartic Functional Equation in Restricted Domains

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Let X be a real normed space and Y a Banach space and $f : X \rightarrow Y$. We prove the Ulam-Hyers stability theorem for the quartic functional equation $f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y) = 0$ in restricted domains. As a consequence we consider a measure zero stability problem of the above inequality when $f : \mathbb{R} \rightarrow Y$.

1. Introduction

Let $\mathbb{R}$, X , and Y be the set of real numbers, a real normed space, and a real Banach space, respectively. A mapping $f : X \rightarrow Y$ is called a *quartic mapping* if f satisfies the functional equation

$$f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y) = 0 \quad (1)$$

for all $x, y \in X$. It is known in [1, Theorem 2.1] that the general solutions f of (1) are given by $f(x) = B(x, x, x, x)$ for all $x \in X$, where $B : X \times X \times X \times X \rightarrow Y$ is a symmetric function which is additive for each variable when the other three variables are fixed. The following is a modified version of [1, Theorem 3.1]. We refer the reader to [2, 3] for the stability of generalized quartic mappings.

Theorem 1. Let $\epsilon \geq 0$ be fixed. Suppose that $f : X \rightarrow Y$ satisfies the cubic functional inequality

$$\|f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y)\| \leq \epsilon \quad (2)$$

for all $x, y \in X$. Then there exists a unique quartic mapping $T : X \rightarrow Y$ such that

$$\|f(x) - T(x)\| \leq \frac{\epsilon}{24} \quad (3)$$

for all $x \in X$.

It is a very natural subject to study functional equations or inequalities satisfied on restricted domains or satisfied under restricted conditions [4–19]. Among the results, Jung [14] and Rassias [17] proved the Ulam-Hyers stability of the quadratic functional equations in a restricted domain. As a refined version of the results in [14, 17] we state the result in [20].

Theorem 2. Let $d > 0$. Suppose that $f : X \rightarrow Y$ satisfies the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta \quad (4)$$

for all $x, y \in D := \{(x, y) \in X \times X : \|x\| + \|y\| \geq d\}$. Then there exists a unique mapping $q : X \rightarrow Y$ satisfying

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad (5)$$

for all $x, y \in X$ such that

$$\|f(x) - q(x)\| \leq \frac{3}{2}\epsilon \quad (6)$$

for all $x \in X$.

Also, Chung-Ju-Rassias [21] proved the following Ulam-Hyers stability of cubic functional equation in restricted domains.

Theorem 3. Let $d > 0$. Suppose that $f : X \rightarrow Y$ satisfies the inequality

$$\|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)\| \leq \epsilon \quad (7)$$

for all $x, y \in D := \{(x, y) \in X \times X : \|x\| + \|y\| \geq d\}$. Then there exists a unique mapping $c : X \rightarrow Y$ satisfying

$$c(2x+y) + c(2x-y) - 2c(x+y) - 2c(x-y) - 12c(x) = 0 \quad (8)$$

for all $x, y \in X$ such that

$$\|f(x) - c(x) - 48f(0)\| \leq \frac{79}{14}\epsilon \quad (9)$$

for all $x \in X$.

In this paper, we consider the Ulam-Hyers stability of quartic functional equation (1) in some restricted domains $\Omega \subset X \times X$. First, imposing condition (C) on Ω (see Section 2) we prove that if $f : X \rightarrow Y$ satisfies inequality (2) for all $(x, y) \in \Omega$, then there exists a unique quartic mapping $T : X \rightarrow Y$ such that

$$\|f(x) - T(x) + 231f(0)\| \leq \frac{509}{36}\epsilon \quad (10)$$

for all $x \in X$. Since $\Omega = \{(x, y) \in X \times X : \|x\| + \|y\| \geq d\}$ satisfies condition (C), we obtain the parallel result for quartic functional equation as Theorem 2 for a quadratic functional equation and Theorem 3 for a cubic functional equation.

Secondly, constructing a subset $\Gamma \subset \mathbb{R}^2$ of measure 0 satisfying condition (C) we consider a measure zero stability problem of quartic functional equation (1) for all $x, y \in \Gamma$, where $f : \mathbb{R} \rightarrow Y$ and $\Gamma \subset \{(x, y) \in \mathbb{R}^2 : |x| + |y| \geq d\}$ has 2-dimensional Lebesgue measure 0.

As an application we consider an asymptotic behavior of $f : \mathbb{R} \rightarrow Y$ satisfying the weak condition

$$\|f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y)\| \rightarrow 0 \quad (11)$$

as $|x| + |y| \rightarrow \infty$ only for $(x, y) \in \Gamma$, where Γ has 2-dimensional Lebesgue measure 0.

2. Stability of the Quartic Functional Equation in Restricted Domain

Throughout this section we assume that $\Omega \subset X \times X$ satisfies the following condition:

For given $x, y \in X$ there exists $t \in X$ such that

$$(C) P_{x,y,t} := \{(x-t, y+2t), (x+t, y+2t), (x-t, -y+2t), (x+t, -y+2t), (x, y+4t), (x, -y+4t), (-t, x+y+2t), (t, x-y-2t), (-t, x-y+2t), (t, x+y-2t), (-t, x+y+2t), (t, x-y-t), (-t, x-y+t), (t, x+y-t), (t, x+y), (x, y+2t), (x, -y+2t), (0, x+y+4t), (0, x+y-4t), (0, x-y+4t), (0, x-y-4t), (0, x+y+3t), (0, x+y-3t), (0, x-y+3t), (0, x-y-3t), (0, x+y+2t), (0, x+y-2t), (0, x-y+2t), (0, x-y-2t), (0, x+y-t), (0, x+y+t), (0, x-y-t), (0, x-y+t), (2t, y), (t, 0), (0, t)\} \subset \Omega.$$

Theorem 4. Let $\epsilon \geq 0$ be fixed. Suppose that $f : X \rightarrow Y$ satisfies inequality (2) for all $(x, y) \in \Omega$. Then there exists a unique quartic mapping $T : X \rightarrow Y$ satisfying (10) for all $x \in X$.

Proof. Let

$$F(x, y) = f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y) \quad (12)$$

for all $x, y \in X$. Then

$$\begin{aligned} F(x-t, y+2t) &= f(2x+y) + f(2x-y-4t) \\ &\quad - 4f(x+y+t) \\ &\quad - 4f(x-y-3t) \\ &\quad - 24f(x-t) + 6f(y+2t), \end{aligned}$$

$$\begin{aligned} F(x+t, y+2t) &= f(2x+y+4t) + f(2x-y) \\ &\quad - 4f(x+y+3t) \\ &\quad - 4f(x-y-t) \\ &\quad - 24f(x+t) + 6f(y+2t), \end{aligned}$$

$$\begin{aligned} F(x-t, -y+2t) &= f(2x-y) + f(2x+y-4t) \\ &\quad - 4f(x-y+t) \\ &\quad - 4f(x+y-3t) \\ &\quad - 24f(x-t) + 6f(-y+2t), \end{aligned}$$

$$\begin{aligned} F(x+t, -y+2t) &= f(2x-y+4t) + f(2x+y) \\ &\quad - 4f(x-y+3t) \\ &\quad - 4f(x+y-t) \\ &\quad - 24f(x+t) + 6f(-y+2t), \end{aligned}$$

$$\begin{aligned} F(x, y+4t) &= f(2x+y+4t) \\ &\quad + f(2x-y-4t) \\ &\quad - 4f(x+y+4t) \\ &\quad - 4f(x-y-4t) - 24f(x) \\ &\quad + 6f(y+4t), \end{aligned}$$

$$\begin{aligned}
F(x, -y + 4t) &= f(2x - y + 4t) \\
&\quad + f(2x + y - 4t) \\
&\quad - 4f(x - y + 4t) \\
&\quad - 4f(x + y - 4t) - 24f(x) \\
&\quad + 6f(-y + 4t), \\
F(-t, x + y + 2t) &= f(x + y) + f(-x - y - 4t) \\
&\quad - 4f(x + y + t) \\
&\quad - 4f(-x - y - 3t) \\
&\quad - 24f(-t) + 6f(x + y + 2t), \\
F(t, x - y - 2t) &= f(x - y) + f(-x + y + 4t) \\
&\quad - 4f(x - y - t) \\
&\quad - 4f(-x + y + 3t) - 24f(t) \\
&\quad + 6f(x - y - 2t), \\
F(-t, x - y + 2t) &= f(x - y) + f(-x + y - 4t) \\
&\quad - 4f(x - y + t) \\
&\quad - 4f(-x + y - 3t) \\
&\quad - 24f(-t) + 6f(x - y + 2t), \\
F(t, x + y - 2t) &= f(x + y) + f(-x - y + 4t) \\
&\quad - 4f(x + y - t) \\
&\quad - 4f(-x - y + 3t) - 24f(t) \\
&\quad + 6f(x + y - 2t), \\
F(-t, x + y + t) &= f(x + y - t) + f(-x - y - 3t) \\
&\quad - 4f(x + y) \\
&\quad - 4f(-x - y - 2t) \\
&\quad - 24f(-t) + 6f(x + y + t), \\
F(t, x - y - t) &= f(x - y + t) + f(-x + y + 3t) \\
&\quad - 4f(x - y) \\
&\quad - 4f(-x + y + 2t) - 24f(t) \\
&\quad + 6f(x - y - t), \\
F(-t, x - y + t) &= f(x - y - t) + f(-x + y - 3t) \\
&\quad - 4f(x - y) \\
&\quad - 4f(-x + y - 2t) \\
&\quad - 24f(-t) + 6f(x - y + t),
\end{aligned}$$

$$\begin{aligned}
F(t, x + y - t) &= f(x + y + t) + f(-x - y + 3t) \\
&\quad - 4f(x + y) \\
&\quad - 4f(-x - y + 2t) - 24f(t) \\
&\quad + 6f(x + y - t), \\
F(t, x + y) &= f(x + y + 2t) + f(-x - y + 2t) \\
&\quad - 4f(x + y + t) \\
&\quad - 4f(-x - y + t) - 24f(t) \\
&\quad + 6f(x + y), \\
F(-t, x - y) &= f(x - y - 2t) + f(-x + y - 2t) \\
&\quad - 4f(x - y - t) \\
&\quad - 4f(-x + y - t) - 24f(-t) \\
&\quad + 6f(x - y), \\
F(t, x - y) &= f(x - y + 2t) + f(-x + y + 2t) \\
&\quad - 4f(x - y + t) \\
&\quad - 4f(-x + y + t) - 24f(t) \\
&\quad + 6f(x - y), \\
F(-t, x + y) &= f(x + y - 2t) + f(-x - y - 2t) \\
&\quad - 4f(x + y - t) \\
&\quad - 4f(-x - y - t) - 24f(-t) \\
&\quad + 6f(x + y), \\
F(x + t, y) &= f(2x + y + 2t) \\
&\quad + f(2x - y + 2t) \\
&\quad - 4f(x + y + t) \\
&\quad - 4f(x - y + t) \\
&\quad - 24f(x + t) + 6f(y), \\
F(x - t, y) &= f(2x + y - 2t) \\
&\quad + f(2x - y - 2t) \\
&\quad - 4f(x + y - t) \\
&\quad - 4f(x - y - t) \\
&\quad - 24f(x - t) + 6f(y), \\
F(x, y + 2t) &= f(2x + y + 2t) \\
&\quad + f(2x - y - 2t) \\
&\quad - 4f(x + y + 2t)
\end{aligned}$$

$$\begin{aligned}
& -4f(x-y-2t) - 24f(x) \\
& + 6f(y+2t), \\
F(x, -y+2t) &= f(2x-y+2t) \\
& + f(2x+y-2t) \\
& - 4f(x-y+2t) \\
& - 4f(x+y-2t) - 24f(x) \\
& + 6f(-y+2t), \\
F(0, x+y+4t) &= 3f(x+y+4t) \\
& - 3f(-x-y-4t) \\
& - 24f(0), \\
F(0, x+y-4t) &= 3f(x+y-4t) \\
& - 3f(-x-y+4t) \\
& - 24f(0), \\
F(0, x-y+4t) &= 3f(x-y+4t) \\
& - 3f(-x+y-4t) \\
& - 24f(0), \\
F(0, x-y-4t) &= 3f(x-y-4t) \\
& - 3f(-x+y+4t) \\
& - 24f(0), \\
F(0, x+y+3t) &= 3f(x+y+3t) \\
& - 3f(-x-y-3t) \\
& - 24f(0), \\
F(0, x+y-3t) &= 3f(x+y-3t) \\
& - 3f(-x-y+3t) \\
& - 24f(0), \\
F(0, x-y+3t) &= 3f(x-y+3t) \\
& - 3f(-x+y-3t) \\
& - 24f(0), \\
F(0, x-y-3t) &= 3f(x-y-3t) \\
& - 3f(-x+y+3t) \\
& - 24f(0), \\
F(0, x+y+2t) &= 3f(x+y+2t) \\
& - 3f(-x-y-2t) \\
& - 24f(0), \\
F(0, x+y-2t) &= 3f(x+y-2t) \\
& - 3f(-x-y+2t) \\
& - 24f(0), \\
F(0, x-y+2t) &= 3f(x-y+2t) \\
& - 3f(-x+y-2t) \\
& - 24f(0), \\
F(0, x-y-2t) &= 3f(x-y-2t) \\
& - 3f(-x+y+2t) \\
& - 24f(0), \\
F(2t, y) &= f(y+4t) + f(-y+4t) \\
& - 4f(y+2t) - 4f(-y+2t) \\
& - 24f(2t) + 6f(y), \\
F(t, 0) &= 2f(2t) - 32f(t) + 6f(0), \\
F(0, t) &= 3f(t) - 3f(-t) - 24f(0).
\end{aligned} \tag{13}$$

Thus, we get the functional identity

$$\begin{aligned}
& 3F(x-t, y+2t) + 3F(x+t, y+2t) \\
& + 3F(x-t, -y+2t) + 3F(x+t, -y+2t) \\
& - 3F(x, y+4t) - 3F(x, -y+4t) \\
& - 12F(-t, x+y+2t) - 12F(t, x-y-2t) \\
& - 12F(-t, x-y+2t) - 12F(t, x+y-2t) \\
& - 36F(-t, x+y+t) - 36F(t, x-y-t) \\
& - 36F(-t, x-y+t) - 36F(t, x+y-t) \\
& - 24F(t, x+y) - 24F(-t, x-y) \\
& - 24F(t, x-y) - 24F(-t, x+y) \\
& - 6F(x+t, y) - 6F(x-t, y) + 6F(x, y+2t)
\end{aligned}$$

$$\begin{aligned}
& + 6F(x, -y + 2t) - 4F(0, x + y + 4t) \\
& - 4F(0, x + y - 4t) - 4F(0, x - y + 4t) \\
& - 4F(0, x - y - 4t) + 4F(0, x + y + 3t) \\
& + 4F(0, x + y - 3t) + 4F(0, x - y + 3t) \\
& + 4F(0, x - y - 3t) + 40F(0, x + y + 2t) \\
& + 40F(0, x + y - 2t) + 40F(0, x - y + 2t) \\
& + 40F(0, x - y - 2t) + 32F(0, x + y + t) \\
& + 32F(0, x + y - t) + 32F(0, x - y + t) \\
& + 32F(0, x - y - t) + 18F(2t, y) + 216F(t, 0) \\
& + 1152F(0, t) \\
& = 6f(2x + y) + 6f(2x - y) - 24f(x + y) \\
& - 24f(x - y) - 144f(x) + 36f(y) \\
& - 33264f(0).
\end{aligned} \tag{14}$$

Since Ω satisfies condition (C), for given $x, y \in X$, there exists $t \in X$ such that

$$\|F(u, v)\| \leq \epsilon \tag{15}$$

for all $(u, v) \in P_{x,y,t}$. Thus, dividing (14) by 6 and using the triangle inequality and (15) we have

$$\begin{aligned}
& \|f(2x + y) + f(2x - y) - 4f(x + y) - 4f(x - y) \\
& - 24f(x) + 6f(y) - 5544f(0)\| \leq \frac{1018}{3}\epsilon
\end{aligned} \tag{16}$$

for all $x, y \in X$. Let $g(x) = f(x) + 231f(0)$ for all $x \in X$. Then from (16) we have

$$\begin{aligned}
& \|g(2x + y) + g(2x - y) - 4g(x + y) - 4g(x - y) \\
& - 24g(x) + 6g(y)\| \leq \frac{1018}{3}\epsilon
\end{aligned} \tag{17}$$

for all $x, y \in X$. Using Theorem 1 with (17), we get (10). This completes the proof. $\square$

Remark 5. Letting $x = 0$ in (10) and dividing the result by 232 we have $\|f(0)\| \leq 509\epsilon/(232 \times 36)$. Thus, inequality (10) implies

$$\|f(x) - T(x)\| \leq 231\|f(0)\| + \frac{509}{36}\epsilon \leq \frac{235667}{8352}\epsilon \tag{18}$$

for all $x \in X$.

Let $d \geq 0$. It is easy to see that $\{(x, y) \in X \times X : \|x\| + \|y\| \geq d\}$ satisfies condition (C). Indeed, for given $x, y \in X$ if we choose $t \geq \|x\| + \|y\| + d$, then $P_{x,y,t} \subset \{(x, y) \in X \times X : \|x\| + \|y\| \geq d\}$. Thus, as a direct consequence of Theorem 4 we obtain the following result (see [14, 16, 17] for similar results).

Corollary 6. Let $\epsilon, d \geq 0$ be fixed. Suppose that $f : X \rightarrow Y$ satisfies inequality (2) for all $x, y \in X$ such that $\|x\| + \|y\| \geq d$. Then there exists a unique quartic mapping $T : X \rightarrow Y$ satisfying (10) for all $x \in X$.

In particular, if $\epsilon = 0$, we have the following.

Corollary 7. Suppose that $f : X \rightarrow Y$ (1) for all $(x, y) \in \Omega$. Then, (1) holds for all $x, y \in X$.

3. Stability Problem in a Set of Lebesgue Measure Zero

In this section, we show that even a set Γ of Lebesgue measure zero can satisfy condition (C) when $X = \mathbb{R}$. From now on, we identify $\mathbb{R}^2$ with $\mathbb{C}$.

Lemma 8. Let $P = \{(p_j + a_j t, q_j + b_j t) : j = 1, 2, \dots, r\}$, where $p_j, q_j, a_j, b_j \in \mathbb{R}$ with $a_j^2 + b_j^2 \neq 0$ for all $j = 1, 2, \dots, r$. Then there exists $\theta \in [0, 2\pi]$ such that $e^{i\theta}P := \{(p'_j + a'_j t, q'_j + b'_j t) : j = 1, 2, \dots, r\}$ satisfies $a'_j b'_j \neq 0$ for all $j = 1, 2, \dots, r$.

Proof. The coefficients a'_j and b'_j are given by

$$\begin{aligned}
a'_j &= a_j \cos \theta - b_j \sin \theta, \\
b'_j &= a_j \sin \theta - b_j \cos \theta
\end{aligned} \tag{19}$$

for all $j = 1, 2, \dots, r$. Now, the equation

$$\prod_{j=1}^r (a_j \cos \theta - b_j \sin \theta) (a_j \sin \theta - b_j \cos \theta) = 0 \tag{20}$$

has only a finite number of zeros in $[0, 2\pi]$. Thus, we can choose $\theta \in [0, 2\pi]$ such that $\prod_{j=1}^r a'_j b'_j \neq 0$. This completes the proof. $\square$

Lemma 9. One can find a set $K \subset \mathbb{R}$ of Lebesgue measure 0 such that, for any countable subsets $U \subset \mathbb{R}$, $V \subset \mathbb{R} \setminus \{0\}$, and $M > 0$, there exists $\lambda \geq M$ satisfying

$$U + \lambda V = \{u + \lambda v : u \in U, v \in V\} \subset K. \tag{21}$$

Proof. It is shown in [22, Theorem 1.6] that there exists a set $K \subset \mathbb{R}$ of Lebesgue measure 0 such that $\mathbb{R} \setminus K$ is of the first Baire category; that is, $\mathbb{R} \setminus K$ is a countable union of nowhere dense subsets of $\mathbb{R}$. Let $U = \{u_1, u_2, u_3, \dots\}$ and $V = \{v_1, v_2, v_3, \dots\}$ and $K_{m,n}^c = v_m^{-1}(K^c - u_n)$, $m, n = 1, 2, 3, \dots$. Then, since K^c is of the first Baire category, $K_{m,n}^c$ are also of the first Baire category for all $m, n = 1, 2, 3, \dots$. Since each $K_{m,n}^c$ consists of a countable union of nowhere dense subsets, by the Baire category theorem, countably many of them cannot cover $[M, \infty)$; that is,

$$\bigcup_{m,n=1}^{\infty} K_{m,n}^c \not\supset [M, \infty). \tag{22}$$

Thus, there exists $\lambda \geq M$ such that $\lambda \notin K_{m,n}^c$ for all $m, n = 1, 2, 3, \dots$. This means that $u_n + v_m \lambda \in K$ for all $m, n = 1, 2, 3, \dots$. This completes the proof. $\square$

Theorem 10. *Let $d \geq 0$ be fixed. Then there exists a set $\Gamma_d \subset \{(x, y) : |x| + |y| \geq d\}$ of 2-dimensional Lebesgue measure 0 which satisfies condition (C).*

Proof. Let $P_{x,y,t}$ be the set in condition (C). Then by Lemma 8 we can choose $\theta \in [0, 2\pi]$ such that $e^{i\theta} P_{x,y,t} := \{(p'_j + a'_j t, q'_j + b'_j t) : j = 1, 2, \dots, r\}$ satisfies $a'_j b'_j \neq 0$ for all $j = 1, 2, \dots, r$. Let K be the set in Lemma 9. Then $\Gamma_d = e^{-i\theta} (K \times K) \cap \{(x, y) : |x| + |y| \geq d\}$ has 2-dimensional Lebesgue measure 0. Now, we show that Γ satisfies condition (C); that is, for given $x, y \in \mathbb{R}$, there exists $t \in \mathbb{R}$ satisfying the conditions

$$\begin{aligned} e^{i\theta} P_{x,y,t} &\subset K \times K, \\ P_{x,y,t} &\subset \{(x, y) : |x| + |y| \geq d\}. \end{aligned} \quad (23)$$

Let $e^{i\theta} P_{x,y,t} := \{(p'_j + a'_j t, q'_j + b'_j t) : j = 1, 2, \dots, r\}$, $U = \{p'_j, q'_j : j = 1, 2, \dots, r\}$, and $V = \{a'_j, b'_j : j = 1, 2, \dots, r\}$. Then we have

$$\{u, v : (u, v) \in e^{i\theta} P_{x,y,t}\} \subset U + tV. \quad (24)$$

Now, by Lemma 9, for given $x, y \in \mathbb{R}$ there exists $t \geq |x| + |y| + d$ such that

$$U + tV \subset K. \quad (25)$$

From (24) and (25) we have

$$e^{i\theta} P_{x,y,t} \subset K \times K. \quad (26)$$

Since $t \geq |x| + |y| + d$, using the triangle inequality we have $|u| + |v| \geq d$ for all $(u, v) \in P_{x,y,t}$. Thus, Γ satisfies (C). This completes the proof. $\square$

Now, as a direct consequence of Theorems 4 and 10 we have the following.

Corollary 11. *Let $\epsilon, d \geq 0$ be fixed. Suppose that $f : \mathbb{R} \rightarrow Y$ satisfies inequality (2) for all $(x, y) \in \Gamma_d$. Then there exists a unique quartic mapping $T : X \rightarrow Y$ satisfying (10) for all $x \in \mathbb{R}$.*

As a consequence of Corollary 11 we obtain an asymptotic behavior of f satisfying condition (11) as $|x| + |y| \rightarrow \infty$ only for $(x, y) \in \Gamma_0$.

Corollary 12. *Suppose that $f : \mathbb{R} \rightarrow Y$ satisfies condition (11). Then f is a quartic mapping.*

Proof. Condition (11) implies that, for each $n \in \mathbb{N}$, there exists $d_n > 0$ such that

$$\begin{aligned} &\|f(2x + y) + f(2x - y) - 4f(x + y) - 4f(x - y) \\ &\quad - 24f(x) + 6f(y)\| \leq \frac{1}{n} \end{aligned} \quad (27)$$

for all $(x, y) \in \Gamma_{d_n}$. By Corollary 11, there exists a unique quartic mapping $T_n : X \rightarrow Y$ such that

$$\|f(x) - T_n(x) + 231f(0)\| \leq \frac{509}{36n} \quad (28)$$

for all $x \in X$. Replacing n by m in (28) and using the triangle inequality we have

$$\|T_m(x) - T_n(x)\| \leq \frac{509}{36n} + \frac{509}{36m} \leq \frac{509}{18} \quad (29)$$

for all $x \in X$. Let $T_{m,n}(x) = T_m(x) - T_n(x)$ for all $x \in X$. Then by (29), $T_{m,n}$ is a bounded quartic mapping. Thus, we have $T_{m,n} = 0$ and hence $T_m = T_n := T$ for all $m, n \in \mathbb{N}$. Letting $n \rightarrow \infty$ in (28) we have $f(0) = 0$ and hence $f(x) = T(x)$ for all $x \in X$. This completes the proof. $\square$

Remark 13. If we define $\Gamma \subset \mathbb{R}^{2n}$ as an appropriate rotation of $2n$ -product K^{2n} of K , then Γ_d has $2n$ -dimensional measure 0 and satisfies condition (C). Consequently, we obtain the following.

Theorem 14. *Suppose that $f : \mathbb{R}^n \rightarrow Y$ satisfies inequality (2) for all $(x, y) \in \Gamma_d$. Then there exists a unique quartic mapping $T : \mathbb{R}^n \rightarrow Y$ satisfying (10) for all $x \in \mathbb{R}^n$.*

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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