Research Article

Estimates of Fractional Integral Operators on Variable Exponent Lebesgue Spaces

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By some estimates for the variable fractional maximal operator, the authors prove that the fractional integral operator is bounded and satisfies the weak-type inequality on variable exponent Lebesgue spaces.

1. Introduction

It is well known that the boundedness of fractional integral operators is the focus of study in the classical Lebesgue spaces. Sobolev [1] showed that the fractional integral operator is bounded from the classical Lebesgue space $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. In 1999, Kenig and Stein [2] have obtained the boundedness of multilinear fractional integral operator $I_{\alpha,k}$ from $L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n) \times \cdots \times L^{r_k}(\mathbb{R}^n)$ to $L^s(\mathbb{R}^n)$. During the recent three decades, variable exponent function spaces have been studied extensively; see, for example, [3–15]. The characterization and the boundedness of the classical operators on variable exponent function spaces were systemically studied; see [5, 6, 16–21]. For example, Capone et al. [22] proved that Riesz fractional integral I_α satisfies the weak-type inequality on variable exponent Lebesgue spaces in 2007.

Motivated by the aforementioned results, we will consider the variable fractional integral operator $I_{\alpha(\cdot)}$ in this paper. Before stating our results, we need to recall some notions firstly.

Given a bounded open set $\Omega \subset \mathbb{R}^n$ and a measurable function $p(\cdot) : \Omega \rightarrow [1, \infty)$. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is

$$L^{p(\cdot)}(\Omega) := \left\{ f \text{ is measurable} : \int_{\Omega} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty \text{ for some } \lambda > 0 \right\} \quad (1)$$

equipped with the norm

$$\|f\|_{p(\cdot), \Omega} := \inf \left\{ \lambda > 0 : \int_{\Omega} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}. \quad (2)$$

Denote by $\mathcal{P}(\Omega)$ the set of measurable functions p on Ω with values in $[1, \infty)$ satisfying

$$1 < p_- \leq p(x) \leq p_+ < \infty, \quad (3)$$

where we use the standard notation

$$\begin{aligned} p_- &:= \operatorname{ess\,inf}_{x \in \Omega} p(x), \\ p_+ &:= \operatorname{ess\,sup}_{x \in \Omega} p(x). \end{aligned} \quad (4)$$

Definition 1. (i) If $p(\cdot)$ satisfies

$$|p(x) - p(y)| \leq \frac{C}{\log(e + 1/|x - y|)}, \quad (5)$$

$$x, y \in \mathbb{R}^n, \quad |x - y| < \frac{1}{2},$$

then we say $p(\cdot)$ satisfies the Log-Hölder condition.

(ii) If $p(\cdot)$ satisfies

$$|p(x) - p(y)| \leq \frac{C}{\log(e + |x|)}, \quad \forall |y| \geq |x|, \quad (6)$$

then we say $p(\cdot)$ satisfies the Log-Hölder decay condition.

Given $\alpha(x)$, $0 < \alpha(x) < n$, the variable fractional operator can be defined by

$$I^{\alpha(x)} f(x) = \int_{\Omega} \frac{f(y)}{|x-y|^{n-\alpha(x)}} dy. \quad (7)$$

Now the first result is the following theorem.

Theorem 2. *Given an open set $\Omega \subset \mathbb{R}^n$ and $\alpha(\cdot) : 0 < \alpha_- \leq \alpha_+ < n$, let $p(\cdot)$ be a function from Ω to $(1, \infty)$ such that $1 < p_- \leq p_+ < \infty$ and $(\alpha(\cdot)p(\cdot))_+ < n$. Suppose further that $\alpha(\cdot)$ satisfies condition (5) and $p(\cdot)$ satisfies conditions (3), (5), and (6). Define $q(\cdot)$ by*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha(x)}{n}, \quad x \in \Omega. \quad (8)$$

Then, the fractional integral operator $I_{\alpha(\cdot)}$ satisfies the weak-type inequality

$$\sup_{t>0} t \left\| \chi_{\{|I_{\alpha(\cdot)} f(x)| > t\}} \right\|_{q(\cdot), \Omega} \leq C \|f\|_{p(\cdot), \Omega}. \quad (9)$$

Remark 3. If $\alpha(x) \equiv \alpha$, Theorem 2 is consistent with Theorem 1.9 in [22].

Given $\alpha(\cdot)$ with $0 < \alpha_- \leq \alpha_+ < kn$, the multilinear variable fractional integral operators can be defined by

$$\begin{aligned} I_{\alpha(\cdot), k}(f_1, \dots, f_k)(x) \\ = \int_{\Omega^k} \frac{f_1(x-y_1) f_2(x-y_2) \cdots f_k(x-y_k)}{|(y_1, y_2, \dots, y_k)|^{kn-\alpha(x)}} dy_1 \cdots dy_k. \end{aligned} \quad (10)$$

The following is our second result.

Theorem 4. *Let $k \in \mathbb{N}$ and $\alpha(\cdot)$ satisfy the Log-Hölder condition, $0 < \alpha_- \leq \alpha_+ < kn$. If $r_i(\cdot)$ satisfies the Log-Hölder condition and $1 \leq (r_i)_- \leq (r_i)_+ \leq \infty$, for $i = 1, \dots, k$, and*

$$\frac{1}{s(x)} = \frac{1}{r_1(x)} + \frac{1}{r_2(x)} + \cdots + \frac{1}{r_k(x)} - \frac{\alpha(x)}{n} > 0, \quad (11)$$

then if each $r_i > 1$,

$$\|I_{\alpha(\cdot), k}(f_1, \dots, f_k)\|_{s(\cdot), \Omega} \leq C \prod_{i=1}^k \|f_i\|_{r_i(\cdot), \Omega}. \quad (12)$$

The following sections are the proofs of Theorems 2 and 4. Here we point out that C denotes a positive constant, but it may vary from line to line.

2. Proof of Theorem 2

All cubes are assumed to have their sides parallel to the coordinate axes; $Q = Q(x_0, r)$ denotes that a cube is centered at x_0 with side length r . $|Q|$ is the Lebesgue measure of Q . $B(x, r)$ denotes that a ball is centered at x with radius r .

Define the variable fractional maximal operator by

$$M^{\alpha(\cdot)} f(x) := \sup_{x \in Q} |Q|^{\alpha(x)/n-1} \int_Q |f(y)| dy. \quad (13)$$

If $\alpha(\cdot) = 0$, $M^{\alpha(\cdot)} f(x)$ becomes

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy \quad (14)$$

which is the usual Hardy-Littlewood maximal function.

Given a function f , denote

$$|f|_{p(\cdot), \Omega} = \int_{\Omega} |f(y)|^{p(y)} dy. \quad (15)$$

Lemma 5 (see [4]). *Given $p(\cdot) : \Omega \rightarrow [1, \infty)$ such that $p_+ < \infty$, then $\|f\|_{p(\cdot), \Omega} < C_1$ if and only if $|f|_{p(\cdot), \Omega} < C_2$. In particular, if either constant equals 1, we can take the other equal to 1 as well.*

Lemma 6 (see [4]). *Given a set Ω with finite measure and exponent functions $p(\cdot), q(\cdot) : \Omega \rightarrow [1, \infty)$ such that $p(x) \leq q(x)$,*

$$\|f\|_{p(\cdot), \Omega} \leq C(1 + |\Omega|) \|f\|_{q(\cdot), \Omega}. \quad (16)$$

The next lemma is the generalized Hölder inequality on variable exponent Lebesgue spaces.

Lemma 7 (see [12]). *If $p(\cdot) \in \mathcal{P}(\Omega)$, there is a constant C such that, for all $f \in L^{p(\cdot)}(\Omega)$ and all $g \in L^{p'(\cdot)}(\Omega)$,*

$$\int_{\Omega} |f(x) g(x)| dx \leq C \|f\|_{p(\cdot), \Omega} \|g\|_{p'(\cdot), \Omega}, \quad (17)$$

where $1/p'(\cdot) + 1/p(\cdot) = 1$.

Lemma 8 (see [23]). *Given an open set Ω and a function $p(\cdot) : \Omega \rightarrow [1, \infty)$ which satisfies the Log-Hölder condition, then, for any ball B such that $|B \cap \Omega| > 0$,*

$$|B|^{p_-(B \cap \Omega) - p_+(B \cap \Omega)} \leq C. \quad (18)$$

Lemma 9 (see [22]). *Given an open set Ω and a function $p(\cdot) : \Omega \rightarrow [1, \infty)$, $p_+ < \infty$ which satisfies the Log-Hölder condition, then, for any ball B such that $|B \cap \Omega| > 0$ and $|B| \leq 1$,*

$$\|\chi_{B \cap \Omega}\|_{p(\cdot), \Omega} \leq C |B|^{1/p(x)}. \quad (19)$$

Here and below, for $t > 0$, let

$$R_t(x) = (e + |x|)^{-tn}. \quad (20)$$

Lemma 10 (see [22]). *Given a set G and two nonnegative functions $r(\cdot)$ and $s(\cdot)$, suppose that, for each $y \in G$,*

$$|s(y) - r(y)| \leq \frac{C}{\log(e + |y|)}. \quad (21)$$

Then, there exists a constant C_t such that, for every function f obeying $|f(y)| \leq 1$, $y \in G$,

$$\int_G |f(y)|^{r(y)} dy \leq C_t \int_G |f(y)|^{s(y)} dy + \int_G R_t(y)^{r-(G)} dy. \quad (22)$$

Lemma 11. Given $\alpha(\cdot) : 0 < \alpha_- \leq \alpha_+ < n$, fix ϵ , $0 < \epsilon < \max(\alpha_-, n - \alpha_+)$. Then, there exists a constant $C = C(\alpha_-, n, \epsilon)$ such that, for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$,

$$|I_{\alpha(\cdot)} f(x)| \leq C [M^{\alpha(\cdot)-\epsilon} f(x)]^{1/2} [M^{\alpha(\cdot)+\epsilon} f(x)]^{1/2}. \quad (23)$$

Proof. Let $\delta > 0$ be given. Then,

$$\begin{aligned} I_{\alpha(\cdot)} f(x) &= \int_{|x-y|<\delta} f(y) |x-y|^{\alpha(x)-n} dy \\ &\quad + \int_{|x-y|\geq\delta} f(y) |x-y|^{\alpha(x)-n} dy \\ &= I_1(x) + I_2(x). \end{aligned} \quad (24)$$

Let $R_i = \{y \mid 2^{i-1}\delta \leq |x-y| < 2^i\delta\}$; we have

$$\begin{aligned} |I_1(x)| &\leq \sum_{i=0}^{\infty} \int_{R_i} \frac{|f(y)|}{|x-y|^{n-\alpha(x)}} dy \\ &\leq \sum_{i=0}^{\infty} \frac{1}{(2^{i-1}\delta)^{n-\alpha(x)}} \int_{R_i} |f(y)| dy \\ &\leq \sum_{i=0}^{\infty} 2^{n-\alpha_-} (2^{-i})^{\epsilon} \delta^{\epsilon} \frac{1}{(2^{-i}\delta)^{n-\alpha(x)+\epsilon}} \int_{B_i} |f(y)| dy, \end{aligned} \quad (25)$$

where $B_i = \{y \mid |x-y| \leq 2^{-i}\delta\}$. Hence, $I_1(x) \leq C_{\epsilon} \delta^{\epsilon} (M^{\alpha(\cdot)-\epsilon} f)(x)$. Similarly,

$$\begin{aligned} I_2(x) &\leq \sum_{i=1}^{\infty} (2^{i-1}\delta)^{\alpha(x)-n} \int_{B_i} |f(y)| dy \\ &\leq C'_{\epsilon} \delta^{-\epsilon} (M^{\alpha(\cdot)+\epsilon} f)(x). \end{aligned} \quad (26)$$

By choosing $\delta^{\epsilon} = [M^{\alpha(\cdot)+\epsilon} f(x)/M^{\alpha(\cdot)-\epsilon} f(x)]^{1/2}$, one finds

$$|I_{\alpha(\cdot)} f(x)| \leq C [M^{\alpha(\cdot)-\epsilon} f(x)]^{1/2} [M^{\alpha(\cdot)+\epsilon} f(x)]^{1/2}. \quad (27)$$

□

Lemma 12. Given an open set $\Omega \subset \mathbb{R}^n$ and $\alpha(\cdot) : 0 < \alpha_- \leq \alpha_+ < n$, let $p(\cdot) : \Omega \rightarrow [1, \infty)$ be such that $1 = p_- \leq p_+$, $(\alpha(\cdot)p(\cdot))_+ < n$, and such that $p(\cdot)$ satisfies the Log-Hölder condition. Define $q(\cdot)$ by

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha(x)}{n}, \quad x \in \Omega. \quad (28)$$

Then, for all $f \in L^{p(\cdot)}(\Omega)$ such that $\|f\|_{p(\cdot), \Omega} \leq 1$ and such that $|f(x)| \geq 1$ or $f(x) = 0$, $x \in \Omega$,

$$M^{\alpha(\cdot)} f(x) \leq CMf(x)^{p(x)/q(x)}. \quad (29)$$

Proof. Fix $x \in \Omega$, and fix a ball B containing x . Then, by the definition of $q(\cdot)$,

$$\begin{aligned} \frac{1}{|B|^{1-\alpha(x)/n}} \int_{B \cap \Omega} |f(y)| dy &= |B|^{\alpha(x)/n} \\ &\cdot \left(\frac{1}{|B|} \int_{B \cap \Omega} |f(y)| dy \right)^{\alpha(x)p(x)/n} \\ &\cdot \left(\frac{1}{|B|} \int_{B \cap \Omega} |f(y)| dy \right)^{1-\alpha(x)p(x)/n} \leq |B|^{\alpha(x)/n} \\ &\cdot \left(\frac{1}{|B|} \int_{B \cap \Omega} |f(y)| dy \right)^{\alpha(x)p(x)/n} Mf(x)^{p(x)/q(x)}. \end{aligned} \quad (30)$$

To complete the proof we will show that

$$|B|^{\alpha(x)/n} \left(\frac{1}{|B|} \int_{B \cap \Omega} |f(y)| dy \right)^{\alpha(x)p(x)/n} \leq C. \quad (31)$$

We consider it in two cases depending on the size of B . Suppose first that $|B| > 1$. Let $\Omega_f = \text{supp } f$. By Chebyshev's inequality and Lemma 5,

$$|\Omega_f| \leq |f|_{p(\cdot), \Omega} \leq \|f\|_{p(\cdot), \Omega} < 1. \quad (32)$$

Therefore, since $p(x) \geq 1$, by Lemma 6,

$$\begin{aligned} |B|^{\alpha(x)/n} \left(\frac{1}{|B|} \int_{B \cap \Omega} |f(y)| dy \right)^{\alpha(x)p(x)/n} \\ \leq \left(\int_{B \cap \Omega} |f(y)| dy \right)^{\alpha(x)p(x)/n} \leq \|f\|_{1, \Omega_f}^{\alpha(x)p(x)/n} \\ \leq C (1 + |\Omega_f|)^{(\alpha p)_+/n} \|f\|_{p(\cdot), \Omega_f}^{(\alpha p)_-/n} \leq C. \end{aligned} \quad (33)$$

Now suppose $|B| \leq 1$. If $p_-(B \cap \Omega) > 1$, then $(p')_+(B \cap \Omega) < \infty$ and $p'(\cdot)$ satisfies the Log-Hölder condition, so by Lemmas 7 and 9

$$\begin{aligned} |B|^{\alpha(x)/n} \left(\frac{1}{|B|} \int_{B \cap \Omega} |f(y)| dy \right)^{\alpha(x)p(x)/n} \\ \leq |B|^{\alpha(x)/n - \alpha(x)p(x)/n} \|\chi_{B \cap \Omega}\|_{p'(\cdot), \Omega}^{\alpha(x)p(x)/n} \|f\|_{p(\cdot), \Omega}^{\alpha(x)p(x)/n} \\ \leq |B|^{\alpha(x)/n - \alpha(x)p(x)/n} \|\chi_{B \cap \Omega}\|_{p'(\cdot), \Omega}^{\alpha(x)p(x)/n} \\ \leq C |B|^{\alpha(x)/n - \alpha(x)p(x)/n} |B|^{\alpha(x)p(x)/n p'(\cdot)} \leq C. \end{aligned} \quad (34)$$

The argument is the same when $p_-(B \cap \Omega) = 1$ except that instead of applying Lemma 9 we note that, by Lemma 6,

$$\begin{aligned} \|\chi_{B \cap \Omega}\|_{p'(\cdot), \Omega}^{\alpha(x)p(x)/n} &= \|\chi_{B \cap \Omega}\|_{p'(\cdot), B \cap \Omega}^{\alpha(x)p(x)/n} \\ &\leq C (1 + |B|) \|\chi_{B \cap \Omega}\|_{\infty, B \cap \Omega}^{\alpha(x)p(x)/n} \leq C. \end{aligned} \quad (35)$$

Then, by Lemma 8,

$$\begin{aligned} |B|^{\alpha(x)/n - \alpha(x)p(x)/n} &\leq |B|^{(1-p(x))(\alpha_+/n)} \\ &\leq |B|^{(p_-(B \cap \Omega) - p_+(B \cap \Omega))(\alpha_+/n)} \leq C. \end{aligned} \quad (36)$$

This completes the proof. □

Lemma 13. Given an open set $\Omega \subset \mathbb{R}^n$ and $\alpha(\cdot) : 0 < \alpha_- \leq \alpha_+ < n$, let $p(\cdot) \in \mathcal{P}(\Omega)$ be such that $1 < p_- \leq p_+$, $(\alpha(\cdot)p(\cdot))_+ < n$, and satisfy the Log-Hölder decay condition. Define $q(\cdot)$ by

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha(x)}{n}, \quad x \in \Omega. \quad (37)$$

Then, for all $f \in L^{p(\cdot)}(\Omega)$ such that $\|f\|_{p(\cdot),\Omega} \leq 1$ and $|f(x)| \leq 1$, $x \in \Omega$,

$$M^{\alpha(\cdot)} f(x) \leq CMf(x)^{p(x)/I_q(x)}, \quad (38)$$

where $I_r(x) = \sup_{|y|>|x|} r(y)$.

Proof. Fix $x \in \Omega$ and let B be any ball containing x . It will suffice to show that

$$\frac{1}{|B|^{1-\alpha(x)/n}} \int_{B \cap \Omega} |f(y)| dy \leq CMf(x)^{p(x)/I_q(x)}, \quad (39)$$

where C is independent of B .

It follows at once from the definition of $q(\cdot)$ that $1/I_p(x) - 1/I_q(x) = \alpha(x)/n$. Therefore, by Hölder's inequality,

$$\begin{aligned} \frac{1}{|B|^{1-\alpha(x)/n}} \int_{B \cap \Omega} |f(y)| dy &= |B|^{\alpha(x)/n} \\ &\cdot \left(\frac{1}{|B|} \int_{B \cap \Omega} |f(y)| dy \right)^{\alpha(x)I_p(x)/n} \\ &\cdot \left(\frac{1}{|B|} \int_{B \cap \Omega} |f(y)| dy \right)^{1-\alpha(x)I_p(x)/n} \\ &\leq \left(\int_{B \cap \Omega} |f(y)|^{I_p(x)} dy \right)^{\alpha(x)/n} Mf(x)^{I_p(x)/I_q(x)}. \end{aligned} \quad (40)$$

Since $|f(x)| \leq 1$, $Mf(x) \leq 1$, so $Mf(x)^{I_p(x)/I_q(x)} \leq Mf(x)^{p(x)/I_q(x)}$. Therefore, to complete the proof we only need to show that

$$\left(\int_{B \cap \Omega} |f(y)|^{I_p(x)} dy \right)^{\alpha(x)/n} \leq C. \quad (41)$$

For

$$\begin{aligned} &\min \left\{ \left(\int_{B \cap \Omega} |f(y)|^{I_p(x)} dy \right)^{\alpha_-/n}, \right. \\ &\left. \left(\int_{B \cap \Omega} |f(y)|^{I_p(x)} dy \right)^{\alpha_+/n} \right\} \\ &\leq \left(\int_{B \cap \Omega} |f(y)|^{I_p(x)} dy \right)^{\alpha(x)/n} \\ &\leq \max \left\{ \left(\int_{B \cap \Omega} |f(y)|^{I_p(x)} dy \right)^{\alpha_-/n}, \right. \\ &\left. \left(\int_{B \cap \Omega} |f(y)|^{I_p(x)} dy \right)^{\alpha_+/n} \right\}, \quad 0 < \alpha_- \leq \alpha_+ < n, \end{aligned} \quad (42)$$

it suffices to prove

$$\int_{B \cap \Omega} |f(y)|^{I_p(x)} dy \leq C. \quad (43)$$

Define the sets

$$\begin{aligned} D_x &:= \{y \in B \cap \Omega, |y| \leq |x|\}, \\ D^x &:= \{y \in B \cap \Omega, |y| > |x|\}. \end{aligned} \quad (44)$$

Then,

$$\begin{aligned} \int_{B \cap \Omega} |f(y)|^{I_p(x)} dy &= \int_{D_x} |f(y)|^{I_p(x)} dy \\ &+ \int_{D^x} |f(y)|^{I_p(x)} dy. \end{aligned} \quad (45)$$

The integral in the right-hand side is easy to estimate. If $y \in D^x$, then $I_p(x) \geq p(y)$. Thus, since $|f(x)| \leq 1$, by Lemma 5,

$$\begin{aligned} \int_{D^x} |f(y)|^{I_p(x)} dy &\leq \int_{D^x} |f(y)|^{p(y)} dy \leq |f|_{p(\cdot),\Omega} \\ &\leq 1. \end{aligned} \quad (46)$$

To estimate the first integral, we will apply Lemma 10. If $y \in D_x$, then, by the Log-Hölder decay condition,

$$|p(y) - I_p(x)| \leq \frac{C}{\log(e + |y|)}. \quad (47)$$

Therefore, for any $t > 1$ and $(I_p)_- \geq 1$, $R_t(y)^{(I_p)_-}$ is integrable, and

$$\begin{aligned} \int_{D_x} |f(y)|^{I_p(x)} dy &\leq C_t \int_{D_x} |f(y)|^{p(y)} dy \\ &+ \int_{D_x} R_t(y)^{(I_p)_-} dy \\ &\leq C |f|_{p(\cdot),\Omega} \leq C. \end{aligned} \quad (48)$$

This completes the proof. $\square$

Lemma 14 (see [22]). Given an open set $\Omega \subset \mathbb{R}^n$, let $p(\cdot) : \Omega \rightarrow [1, \infty)$ satisfy the Log-Hölder decay condition. Suppose $f \in L^{p(\cdot)}(\Omega)$ such that $\|f\|_{p(\cdot),\Omega} \leq 1$. Then, for every ball B and every $x \in B$,

$$\begin{aligned} &\left(\frac{1}{|B|} \int_{B \cap \Omega} |f(y)| dy \right)^{p(x)} \\ &\leq C \left(\frac{1}{|B|} \int_{B \cap \Omega} |f(y)|^{p(y)/p_-} dy \right)^{p_-} + S(x), \end{aligned} \quad (49)$$

where $S(\cdot) \in L^1(\mathbb{R}^n)$ and $C = C(n, p(\cdot))$.

Proof of Theorem 2. Fix $f \in L^{p(\cdot)}(\Omega)$; without loss of generality we may assume that $\|f\|_{p(\cdot),\Omega} = 1$. Since $q_+ < \infty$, by Lemma 5 it will suffice to prove that

$$|t\chi_{|I_{\alpha(\cdot)}f|>t}|_{q(\cdot),\Omega} = \int_{\{x \in \Omega : |I_{\alpha(\cdot)}f|>t\}} t^{q(x)} dx \leq C. \quad (50)$$

Fix ϵ , $0 < \epsilon < \max(\alpha_-, n - \alpha_+)$, such that

$$\frac{p_-}{\epsilon q_+ / n + 1} > 1. \quad (51)$$

Define $r(\cdot) : \Omega \rightarrow (1, \infty)$ by

$$r(x) = \frac{2}{\epsilon q(x) / n + 1}. \quad (52)$$

Since $r_- > 1$ and $(r')_+ < \infty$, by (23) and Young's inequality, for each x ,

$$\begin{aligned} |I_{\alpha(\cdot)} f(x)| &\leq C [M^{\alpha(\cdot)-\epsilon} f(x)]^{1/2} [M^{\alpha(\cdot)+\epsilon} f(x)]^{1/2} \\ &\leq \frac{[CM^{\alpha(\cdot)-\epsilon} f(x)]^{r(x)/2}}{r(x)} \\ &\quad + \frac{[CM^{\alpha(\cdot)+\epsilon} f(x)]^{r'(x)/2}}{r'(x)} \\ &\leq C^{r_+} M^{\alpha(\cdot)-\epsilon} f(x)^{r(x)/2} \\ &\quad + C^{(r')_+} M^{\alpha(\cdot)+\epsilon} f(x)^{r'(x)/2}. \end{aligned} \quad (53)$$

Therefore,

$$\begin{aligned} \int_{\{|I_{\alpha(\cdot)} f| > t\}} t^{q(x)} dx &\leq \int_{\{|M^{\alpha(\cdot)-\epsilon} f(x)|^{r(x)/2} > t/2C\}} t^{q(x)} dx \\ &\quad + \int_{\{|M^{\alpha(\cdot)+\epsilon} f(x)|^{r'(x)/2} > t/2C\}} t^{q(x)} dx \\ &=: I_1 + I_2. \end{aligned} \quad (54)$$

Denote $f = f_1 + f_2$, where $f_1 = f\chi_{\{|f| \geq 1\}}$ and $f_2 = f\chi_{\{|f| < 1\}}$.

Therefore, by (29) and (38),

$$\begin{aligned} M^{\alpha(\cdot)-\epsilon} f(x)^{r(x)/2} &\leq 2^{r_+} (M^{\alpha(\cdot)-\epsilon} f_1(x)^{r(x)/2} + M^{\alpha(\cdot)-\epsilon} f_2(x)^{r(x)/2}) \\ &\leq C (Mf_1(x)^{r(x)p(x)/2q(x)} + Mf_2(x)^{r(x)p(x)/2I_q(x)}), \quad (55) \\ M^{\alpha(\cdot)+\epsilon} f(x)^{r'(x)/2} &\leq C (Mf_1(x)^{r'(x)p(x)/2q(x)} + Mf_2(x)^{r'(x)p(x)/2I_q(x)}). \end{aligned}$$

Therefore,

$$\begin{aligned} I_1 &\leq C \int_{\{Mf_1(x)^{r(x)p(x)/2q(x)} > t/2C\}} t^{q(x)} dx \\ &\quad + C \int_{\{Mf_2(x)^{r(x)p(x)/2I_q(x)} > t/2C\}} t^{q(x)} dx. \end{aligned} \quad (56)$$

We estimate each integral in turn.

For each $N > 0$, define $f_N = f\chi_{B_N(0)}$, where $B_N(0) = \{x \in \mathbb{R}^n : |x| \leq N\}$. Then, for each N the set $\{Mf_N(x)^{r(x)p(x)/2q(x)} >$

$t/2C\}$ is bounded, and by the monotone convergence theorem, the first integral is immediate:

$$\begin{aligned} &\int_{\{Mf_1(x)^{r(x)p(x)/2q(x)} > t/2C\}} t^{q(x)} dx \\ &= \lim_{N \rightarrow \infty} \int_{\{Mf_N(x)^{r(x)p(x)/2q(x)} > t/2C\}} t^{q(x)} dx. \end{aligned} \quad (57)$$

So, it will suffice to prove that

$$\int_{\{Mf_N(x)^{r(x)p(x)/2q(x)} > t/2C\}} t^{q(x)} dx \leq C, \quad (58)$$

where C is independent of N . Fix $N > 0$; for each $x \in \{Mf_N(x)^{r(x)p(x)/2q(x)} > t/2C\}$ there exists a ball B_x containing x such that

$$\begin{aligned} &\left(\frac{2^n}{|2B_x|} \int_{2B_x \cap \Omega} |f(y)| dy \right)^{r(x)p(x)/2q(x)} \\ &\geq \left(\frac{1}{|B_x|} \int_{B_x \cap \Omega} |f_N(y)| dy \right)^{r(x)p(x)/2q(x)} > \frac{t}{2C}. \end{aligned} \quad (59)$$

Therefore, by the Besicovitch covering lemma (see [24]), there exists a sequence $\{B_k\}$ of $\{B_x\}$ such that

$$\left\{ Mf_N(x)^{r(x)p(x)/2q(x)} > \frac{t}{2C} \right\} \subset \bigcup_k 2B_k \quad (60)$$

and such that the balls $\{2B_k\}$ have finite overlap uniformly bounded by a constant depending only on the dimension. Hence,

$$\begin{aligned} &\int_{\{Mf_N(x)^{r(x)p(x)/2q(x)} > t/2C\}} t^{q(x)} dx \leq \sum_k \int_{2B_k \cap \Omega} t^{q(x)} dx \\ &\leq 2^{nr_+ P_+} \sum_k \int_{2B_k \cap \Omega} \left(\frac{1}{|2B_k|} \int_{2B_k \cap \Omega} |f(y)| dy \right)^{r(x)p(x)/2} dx. \end{aligned} \quad (61)$$

For

$$\begin{aligned} &\left| \frac{r(x)p(x)}{2} - \frac{r(y)p(y)}{2} \right| \\ &\leq \frac{r(x)}{2} |p(x) - p(y)| + \frac{p(y)}{2} |r(x) - r(y)| \\ &\leq \frac{r_+}{2} |p(x) - p(y)| \\ &\quad + \frac{(\epsilon/n) P_+}{(1 - ((\alpha_+ - \epsilon)/n) p_+)^2} |p(x) - p(y)| \\ &\leq C |p(x) - p(y)|, \end{aligned} \quad (62)$$

by Lemma 14, we have

$$\begin{aligned}
& \int_{\{Mf_N(x)^{r(x)p(x)/2q(x)} > t/2C\}} t^{q(x)} dx \\
& \leq C \sum_k \int_{2B_k \cap \Omega} \left[\left(\frac{1}{|2B_k|} \int_{2B_k \cap \Omega} |f(y)|^{r(y)p(y)/(rp)_-} dy \right)^{(rp)_-/2} \right. \\
& \quad \left. + S(x) \right] dx \\
& \leq C \sum_k \left[\int_{2B_k \cap \Omega} M(|f(\cdot)|^{r(\cdot)p(\cdot)/(rp)_-})^{(rp)_-/2} dx \right. \\
& \quad \left. + C \int_{2B_k \cap \Omega} S(x) dx \right] \\
& \leq C \sum_k \left[\int_{2B_k \cap \Omega} (|f(x)|^{r(x)p(x)/(rp)_-})^{(rp)_-/2} dx \right. \\
& \quad \left. + C \int_{2B_k \cap \Omega} S(x) dx \right] \leq C \int_{\Omega} |f(x)|^{r(x)p(x)/2} dx \\
& \quad + C \int_{\Omega} S(x) dx,
\end{aligned} \tag{63}$$

where, in the third inequality, since $(rp)_-/2 > 1$, we used the boundedness of maximal operator on $L^{(rp)_-/2}(\Omega)$, and in the last inequality, we used the fact that the balls have uniformly bounded finite overlap.

For $S(\cdot) \in L^1(\mathbb{R}^n)$, $\int_{\Omega} S(x) dx \leq C$; then, it will suffice to prove

$$\int_{\Omega} |f(x)|^{r(x)p(x)/2} dx \leq C. \tag{64}$$

We consider the two cases $|f(x)| \geq 1$ and $|f(x)| < 1$ to estimate it.

If $|f(x)| \geq 1$, since $r(x)/2 = 1/(\epsilon q_+/n + 1) < 1$, therefore

$$\int_{\Omega} |f(x)|^{r(x)p(x)/2} dx < \int_{\Omega} |f(x)|^{p(x)} dx \leq C. \tag{65}$$

If $|f(x)| < 1$, since $2/r_- = (\epsilon/n)q_+ + 1 > 1$, by Hölder's inequality

$$\begin{aligned}
& \int_{\Omega} |f(x)|^{r(x)p(x)/2} dx \\
& \leq C \left(\int_{\Omega} |f(x)|^{r(x)p(x)/2 \cdot 2/r_-} dx \right)^{r_-/2} |\Omega|^{1-r_-/2} \\
& \leq C \left(\int_{\Omega} |f(x)|^{p(x)} dx \right)^{r_-/2} |\Omega|^{1-r_-/2} \leq C |\Omega| \leq C.
\end{aligned} \tag{66}$$

To estimate the second term, we need to show that

$$\int_{\{Mf_2(x)^{r(x)p(x)/2I_q(x)} > t/2C\}} t^{q(x)} dx \leq C. \tag{67}$$

We first need to control the quantity $q(x)/I_q(x)$. Fix $\epsilon > 0$; then, there exists y , $|y| > |x|$, such that $I_q(x) \leq (1 + \epsilon)q(y)$. Then, by the Log-Hölder decay condition, we have

$$\begin{aligned}
0 & \leq I_q(x) - q(x) \leq (1 + \epsilon)q(y) - q(x) \\
& \leq |q(y) - q(x)| + \epsilon q_+ \leq \frac{C}{\log(e + |x|)} + \epsilon q_+.
\end{aligned} \tag{68}$$

Since $\epsilon > 0$ is arbitrary and $q_- \geq 1$, it follows that

$$0 \leq \frac{I_q(x)}{q(x)} - 1 \leq \frac{C}{\log(e + |x|)}. \tag{69}$$

For $R_t(x) = (e + |x|)^{-tn}$, $R_{2q_+/q_-}(x)$ is integrable, and by Lemma 10

$$\begin{aligned}
& \int_{\{Mf_2(x)^{r(x)p(x)/2I_q(x)} > t/2C\}} t^{q(x)} dx \\
& \leq C \int_{\{Mf_2(x)^{r(x)p(x)/2I_q(x)} > t/2C\}} (t^{q(x)})^{I_q(x)/q(x)} dx \\
& \quad + C \int_{\{Mf_2(x)^{r(x)p(x)/2I_q(x)} > t/2C\}} R_{2q_+/q_-}(x) dx \\
& \leq C \int_{\{Mf_2(x)^{r(x)p(x)/2I_q(x)} > t/2C\}} t^{I_q(x)} dx.
\end{aligned} \tag{70}$$

The proof now proceeds as it does for the first term.

We estimate I_2 as we did before; here we omit the detail. Then, we finish the proof of Theorem 2. $\square$

3. Proof of Theorem 4

To prove Theorem 4, we need the following lemma.

Lemma 15 (see [16]). *Let Ω be bounded. Suppose that $p(\cdot)$ satisfies conditions (3) and (5) and $\alpha(\cdot)$ also satisfies the Log-Hölder condition:*

$$\inf_{x \in \Omega} \alpha(x) > 0, \tag{71}$$

$$\sup_{x \in \Omega} (\alpha(x)p(x)) < n.$$

If $1/q(x) = 1/p(x) - \alpha(x)/n$, then there exists a positive constant C such that for all $f \in L^{q(\cdot)}(\Omega)$

$$\|I_{\alpha(\cdot)} f\|_{p(\cdot), \Omega} \leq C \|f\|_{q(\cdot), \Omega}. \tag{72}$$

Proof of Theorem 4. Since $\alpha(x) > 0$, some $r_i < \infty$. If $r_{j+1}(x) = \dots = r_k(x) = \infty$, $1 \leq j < k$, since $\alpha(x)/n < 1/r_1(x) + 1/r_2(x) + \dots + 1/r_j(x) \leq j$, we have $kn - \alpha(x) > (k - j)n$. Then, integration in $y_{j+1}, \dots, y_k$ reduces matters to the case when all $r_i(x)$ are finite (and $k = j$). Thus, we can assume that all $r_i(x) < \infty$.

Now, observe that $0 < \alpha(x) < \sum_{i=1}^k (n/r_i(x))$; we can find $0 < \alpha_i(x) < n/r_i(x)$ such that $\alpha(x) = \sum_{i=1}^k \alpha_i(x)$. Let $1/s_i(x) = 1/r_i(x) - \alpha_i(x)/n$. Since $0 < \alpha_i(x)/n < 1/r_i(x) \leq 1$ and

$$\begin{aligned}
& |y_1|^{n-\alpha_1(x)} |y_2|^{n-\alpha_2(x)} \dots |y_k|^{n-\alpha_k(x)} \\
& \leq |(y_1, y_2, \dots, y_k)|^{kn-\alpha(x)},
\end{aligned} \tag{73}$$

it follows that

$$I_{\alpha(\cdot), k}(f_1, \dots, f_k)(x) \leq \prod_{i=1}^k I_{\alpha_i(\cdot)}(f_i)(x), \tag{74}$$

where $I_{\alpha_i(\cdot)}$ is the variable exponent fractional integral operator.

Because $\sum_{i=1}^k (1/s_i(x)) = 1/s(x)$, by Hölder's inequality and (72), we obtain

$$\begin{aligned} \|I_{\alpha(\cdot),k}(f_1, \dots, f_k)(x)\|_{s(\cdot),\Omega} &\leq \prod_{i=1}^k \|I_{\alpha_i(\cdot)}(f_i)(x)\|_{s_i(\cdot),\Omega} \\ &\leq \prod_{i=1}^k \|f_i(x)\|_{r_i(\cdot),\Omega}. \end{aligned} \quad (75)$$

This finishes the proof. $\square$

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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