Research Article

Hausdorff Dimension of a Random Attractor for Stochastic Boussinesq Equations with Double Multiplicative White Noises

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This paper investigates the existence of random attractor for stochastic Boussinesq equations driven by multiplicative white noises in both the velocity and temperature equations and estimates the Hausdorff dimension of the random attractor.

1. Introduction

In this paper, we consider the following 2D stochastic Boussinesq equation on $D = (0, 1) \times (0, 1) \subset \mathbb{R}^2$:

$$\begin{aligned} dv + [(v \cdot \nabla) v + \nabla p - \mu \Delta v] dt \\ = Te_2 dt + b_1 v \circ dW_1(t), \\ dT + [(v \cdot \nabla) T - k \Delta T] dt = b_2 T \circ dW_2(t), \\ \operatorname{div} v = 0, \\ v|_{t=t_0} = v_0, \\ T|_{t=t_0} = T_0, \end{aligned} \quad (1)$$

where $v = (v_1(x, t), v_2(x, t))$ denotes the velocity of the fluid; $T(x, t)$ is the temperature; and function $p(x, t)$ represents the pressure. We set $(x, t) \in D \times \mathbb{R}$, and $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$. Let us consider $\mu, k > 0$ the Grashof and Prandtl numbers. $W(t) = (W_1(t), W_2(t))^T$ is two-sided real-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, P)$, here

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}^2) : \omega(0) = 0\}. \quad (2)$$

The Borel σ -algebra $\mathcal{F}$ on Ω is generated by the compact open topology, and P is the corresponding Wiener measure; $\circ dW_i(t)$ denotes the Stratonovich form white noise.

Pu and Guo in [1] studied the global well-posedness of stochastic 2D Boussinesq equations with partial viscosity; Li and Guo [2] consider the stochastic Boussinesq equations, which are influenced by multiplicative white noise in the velocity equations. Recently, Zhao and Li [3] established the existence of a random attractor for stochastic Boussinesq equations with double additive white noises.

In many cases, the Hausdorff dimension of a random attractor is finite. Crauel and Flandoli [4] developed a technique to estimate the Hausdorff dimension of attractors for certain dynamical systems. But they required the noise to be bounded. Debussche [5] applied a random squeezing property to estimate the Hausdorff dimension without the assumption of bounded noise. A number of authors [6–13] used this method to estimate the Hausdorff dimension of attractors for many stochastic equations.

As far as we know, there are no results on stochastic 2D Boussinesq equations with double multiplicative white noises on bounded domains. In this paper, we prove the existence of random attractor for the corresponding RDS associated with problem (1). And we give an estimate of Hausdorff dimension for the random attractor.

The paper is organized as follows. In the first section, we recall some latest results and what we want to do for the 2D stochastic Boussinesq equations. We study the RDS determined by (1) in Section 2. In Section 3, we prove the existence of a random attractor. Finally, we estimate the

Hausdorff dimension of the random attractor in Section 4.

2. Random Dynamical System

The random dynamical system generated by stochastic Boussinesq equations with double multiplicative noise will be studied in this section. Thus, we introduce a process, which enables us to transform the stochastic equations into a deterministic equation with a random parameter.

Let

$$\begin{aligned}\alpha_i(t) &= e^{-b_i W_i(t)}, \quad i = 1, 2, \\ \xi &= \alpha_1(t) \nu \\ \eta &= \alpha_2(t) T.\end{aligned}\quad (3)$$

From (1), we get the stochastic Boussinesq equations without white noises:

$$\frac{d\xi}{dt} + \alpha_1^{-1}(t) (\xi \cdot \nabla) \xi - \mu \Delta \xi + \alpha_1(t) \nabla p = \frac{\alpha_1(t)}{\alpha_2(t)} e_2 \eta, \quad (4)$$

$$\frac{d\eta}{dt} + \alpha_1^{-1}(t) (\xi \cdot \nabla) \eta - k \Delta \eta = 0, \quad (5)$$

$$\operatorname{div} \xi = 0, \quad (6)$$

$$\xi|_{t=t_0} = \xi_0, \quad (7)$$

$$\eta|_{t=t_0} = \eta_0.$$

In the present paper, we consider the Hilbert space: $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$, with the scalar product $(\cdot, \cdot)$ and norm $\|\cdot\|$, where

$$\begin{aligned}\mathcal{H}_1 &= \{\xi \in \mathcal{H}_2^2 : \operatorname{div} \xi = 0\}, \\ \mathcal{H}_2 &= L^2(D).\end{aligned}\quad (8)$$

We also consider the subspace: $V = V_1 \times V_2$. Consider

$$\begin{aligned}V_1 &= \{\xi \in V_2^2 : \operatorname{div} \xi = 0\}, \\ V_2 &= \text{closure of } H_0^1(D) \text{ in } L^2(D)\end{aligned}\quad (9)$$

and its scalar product and norms are $((\cdot, \cdot))$ and $\|\cdot\|_1$, respectively, for any $\eta_1, \eta_2 \in V_2$:

$$\begin{aligned}((\eta_1, \eta_2)) &= \int_D \operatorname{grad} \eta_1 \operatorname{grad} \eta_2 \, dx, \\ \|\eta\|_1 &= ((\eta, \eta))^{1/2} = \|\nabla \eta\|.\end{aligned}\quad (10)$$

Now we give some definitions of common operators.

a is the bilinear on V :

$$\begin{aligned}a(u_1, u_2) &= -\mu (\Delta \xi_1, \xi_2) - k (\Delta \eta_1, \eta_2), \\ \forall u_i &= (\xi_i, \eta_i)^T \in V, \quad i = 1, 2.\end{aligned}\quad (11)$$

A is isomorphism from $D(A)$ into $\mathcal{H}$ and from V into the dual space V' , defined by

$$(Au_1, u_2) = a(u_1, u_2), \quad \forall u_i = (\xi_i, \eta_i)^T \in V, \quad i = 1, 2, \quad (12)$$

where $D(A) = D(A_1) \times D(A_2)$, being

$$\begin{aligned}D(A_1) &= \{\xi : \xi \in V_1 \cap (H^2(D))^2\}, \\ D(A_2) &= \{\eta : \eta \in V_2 \cap H^2(D)\}.\end{aligned}\quad (13)$$

We consider trilinear forms b on V defined by

$$\begin{aligned}b(u_1, u_2, u_3) &= -((\xi_1 \cdot \nabla) \xi_2, \xi_3) - ((\xi_1 \cdot \nabla) \eta_2, \eta_3), \\ \forall u_i &= (\xi_i, \eta_i)^T \in V, \quad i = 1, 2.\end{aligned}\quad (14)$$

b is continuous on V and $(\mathcal{H}'(D))^2 \times \mathcal{H}'(D)$.

We also define the bilinear continuous operator B , which maps $V \times V$ into V' and $D(A) \times D(A)$ into $\mathcal{H}$, by

$$\begin{aligned}B(u_1, u_2, u_3) &= b(u_1, u_2, u_3), \\ \forall u_i &= (\xi_i, \eta_i)^T \in V, \quad i = 1, 2, 3.\end{aligned}\quad (15)$$

Finally, we consider a family of linear continuous operators $R(t) = R(t, \omega)$ on $\mathcal{H}$:

$$\begin{aligned}R(t) : u &= (\xi, \eta)^T \longrightarrow \\ Ru &= \left(\frac{\alpha_1(t)}{\alpha_2(t)} e_2 \eta, 0 \right)^T.\end{aligned}\quad (16)$$

Assume that $u = (\xi, \eta)^T$ is a solution of (4)–(7) and $y = (f, g)^T$ is a test function in V . Multiplying (4) by f and (5) by g , integrating over D , and adding the resulting equation, we obtain

$$\begin{aligned}& \left(\frac{d\xi}{dt}, f \right) + \alpha_1^{-1}(t) ((\xi \cdot \nabla) \xi, f) - \mu (\Delta \xi, f) \\ & + \alpha_1(t) (\nabla p, f) + \left(\frac{d\eta}{dt}, g \right) \\ & + \alpha_1^{-1}(t) ((\xi \cdot \nabla) \eta, g) - k (\Delta \eta, g) \\ & = \frac{\alpha_1(t)}{\alpha_2(t)} (e_2 \eta, f).\end{aligned}\quad (17)$$

Note that

$$\begin{aligned}& \left(\frac{d\xi}{dt}, f \right) + \left(\frac{d\eta}{dt}, g \right) = \left(\frac{du}{dt}, y \right) \\ & ((\xi \cdot \nabla) \xi, f) + ((\xi \cdot \nabla) \eta, g) = -b(u, u, y) \\ & -\mu (\Delta \xi, f) - k (\Delta \eta, g) = a(u, y).\end{aligned}\quad (18)$$

On the other hand,

$$\begin{aligned}\operatorname{div} f &= 0, \\ f &= (f_1, f_2)^T.\end{aligned}\quad (19)$$

Then,

$$(\nabla p, f) = 0. \quad (20)$$

Therefore, (9) can also be written as

$$\frac{du}{dt} = -Au + \alpha_1^{-1}(t) B(u, u) + R(t) u. \quad (21)$$

By the Galerkin method (see [14, Theorem 3.1]), we can prove that (21) has a unique solution $u(x, t)$, which satisfies the following:

(i) For all $t_0 \in \mathbb{R}$, if $u_0 \in \mathcal{H}$, then

$$u \in C([t_0, \infty); \mathcal{H}) \cap L^2_{\text{loc}}([t_0, \infty); V). \quad (22)$$

(ii) Denoting such solution by $u(t, \omega; t_0, u_0)$, for any $t \geq t_0$, the mapping

$$u_0 \longrightarrow u(t, \omega; t_0, u_0) \quad (23)$$

is continuous. Then, we define a stochastic flow $\varphi(t, \omega)$ by

$$\begin{aligned} \varphi(t, \omega) z_0 &= (\xi, \eta)^T = (\alpha_1(t) \mathcal{Q}_{\mathcal{H}_1} z(t, \omega; t_0, z_0), \alpha_2(t) \\ &\quad \cdot \mathcal{Q}_{\mathcal{H}_2} z(t, \omega; t_0, z_0))^T, \end{aligned} \quad (24)$$

where $z_0 = z(t_0) = (v_0, T_0)^T \in \mathcal{H}$ and

$$\mathcal{Q}_{\mathcal{H}_i} : \mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_i, \quad i = 1, 2 \quad (25)$$

are projection operators.

We can prove that $\varphi(t, \omega)$ is a continuous RDS determined by (1) easily.

3. Random Attractor

In this section, we will establish the existence of nonempty compact tempered random attractor for the RDS $\varphi(t, \omega)$ determined by (1). First we prove that $\varphi(t, \omega)$ possesses absorbing set in $\mathcal{H}$.

Lemma 1. Let $u = (\xi, \eta)^T$ be a solution of (4)–(7), for any $t_0 \leq t$, and one has

$$\begin{aligned} \|\xi\|^2 &\leq \|\xi(t_0)\|^2 e^{-(\mu/c_0^2)(t-t_0)} \\ &\quad + \frac{c_0^2}{\mu} \int_{t_0}^t \frac{\alpha_1^2(s)}{\alpha_2^2(s)} \|\eta(s)\|^2 e^{-(\mu/c_0^2)(t-s)} ds, \quad (26) \\ \|\eta\|^2 &\leq \|\eta(t_0)\|^2 e^{-(2k/c_1^2)(t-t_0)}. \end{aligned}$$

Proof. Taking the scalar product of (4) with ξ on $\mathcal{H}_1$

$$\begin{aligned} \left(\frac{d\xi}{dt}, \xi \right) &+ \alpha_1^{-1}(t) ((\xi \cdot \nabla) \xi, \xi) - \mu (\Delta \xi, \xi) \\ &+ \alpha_1(t) (\nabla p, \xi) = \frac{\alpha_1(t)}{\alpha_2(t)} (e_2 \eta, \xi). \end{aligned} \quad (27)$$

By the incompressibility condition (6), we have

$$\begin{aligned} ((\xi \cdot \nabla) \xi, \xi) &= 0, \\ (\nabla p, \xi) &= 0. \end{aligned} \quad (28)$$

From (27), we obtain

$$\frac{1}{2} \frac{d}{dt} \|\xi\|^2 + \mu \|\nabla \xi\|^2 = \frac{\alpha_1(t)}{\alpha_2(t)} (\eta, \xi_2). \quad (29)$$

Using Poincaré inequality, $\|\xi\| \leq c_0 \|\nabla \xi\|$ (c_0 is constant), we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\xi\|^2 + \frac{\mu}{c_0^2} \|\xi\|^2 &\leq \frac{\alpha_1(t)}{\alpha_2(t)} \|\eta\| \|\xi_2\| \leq \frac{\alpha_1(t)}{\alpha_2(t)} \|\eta\| \|\xi\| \\ &\leq \frac{c_0^2 \alpha_1^2(t)}{2\mu \alpha_2^2(t)} \|\eta\|^2 + \frac{\mu}{2c_0^2} \|\xi\|^2. \end{aligned} \quad (30)$$

Then,

$$\frac{d}{dt} \|\xi\|^2 + \frac{\mu}{c_0^2} \|\xi\|^2 \leq \frac{c_0^2 \alpha_1^2(t)}{\mu \alpha_2^2(t)} \|\eta\|^2. \quad (31)$$

By the Gronwall inequality, for any $t_0 \leq t$, we get that

$$\begin{aligned} \|\xi\|^2 &\leq \|\xi(t_0)\|^2 e^{-(\mu/c_0^2)(t-t_0)} \\ &\quad + \frac{c_0^2}{\mu} \int_{t_0}^t \frac{\alpha_1^2(s)}{\alpha_2^2(s)} \|\eta(s)\|^2 e^{-(\mu/c_0^2)(t-s)} ds. \end{aligned} \quad (32)$$

Taking the scalar product of (5) with η on $\mathcal{H}_2$ as well

$$\left(\frac{d\eta}{dt}, \eta \right) + \frac{1}{\alpha_1(t)} ((\xi \cdot \nabla) \eta, \eta) - k (\Delta \eta, \eta) = 0. \quad (33)$$

Since $((\xi \cdot \nabla) \eta, \eta) = 0$, then

$$\frac{d}{dt} \|\eta\|^2 + 2k \|\nabla \eta\|^2 = 0. \quad (34)$$

Using Poincaré inequality $\|\eta\| \leq c_1 \|\nabla \eta\|$, we have

$$\frac{d}{dt} \|\eta\|^2 + \frac{2k}{c_1^2} \|\eta\|^2 \leq 0. \quad (35)$$

By Gronwall inequality, for any $t_0 \leq t$, we obtain that

$$\|\eta\|^2 \leq \|\eta(t_0)\|^2 e^{-(2k/c_1^2)(t-t_0)}. \quad (36)$$

□

Lemma 2. There exist two positive random variables $r(\omega)$ and $C(\omega)$, such that, for all $\rho > 0$, $t_0 \leq t(\omega, \rho)$, $t \in [-1, 0]$, and $z_0 = (v_0, T_0)^T \in \mathcal{H}$ with $\|z_0\| < \rho$, the solution $u = (\xi, \eta)^T$ of equations (4)–(7) satisfies

$$\begin{aligned} \|\eta(t, \omega; t_0, \alpha_2(t_0) T_0)\|^2 &\leq C(\omega), \\ \|\xi(t, \omega; t_0, \alpha_1(t_0) v_0)\|^2 &\leq r(\omega). \end{aligned} \quad (37)$$

Proof. For $-1 \leq t \leq 0$, $t_0 \leq -1$, from (26), we have

$$\|\eta\|^2 \leq \alpha_2^2(t) \|T(t_0)\|^2 e^{(2k/c_1^2)(1+t_0)}, \quad (38)$$

taking $t_1(\omega, \rho) \leq -1$, such that $\rho \alpha_2^2(t) e^{(2k/c_1^2)(1+t_1)} \leq C(\omega)$.

Then, for any $t_0 \leq t_1(\omega, \rho)$,

$$\|\eta(t, \omega; t_0, \alpha_2(t_0) T_0)\|^2 \leq C(\omega). \quad (39)$$

From (26), we find

$$\begin{aligned} \|\xi\|^2 &\leq \|\xi(t_0)\|^2 e^{(\mu/c_0^2)(1+t_0)} \\ &\quad + \frac{c_0^2}{\mu} \int_{t_0}^t \frac{\alpha_1^2(s)}{\alpha_2^2(s)} \|\eta(s)\|^2 e^{(\mu/c_0^2)(1+s)} ds, \end{aligned} \quad (40)$$

$$\lim_{s \rightarrow -\infty} \frac{W_1(s)}{s} = 0, \quad P\text{-a.s.}$$

Then, $\alpha_1^2(t_0) e^{\mu t_0/c_0^2} \rightarrow 0$, as $t_0 \rightarrow -\infty$.

Thus, there exists $t_2(\omega, \rho) \leq -1$, for $t_0 \leq t_2$:

$$\|v_0\| \alpha_1^2(t_0) e^{\mu t_0/c_0^2} \leq \rho^2 \alpha_1^2(t_0) e^{\mu t_0/c_0^2} \leq 1. \quad (41)$$

By (37), we infer that $\forall t_0 \leq t_1, t \in [-1, 0]$, P -a.s is

$$\begin{aligned} &\int_{t_0}^t \frac{\alpha_1^2(s)}{\alpha_2^2(s)} \|\eta(s)\|^2 e^{(\mu/c_0^2)(1+s)} ds \\ &\leq C^2(\omega) \int_{-\infty}^0 \frac{\alpha_1^2(s)}{\alpha_2^2(s)} e^{(\mu/c_0^2)(1+s)} ds. \end{aligned} \quad (42)$$

Since

$$\frac{\alpha_1^2(s)}{\alpha_2^2(s)} = e^{2b_2 W_2(s) - 2b_1 W_1(s)} = e^{b_3 W(s)}, \quad (43)$$

then $(\alpha_1^2(s)/\alpha_2^2(s)) e^{\mu s/c_0^2} \rightarrow 0, s \rightarrow -\infty$. Therefore,

$$\int_{t_0}^t \frac{\alpha_1^2(s)}{\alpha_2^2(s)} \|\eta(s)\|^2 e^{(\mu/c_0^2)(1+s)} ds < \infty. \quad (44)$$

Put

$$r(\omega) = e^{\mu/c_0^2} \left[1 + \frac{c_0^2}{\mu} \int_{t_0}^t \frac{\alpha_1^2(s)}{\alpha_2^2(s)} \|\eta(s)\|^2 e^{\mu s/c_0^2} ds \right]. \quad (45)$$

Then, $r(\omega)$ exists and it is tempered.

Getting $t(\omega, \rho) = \min\{t_1, t_2\}$, for any $t \geq t_0$,

$$\|\eta(t, \omega; t_0, \alpha_2(t_0) T_0)\|^2 \leq C(\omega), \quad (46)$$

$$\|\xi(t, \omega; t_0, \alpha_1(t_0) v_0)\|^2 \leq r(\omega).$$

Now we are going to prove that $\varphi(t, \omega)$ is asymptotically compact. $\square$

Lemma 3. *There exist two positive random variables $C_1(\omega)$ and $r_1(\omega)$, such that, for all $\rho > 0$, $t_0 \leq t(\omega, \rho)$, $t \in [-1, 0]$, and $z_0 = (v_0, T_0)^T \in \mathcal{H}$ with $\|z_0\| < \rho$, the solution $u = (\xi, \eta)^T$ of equations (4)–(7) satisfies*

$$\begin{aligned} \int_{-1}^0 \|\nabla \eta(s, \omega; t_0, \alpha_2(s) T_0)\|^2 ds &\leq C_1(\omega), \\ \int_{-1}^0 \|\nabla \xi(s, \omega; t_0, \alpha_1(s) v_0)\|^2 ds &\leq r_1(\omega). \end{aligned} \quad (47)$$

Proof. Integrating (34) on $[-1, 0]$, we have

$$\|\eta(0)\|^2 - \|\eta(-1)\|^2 + 2k \int_{-1}^0 \|\nabla \eta\|^2 ds = 0. \quad (48)$$

From Lemma 1, we know that

$$\begin{aligned} \int_{-1}^0 \|\nabla \eta\|^2 ds &\leq \frac{1}{2k} (\|\eta(-1)\|^2 - \|\eta(0)\|^2) \leq \frac{C(\omega)}{k} \\ &= C_1(\omega). \end{aligned} \quad (49)$$

We get from (29),

$$\frac{1}{2} \frac{d}{dt} \|\xi\|^2 + \mu \|\nabla \xi\|^2 \leq \frac{c_0^2 \alpha_1^2(t)}{2\mu \alpha_2^2(t)} \|\eta\|^2 + \frac{\mu}{2} \|\nabla \xi\|^2 \quad (50)$$

then

$$\frac{d}{dt} \|\xi\|^2 + \mu \|\nabla \xi\|^2 \leq \frac{c_0^2 \alpha_1^2(t)}{\mu \alpha_2^2(t)} \|\eta\|^2. \quad (51)$$

Integrating the above formula over $[-1, 0]$,

$$\begin{aligned} \|\xi(0)\|^2 - \|\xi(-1)\|^2 + \mu \int_{-1}^0 \|\nabla \xi(s)\|^2 ds \\ \leq \int_{-1}^0 \frac{c_0^2 \alpha_1^2(s)}{\mu \alpha_2^2(s)} \|\eta(s)\|^2 ds \end{aligned} \quad (52)$$

then

$$\begin{aligned} \int_{-1}^0 \|\nabla \xi(s)\|^2 ds &\leq \frac{1}{\mu} \left(\|\xi(-1)\|^2 - \|\xi(0)\|^2 \right. \\ &\quad \left. + \int_{-1}^0 \frac{c_0^2 \alpha_1^2(s)}{\mu \alpha_2^2(s)} \|\eta(s)\|^2 ds \right). \end{aligned} \quad (53)$$

Let

$$r_1(\omega) = \frac{2r(\omega)}{\mu} + \frac{c_0^2}{\mu^2} \int_{-1}^0 \frac{\alpha_1^2(s)}{\alpha_2^2(s)} \|\eta(s)\|^2 ds \quad (54)$$

then

$$\int_{-1}^0 \|\nabla \xi(s, \omega; t_0, \alpha_1(s) v_0)\|^2 ds \leq r_1(\omega). \quad (55)$$

$\square$

Lemma 4. *There exist two positive random variables $C_2(\omega)$ and $r_2(\omega)$, such that, for all $\rho > 0$, $t_0 \leq t(\omega, \rho)$, $t \in [-1, 0]$, and $z_0 = (v_0, T_0)^T \in \mathcal{H}$ with $\|z_0\| < \rho$, the solution $u = (\xi, \eta)^T$ of (4)–(7) satisfies*

$$\begin{aligned} \|\nabla \eta(0, \omega; t_0, \alpha_2(s) T_0)\|^2 &\leq C_2(\omega), \\ \|\nabla \xi(0, \omega; t_0, \alpha_1(s) v_0)\|^2 &\leq r_2(\omega). \end{aligned} \quad (56)$$

Proof. Taking the scalar product of (5) with $-\Delta \eta$ on $\mathcal{H}_2$, then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \eta\|^2 + k \|\nabla \eta\|^2 &\leq \alpha_1^{-1}(t) |(\xi \cdot \nabla) \eta, -\Delta \eta| \\ &\leq \alpha_1^{-1}(t) \|(\xi \cdot \nabla) \eta\| \|\Delta \eta\|. \end{aligned} \quad (57)$$

Since

$$\begin{aligned} \|(\xi \cdot \nabla) \eta\| &\leq \|\xi\|_{L^4} \|\nabla \eta\|_{L^4} \\ &\leq c_2 \|\nabla \xi\|^{1/2} \|\xi\|^{1/2} \|\nabla \eta\|^{1/2} \|\Delta \eta\|^{1/2}, \end{aligned} \quad (58)$$

then we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \eta\|^2 + k \|\Delta \eta\|^2 &\leq \alpha_1^{-1}(t) c_2 \|\nabla \xi\|^{1/2} \|\xi\|^{1/2} \|\nabla \eta\|^{1/2} \|\Delta \eta\|^{3/2} \\ &\leq \frac{27c_2^4}{256k^3\alpha_1^4(t)} \|\xi\|^2 \|\nabla \xi\|^2 \|\nabla \eta\|^2 + k \|\Delta \eta\|^2. \end{aligned} \quad (59)$$

Hence,

$$\frac{d}{dt} \|\nabla \eta\|^2 \leq \frac{27c_2^4}{128k^3\alpha_1^4(t)} \|\xi\|^2 \|\nabla \xi\|^2 \|\nabla \eta\|^2. \quad (60)$$

Using the Gronwall inequality over an arbitrary interval $[s, 0] \subset [-1, 0]$, we obtain

$$\|\nabla \eta(0)\|^2 \leq \|\nabla \eta(s)\|^2 e^{\int_s^0 h(\tau) d\tau}. \quad (61)$$

Here, $h(\tau) = (27c_2^4/128k^3\alpha_1^4(\tau))\|\xi(\tau)\|^2\|\nabla \xi(\tau)\|^2$.

Integrating the above formula over the interval $[-1, 0]$ for s , we obtain

$$\|\nabla \eta(0)\|^2 \leq e^{\int_{-1}^0 h(\tau) d\tau} \int_{-1}^0 \|\nabla \eta(s)\|^2 ds. \quad (62)$$

Let

$$C_3(\omega) = e^{\int_{-1}^0 h(\tau) d\tau} C_1(\omega). \quad (63)$$

Thus, $C_2(\omega)$ exists and it is tempered and we have

$$\|\nabla \eta(0, \omega; t_0, \alpha_2(t_0) T_0)\|^2 \leq C_2(\omega). \quad (64)$$

Taking the scalar product of (4) with $-\Delta \xi$ on $\mathcal{H}_1$

$$\begin{aligned} \left(\frac{d\xi}{dt}, -\Delta \xi \right) + \alpha_1(t)^{-1} ((\xi \cdot \nabla) \xi, -\Delta \xi) - \mu (\Delta \xi, -\Delta \xi) \\ + \alpha_1(t) (\nabla p, -\Delta \xi) = \left(\frac{\alpha_1(t)}{\alpha_2(t)} e_2 \eta, -\Delta \xi \right). \end{aligned} \quad (65)$$

Since $(\nabla p, -\Delta \xi) = 0$, then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \xi\|^2 + \mu \|\Delta \xi\|^2 &\leq \alpha_1^{-1}(t) ((\xi \cdot \nabla) \xi, \Delta \xi) + \frac{\alpha_1(t)}{\alpha_2(t)} (e_2 \eta, \Delta \xi) \\ &\leq \frac{c_3}{\alpha_1(t)} \|\xi\|^{1/2} \|\nabla \xi\| \|\Delta \xi\|^{3/2} + \frac{\alpha_1(t)}{\alpha_2(t)} \|\eta\| \|\Delta \xi\| \\ &\leq \frac{27c_3^4}{32\alpha_1^4(t)\mu^3} \|\xi\|^2 \|\nabla \xi\|^4 + \frac{\alpha_1^2(t)}{2\mu\alpha_2^2(t)} \|\eta\|^2 \\ &\quad + \mu \|\Delta \xi\|^2. \end{aligned} \quad (66)$$

Namely,

$$\begin{aligned} \frac{d}{dt} \|\nabla \xi\|^2 &\leq \frac{27c_3^4}{16\alpha_1^4(t)\mu^3} \|\xi\|^2 \|\nabla \xi\|^4 + \frac{\alpha_1^2(t)}{\mu\alpha_2^2(t)} \|\eta\|^2 \\ &\leq G_1(t) \|\nabla \xi\|^2 + G_2(t), \end{aligned} \quad (67)$$

where $G_1(t) = (27c_3^4/16\alpha_1^4(t)\mu^3)\|\xi\|^2\|\nabla \xi\|^2$ and $G_2(t) = (\alpha_1^2(t)/\mu\alpha_2^2(t))\|\eta\|^2$.

By the Gronwall inequality over an arbitrary interval $[s, 0] \subset [-1, 0]$, we obtain

$$\begin{aligned} \|\nabla \xi(0)\|^2 &\leq e^{\int_s^0 G_1(\sigma) d\sigma} \|\nabla \xi(s)\|^2 \\ &\quad + \int_s^0 G_2(\tau) e^{\int_\tau^0 G_1(\sigma) d\sigma} d\tau. \end{aligned} \quad (68)$$

Integrating the above formula over the interval $[-1, 0]$ for s , we have

$$\begin{aligned} \|\nabla \xi(0)\|^2 &\leq e^{\int_{-1}^0 G_1(\sigma) d\sigma} \int_{-1}^0 \|\nabla \xi(s)\|^2 ds \\ &\quad + \int_{-1}^0 G_2(\tau) e^{\int_{-1}^0 G_1(\sigma) d\sigma} d\tau. \end{aligned} \quad (69)$$

Since

$$\begin{aligned} \int_{-1}^0 G_1(\sigma) d\sigma &\leq \frac{27c_3^4}{16\alpha_1^4(t)\mu^3} r(\omega) \int_{-1}^0 \|\nabla \xi(\sigma)\|^2 d\sigma \\ &\leq \frac{27c_3^4}{16\alpha_1^4(t)\mu^3} r(\omega) r_1(\omega), \end{aligned} \quad (70)$$

let

$$\begin{aligned} r_2(\omega) &= e^{(27c_3^4/16\alpha_1^4(t)\mu^3)r(\omega)r_1(\omega)} \left[r_1(\omega) + \int_{-1}^0 G_2(\tau) d\tau \right]. \end{aligned} \quad (71)$$

Thus, $r_2(\omega)$ exists and it is tempered. Then,

$$\|\nabla \xi(0, \omega; t_0, \alpha_1(t_0) u_0)\|^2 \leq r_2(\omega). \quad (72)$$

Let $\mathcal{B}(\omega)$ be a random closed ball centered at 0 with radius $\sqrt{C_2(\omega) + r_2(\omega)}$.

Then, $\mathcal{B}(\omega)$ is a random absorbing set for $\varphi(t, \omega)$ in V . Since V embedding $\mathcal{H}$ is compact, then RDS $\varphi(t, \omega)$ is asymptotically compact in $\mathcal{H}$. Thus, the RDS determined by (1) possesses a random attractor. $\square$

4. Hausdorff Dimensions of Random Attractors

In this section, we will estimate the Hausdorff dimensions of $\mathcal{A}(\omega)$. Now, we assume that $u_0, u_1(t), u_2(t), \chi$ are $\mathcal{F}_t$ measurable and prove that the RDS $\varphi(t, \omega)$ is differentiable on $\mathcal{A}(\omega)$.

Theorem 5. *The RDS $\varphi(t, \omega)$ determined by (21) is almost surely uniformly differentiable on $\mathcal{A}(\omega)$, $\omega \in \Omega$, and its derivative is $D\varphi(t, t_0; \omega; u_0)\chi = U(t)$, $t_0 \leq t$, which is the solution of the following variational equation:*

$$\begin{aligned} \frac{dU}{dt} &= -AU + F(t, u)U, \\ U(t_0) &= \chi, \end{aligned} \quad (73)$$

where $F(t, u)U = \alpha_1^{-1}(t)[B(u, U) + B(U, u)] + R(t)U$, $u = \varphi(t, t_0; \omega)u_0$, $t_0 \leq t$.

Proof. Let $u_1(t) = \varphi(t, t_0; \omega)u_0^{(1)}$, $u_2(t) = \varphi(t, t_0; \omega)u_0^{(2)}$, and $\delta(t) = u_1(t) - u_2(t)$, and we have

$$\begin{aligned} \frac{d}{dt}\delta(t) &= -A\delta(t) \\ &\quad + \alpha_1^{-1}(t)[B(u_1, \delta(t)) + B(\delta(t), u_2)] \\ &\quad + R(t)\delta(t), \\ \frac{dU}{dt} &= -A(U) + \alpha_1^{-1}(t)[B(u_1, U) + B(U, u_2)] \\ &\quad + R(t)U. \end{aligned} \quad (74)$$

We also set $l(t) = \delta(t) - U(t)$, and then

$$\begin{aligned} \frac{dl(t)}{dt} &= -Al(t) + \alpha_1^{-1}(t) \\ &\quad \cdot [B(u_2, l(t)) + B(l(t), u_2) + B(\delta(t), \delta(t))] \\ &\quad + R(t)l(t). \end{aligned} \quad (75)$$

Taking the scalar product of (75) with $l(t)$ on $\mathcal{H}$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|l(t)\|^2 &\leq -\|\nabla l(t)\|^2 + \alpha_1^{-1}(t) \\ &\quad \cdot [(B(l(t), u_2), l(t)) + (B(\delta(t), \delta(t)), l(t))] \\ &\quad + R(t)\|l(t)\|^2 \\ &\leq \frac{c_4}{\alpha_1(t)} (\|\nabla u_2\| \|l(t)\|^2 + \|\delta(t)\| \|\nabla \delta(t)\| \|l(t)\|) \\ &\quad + R(t)\|l(t)\|^2 \leq K(t)\|l(t)\|^2 + c_5 \|\delta(t)\|^2 \\ &\quad \cdot \|\nabla \delta(t)\|^2. \end{aligned} \quad (76)$$

Here, $K(t)$ is the function about t . By the Gronwall inequality, we obtain

$$\begin{aligned} \|l(t)\|^2 &\leq \int_{t_0}^t e^{\int_s^t 2K(\tau)d\tau} (2c_5 \|\delta(s)\|^2 \|\nabla \delta(s)\|^2) ds \\ &\leq K_1(t) \int_{t_0}^t \|\delta(s)\|^2 \|\nabla \delta(s)\|^2 ds. \end{aligned} \quad (77)$$

Taking the scalar product of (74) with $\delta(t)$ on $\mathcal{H}$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\delta(t)\|^2 &\leq -\|\nabla \delta(t)\|^2 \\ &\quad + \alpha_1^{-1}(t) (B(\delta(t), u_2), \delta(t)) \\ &\quad + R(t)\|\delta(t)\|^2 \\ &\leq -\|\nabla \delta(t)\|^2 \\ &\quad + \frac{c_6}{\alpha_1(t)} \|\delta(t)\| \|\nabla u_2\| \|\nabla \delta(t)\| \\ &\quad + R(t)\|\delta(t)\|^2 \\ &\leq -\|\nabla \delta(t)\|^2 + \frac{1}{2} \|\nabla \delta(t)\|^2 \\ &\quad + \frac{c_6}{2\alpha_1^2(t)} \|\delta(t)\|^2 \|\nabla u_2\|^2 \\ &\quad + R(t)\|\delta(t)\|^2 \\ &\leq -\frac{1}{2} \|\nabla \delta(t)\|^2 + K_2(t)\|\delta(t)\|^2. \end{aligned} \quad (78)$$

Then,

$$\frac{d}{dt} \|\delta(t)\|^2 \leq 2K_2(t)\|\delta(t)\|^2. \quad (79)$$

By the Gronwall inequality, we get

$$\begin{aligned} \|\delta(t)\|^2 &\leq e^{\int_{t_0}^t 2K_2(s)ds} \|u_0^{(1)} - u_0^{(2)}\|^2 \\ &\leq K_3(t) \|u_0^{(1)} - u_0^{(2)}\|^2. \end{aligned} \quad (80)$$

We get from (80) that

$$\begin{aligned} \int_{t_0}^t \|\nabla \delta(s)\|^2 ds &\leq \|u_0^{(1)} - u_0^{(2)}\|^2 \\ &\quad + \int_{t_0}^t 2K_2(s) \|\delta(s)\|^2 ds \\ &\leq \|u_0^{(1)} - u_0^{(2)}\|^2 \\ &\quad + \|u_0^{(1)} - u_0^{(2)}\|^2 \int_{t_0}^t 2K_2(s) K_3(s) ds \\ &\leq K_4(t) \|u_0^{(1)} - u_0^{(2)}\|^2. \end{aligned} \quad (81)$$

From (80) to (81), we have

$$\|l(t)\|^2 \leq K_5(t) \|u_0^{(1)} - u_0^{(2)}\|^4. \quad (82)$$

The proof is completed. $\square$

Theorem 6. Set

$$d = \min \left\{ d : -\sum_{j=1}^d \lambda_j + c_9 d < 0, \quad c_9 \in R^+ \right\}. \quad (83)$$

Then the Hausdorff dimension of random attractor $\mathcal{A}(t, \omega)$ satisfies

$$\dim_H(\mathcal{A}(t, \omega)) \leq d, \quad \forall t \in R. \quad (84)$$

Proof. Taking the scalar product of (74) with $U(t)$ on $\mathcal{H}$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|U\|^2 &= -\|\nabla U\|^2 + \alpha_1^{-1}(t) (B(U, u), U) \\ &\quad + R(t) \|U\|^2 \\ &\leq \alpha_1^{-1}(t) (B(U, u), U) + R(t) \|U\|^2 \\ &\leq \frac{c_7}{\alpha_1(t)} \|U\|^2 \|\nabla u\| + R(t) \|U\|^2. \end{aligned} \quad (85)$$

Then,

$$\frac{d}{dt} \|U\|^2 \leq 2K_6(t) \|U\|^2. \quad (86)$$

By the Gronwall inequality, we get

$$\|U(t)\|^2 \leq e^{\int_{t_0}^t 2K_6(s)ds} \|U(t_0)\|^2 \quad (87)$$

which implies that

$$\|D\varphi(t, t_0, u_0; \omega)\| \leq e^{\int_{t_0}^t 2K_6(s)ds}. \quad (88)$$

Since

$$a_1(D\varphi(t, t_0, u_0; \omega)) = \|D\varphi(t, t_0, u_0; \omega)\|, \quad (89)$$

then

$$a_1(D\varphi(t, t_0, u_0; \omega)) \leq e^{\int_{t_0}^t 2K_6(s)ds}. \quad (90)$$

Taking $a'(\omega) = \max\{1, e^{\int_{t_0}^t 2K_6(s)ds}\}$, then we have

$$\begin{aligned} a_1(\omega) (D\varphi(t, 0, u_0; \omega)) &\leq a'_1(\omega), \\ E(\ln a'_1(\omega)) &< \infty. \end{aligned} \quad (91)$$

Assume that $U_j(t) = D\varphi(t, t_0, u_0; \omega)\chi_j$, here $\chi_j \in \mathcal{H}$, and $i = 1, 2, \dots, d$. $Q_d(t)$ is the orthogonal projection in $\mathcal{H}$ onto

the space spanned by $U_1(t), U_2(t), \dots, U_d(t)$. Let $\{\Psi_j\}$, $j = 1, 2, \dots, d$, be an orthonormal basis in $Q_d(t)\mathcal{H}$. Then,

$$\begin{aligned} \text{Tr} \{-A + F(t, u)\} \circ Q_d(t) \\ &= \sum_{j=1}^d \left(\{-A + F(t, u)\} \Psi_j, \Psi_j \right) \leq \sum_{j=1}^d \left\{ (-A \Psi_j, \Psi_j) \right. \\ &\quad \left. + \alpha_1^{-1}(t) (B(\Psi_j, u(t)), \Psi_j) + R(t) (\Psi_j, \Psi_j) \right\} \\ &\leq -\sum_{j=1}^d \lambda_j \|\Psi_j\|^2 + \sum_{j=1}^d \left(c_8 e^{b_1 W_1(t)} \|\Psi_j\|^2 \right. \\ &\quad \left. + e^{b_3 W_3(t)} \|\Psi_j\|^2 \right) \leq -\sum_{j=1}^d \lambda_j + \sum_{j=1}^d \left(c_8 e^{b_1 W_1(t)} \right. \\ &\quad \left. + e^{b_3 W_3(t)} \right). \end{aligned} \quad (92)$$

Set

$$\begin{aligned} \omega_d(D\varphi(t, 0, u_0; \omega)) \\ \leq \exp \left\{ -\sum_{j=1}^d \lambda_j + \sum_{j=1}^d \left(c_8 e^{b_1 W_1(t)} + e^{b_3 W_3(t)} \right) \right\}. \end{aligned} \quad (93)$$

Then,

$$\begin{aligned} E \left[\ln \left(\sup_{u_0 \in \mathcal{A}(\omega)} \omega(D\varphi(t, t_0, u_0; \omega)) \right) \right] \\ \leq -\sum_{j=1}^d \lambda_j \\ + \int_0^1 \left\{ \sum_{j=1}^d \left(c_8 E[e^{b_1 W_1(t)}] + E[e^{b_3 W_3(t)}] \right) \right\} dt \\ \leq -\sum_{j=1}^d \lambda_j + c_9 d. \end{aligned} \quad (94)$$

We can get that

$$E \left[\ln \left(\sup_{u_0 \in \mathcal{A}(\omega)} \omega(D\varphi(t, t_0, u_0; \omega)) \right) \right] < 0 \quad (95)$$

when

$$-\sum_{j=1}^d \lambda_j + c_9 d < 0, \quad c_9 \in R^+. \quad (96)$$

In this case, the Hausdorff dimension of random attractor $\mathcal{A}(t, \omega)$ satisfies

$$\dim_H(\mathcal{A}(t, \omega)) \leq d, \quad \forall t \in R. \quad (97)$$

$\square$

Competing Interests

The authors declare that they have no competing interests.

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