A Remark on the Stability of Approximative Compactness

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We study the stability of approximative \( \tau \)-compactness, where \( \tau \) is the norm or the weak topology. Let \( \Lambda \) be an index set and for every \( \lambda \in \Lambda \), let \( Y_\lambda \) be a subspace of a Banach space \( X_\lambda \). For \( 1 \leq p < \infty \), let \( X = \oplus_{\lambda} X_\lambda \) and \( Y = \oplus_{\lambda} Y_\lambda \). We prove that \( Y \) (resp., \( B_Y \)) is approximatively \( \tau \)-compact in \( X \) if and only if, for every \( \lambda \in \Lambda \), \( Y_\lambda \) (resp., \( B_{Y_\lambda} \)) is approximatively \( \tau \)-compact in \( X_\lambda \).

1. Introduction

Let \( X \) be a real Banach space and let \( K \) be a subset of \( X \). We denote by \( \tau \) either the norm or the weak topology on \( X \). The metric projection of \( X \) onto \( K \) is the set valued map defined by \( P_\tau(x) = \{ y \in K : \| x - y \| = d(x, K) \} \) for \( x \in X \), where \( d(x, K) \) denotes the distance from \( x \) to \( K \). If, for every \( x \in X \), \( P_\tau(x) \neq \emptyset \), we say that \( K \) is a proximinal subset of \( X \). A sequence \( \{ y_n \} \subset K \) is called minimizing for \( x \in X \), if \( \| x - y_n \| \to d(x, K) \).

The notion of approximative compactness was introduced by Efimov and Stechkin [1] in connection with the study of Chebyshev sets in Banach spaces and plays an important role in approximation theory (see, e.g., [2, 3]). Deutsch [4] extended this notion to define approximative \( \tau \)-compactness.

Definition 1. Let \( K \) be a \( \tau \)-closed subset of \( X \) and \( x_0 \in X \). We say that \( K \) is approximatively \( \tau \)-compact for \( x_0 \) if every minimizing sequence \( \{ y_n \} \subset K \) for \( x_0 \) has a \( \tau \)-convergent subsequence. If \( K \) is approximatively \( \tau \)-compact for every \( x \in X \), we say that \( K \) is approximatively \( \tau \)-compact in \( X \).

It is easy to verify that approximative \( \tau \)-compactness implies proximinality. Clearly, compact sets or finite-dimensional subspaces of a Banach space are approximatively compact; weakly compact sets or reflexive subspaces of Banach spaces are approximatively weakly compact. Approximative \( \tau \)-compactness has been studied in detail in [1, 3–7].

When it comes to the stability of approximative \( \tau \)-compactness, we suppose that \( \Lambda \) is an index set and for every \( \lambda \in \Lambda \), \( Y_\lambda \) is a subspace of a Banach space \( X_\lambda \). And let \( X = \oplus_{\lambda} X_\lambda \), \( Y = \oplus_{\lambda} Y_\lambda \), where \( 1 \leq p < \infty \). Bandyopadhyay et al. [5] proved that if \( Y \) is approximatively \( \tau \)-compact in \( X \), then \( Y_\lambda \) is approximatively \( \tau \)-compact in \( X_\lambda \) for every \( \lambda \in \Lambda \). In this paper, we prove that the converse is also true. On the other hand, the proximinality of the unit ball of subspaces has been the subject in many recent papers (see, e.g., [8–11]). In this paper, under the above assumption, we also prove that the unit ball of \( Y \) is approximatively \( \tau \)-compact in \( X \) if and only if, for every \( \lambda \in \Lambda \), the unit ball of \( Y_\lambda \) is approximatively \( \tau \)-compact in \( X_\lambda \).

For a real Banach space \( X \), we denote by \( B_X \) the unit ball of \( X \) and denote by \( X^* \) the dual space of \( X \). Before we prove the main conclusions we first show a simple property on approximative \( \tau \)-compactness of the unit ball of subspaces.

Proposition 2. Let \( Y \) be a subspace of a Banach space \( X \). If \( B_Y \) is approximatively \( \tau \)-compact in \( X \), then so is \( Y \). But the converse is not true.

Proof. Suppose that \( x \in X \) and \( \{ y_n \} \subset Y \) is a minimizing sequence of \( x \) in \( Y \); that is, \( \| x - y_n \| \to d(x, Y) \). Then \( \{ y_n \} \subset \lambda B_Y \) for sufficiently large \( \lambda > 0 \). This means that \( d(x, Y) = d(x, \lambda B_Y) \) and \( \{ y_n \} \) is also a minimizing sequence of \( x \) in \( \lambda B_Y \).

By approximative \( \tau \)-compactness of \( \lambda B_Y \) (which is equivalent to the one of \( B_Y \)), \( \{ y_n \} \) has a \( \tau \)-convergent subsequence.
To illustrate that the converse is not true, first, we show that $B_{r_0}$ is not approximately weakly compact in $C_0$. Take $x = (2, 0, 0, \ldots)$ and for every $n \in \mathbb{N}$, $y_n = (1, 1, 1, \ldots, 0, 0, \ldots)$, where $1$ appears $n$ times. Then $d(x, B_{r_0}) = 1$ and $\{y_n\}$ is a minimizing sequence of $x$ in $B_{r_0}$. But $\{y_n\}$ has no weakly convergent subsequence. Hence $B_{r_0}$ is not approximately weakly compact in $C_0$.

Next, let $X = C[0, 1]$ and $Y = C[0, 1]$, with $1 \leq p < \infty$. For any $\alpha = (z, r) \in X$, it is easy to see that $d(\alpha, Y) = |r|$ and $P_\alpha(\alpha) = \{(z, 0)\}$. Now, suppose $\{\beta_n\} = \{(z_n, 0)\} \subset Y$ is a minimizing sequence of $\alpha$ in $Y$; that is,

$$\|\alpha - \beta_n\| = \|z - z_n\| + |r| \rightarrow d(\alpha, Y) = |r|.$$

This implies that $z_n \rightarrow z$. Hence $\beta_n = (z_n, 0) \rightarrow (z, 0)$. Therefore $Y$ is approximately compact in $X$. But, by the above discussion, $B_Y$ is not approximately weakly compact in $Y$, and not in $X$ either.

In order to prove our conclusions, we need the following lemma.

**Lemma 3.** Let $\{X_i : i \in \mathbb{N}\}$ be a sequence of Banach spaces and let $Y_i$ be a subspace of $X_i$, respectively, for $i \in \mathbb{N}$. Consider $X = X_1$, $X_i$ and $Y = Y_1$, where $1 \leq p < \infty$. Let $x = (x_i) \in X$ and $y = (y_i) \subset B_Y$ be a minimizing sequence of $x$ in $B_Y$. Then, for every $\varepsilon > 0$, there exists some $j \in \mathbb{N}$ such that, for all $n$, $\sum_{i > j} \|y_{n,i}\| < \varepsilon^p$.

**Proof.** If the conclusion does not hold, then, for every $j$, there exists infinitely many $n$ such that $\sum_{i > j} \|y_{n,i}\| \geq \varepsilon^p$. We can choose some $j_0$ such that $\sum_{i > j_0} \|x_i\| < (\varepsilon/3)^p$ and infinite subset $\{n_k\}$ of $\mathbb{N}$ such that $\sum_{i > j_0} \|y_{n_k,i}\| \geq \varepsilon^p$ for every $k$. Therefore for every $k$,

$$\|x - y_{n_k}\| = \sum_{i \leq j_0} \|x_i - y_{n_k,i}\| + \sum_{i > j_0} \|x_i - y_{n_k,i}\|

= \left( \sum_{i \leq j_0} \|x_i - y_{n_k,i}\| \right)^p + \sum_{i > j_0} \|x_i - y_{n_k,i}\|^p$$

$$\geq (d(x, B_Y))^p + \left[ \left( \sum_{i \leq j_0} \|y_{n_k,i}\| \right)^{1/p} - \left( \sum_{i \leq j_0} \|x_i\| \right)^{1/p} \right]^p$$

$$\geq \left( \sum_{i \leq j_0} \|x_i\| \right)^p + \left( \varepsilon \right)^p.$$ 

But $\|x - y_{n_k}\| \rightarrow d(x, B_Y) (n \rightarrow \infty)$; then $\|x - y_{n_k}\|^p < (d(x, B_Y))^p + (\varepsilon/3)^p$ for sufficiently large $n$. This is a contradiction.

**Remark 4.** In Lemma 3, if we replace $B_Y$ by $Y$, that is, $\{y_n = (y_{n,i})\} \subset Y$ is a minimizing sequence of $x$ in $Y$, then the conclusion still holds.

**Lemma 5.** Under the assumption in Lemma 3, if, moreover, $\lim_{n \rightarrow \infty} \|y_n\| = r$ and for every $i \in \mathbb{N}$, $\lim_{n \rightarrow \infty} \|y_{n,i}\| = r_i$, then

1. $r = (\sum_{i \in \mathbb{N}} r_i^p)^{1/p}$;
2. $d(x, B_Y) = d(x, \oplus_{i \in \mathbb{N}} r_i B_Y) = (\sum_{i \in \mathbb{N}} [d(x, B_Y)]^p)^{1/p}$.

**Proof.** (1) For every $\varepsilon > 0$, by Lemma 3, there exists $j \in \mathbb{N}$ such that, for all $n$, $\sum_{i > j} \|y_{n,i}\|^p < \varepsilon^p$. For every fixed $j > j$, we can choose $\varepsilon' > 0$ such that $\sum_{i \leq j} \|y_{n,i}'\|^p - r_j^p < \varepsilon^p$ and $\sum_{i \geq j} \|y_{n,i}'\|^p < \varepsilon^p$. Then

$$\left| \sum_{i \leq j} r_j^p - \sum_{i \geq j} \|y_{n,i}'\|^p \right| < \varepsilon^p.$$

By the arbitrariness of $\varepsilon$, we have $r = (\sum_{i \in \mathbb{N}} r_i^p)^{1/p}$.

(2) Note that $r \leq 1$; hence $\oplus_{i \in \mathbb{N}} r_i B_Y \subset B_Y$. This implies that $d(x, B_Y) \leq d(x, \oplus_{i \in \mathbb{N}} r_i B_Y)$. To prove that $d(x, B_Y) \geq d(x, \oplus_{i \in \mathbb{N}} r_i B_Y)$, for every $n$, we define $z_n = (z_{n,i})$, where $z_{n,i} = y_{n,i}$ for $\|y_{n,i}\| < r_i$, and $z_{n,i} = (r_i/\|y_{n,i}\|) y_{n,i}$ for $\|y_{n,i}\| > r_i$. Then $\{z_n\} \subset \oplus_{i \in \mathbb{N}} r_i B_Y$.

For arbitrary $\varepsilon > 0$, by Lemma 3, there exists $j$ such that $\sum_{i > j} \|y_{n,i}'\|^p < \varepsilon^p$, and for all $n$, $\sum_{i \leq j} \|y_{n,i}'\|^p < \varepsilon^p$. Further, we can choose some $n_0$ such that, for all $n > n_0$, $\sum_{i \leq j} \|y_{n,i}'\| - r_i^p < \varepsilon^p$. Then for all $n > n_0$, we have

$$\|y_n - z_n\| = \left( \sum_{i \in \mathbb{N}} \|y_{n,i} - z_{n,i}\|^p \right)^{1/p}$$

$$\leq \left( \sum_{i \in \mathbb{N}} \|y_{n,i}\| - r_i^p \right)^{1/p}$$

$$\leq \left( \sum_{i \leq j} \|y_{n,i}\| - r_i^p \right)^{1/p} + \left( \sum_{i > j} \|y_{n,i}'\| \right)^{1/p}$$

$$\leq \left( \sum_{i \leq j} \|y_{n,i}\| - r_i^p \right)^{1/p} + \left( \sum_{i \geq j} \|y_{n,i}'\| \right)^{1/p}$$

$$+ \left( \sum_{i \geq j} \|y_{n,i}'\| \right)^{1/p} < 3\varepsilon.$$
By the arbitrariness of \( \varepsilon \), we have \( \|y_n - z_n\| \rightarrow 0 \). This implies that

\[
\lim_{n \to \infty} \|x - z_n\| = \lim_{n \to \infty} \|x - y_n\| = d(x, B_Y).
\]

(5)

Therefore \( d(x, B_Y) \geq d(x, \Phi_x r_B Y) \). So we have \( d(x, B_Y) = d(x, \Phi_x r_B Y) \).

For the second equality, first, it is obvious that

\[
d \left( x, \Phi_x r_B Y \right) = \left( \sum_{i \in \mathbb{N}} d( x_i, r_B Y_{\tau_i}) \right)^{1/p}.
\]

(6)

On the other hand, let \( \varepsilon > 0 \) be given. For every \( i \), we can choose \( z_i \in r_B Y_{\tau_i} \) such that \( \|x_i - z_i\|^p < \|d(x_i, r_B Y_{\tau_i})\|^p + \varepsilon/2^i \). Let \( z = (z_i) \in \Phi_x r_B Y \); then

\[
\|x - z\| = \left( \sum_{i \in \mathbb{N}} \|x_i - z_i\|^p \right)^{1/p} < \left( \sum_{i \in \mathbb{N}} \|d(x_i, r_B Y_{\tau_i})\|^p + \varepsilon \right)^{1/p}.
\]

(7)

By the arbitrariness of \( \varepsilon \), we have \( d(x, \Phi_x r_B Y) \leq \left( \sum_{i \in \mathbb{N}} \|d(x_i, r_B Y_{\tau_i})\|^p \right)^{1/p} \). Therefore the second equality holds.

The following is our main result.

**Theorem 6.** Let \( \Lambda \) be an index set. For every \( \lambda \in \Lambda \), let \( Y_\lambda \) be a subspace of a Banach space \( X_\lambda \). For \( 1 \leq p < \infty \), let \( X = \Phi_x r_B X_\lambda \) and \( Y = \Phi_y r_B Y_\lambda \). Then

1. \( Y \) is approximatively \( \tau \)-compact in \( X \) if and only if, for every \( \lambda \in \Lambda \), \( Y_\lambda \) is approximatively \( \tau \)-compact in \( X_\lambda \).
2. \( B_\lambda \) is approximatively \( \tau \)-compact in \( X \) if and only if, for every \( \lambda \in \Lambda \), \( B_\lambda \) is approximatively \( \tau \)-compact in \( X_\lambda \).

**Proof.** (1) Necessity has been proven in [5].

Sufficiency: let \( x \in X \) and \( \{y_n\} \subset Y \) be a minimizing sequence for \( x \). We will show that \( \{y_n\} \) has a \( \tau \)-convergent subsequence. Without loss of generality, we can assume \( \Lambda = \mathbb{N} \) and \( x = (x_i), y_n = (y_n) \).

First, notice that if \( z_i \in P_{Y_i}(x_i) \) for every \( i \), then \( z = (z_i) \in P_Y(x) \). Hence \( d(x, Y_i)^p = \sum_{i \in \mathbb{N}} d(x, Y_{\tau_i})^p \). And for every \( i \),

\[
\|x_i - y_{n_i}\|^p + \sum_{j \neq i} \|d(x, Y_j)^p \leq \|x - y\|^p
\]

\[
= (d(x, Y_i)^p + \|x - y\|^p - (d(x, Y))^p)
\]

\[
= \sum_{j \in \mathbb{N}} \|d(x, Y_j)^p + \|x - y\|^p - (d(x, Y))^p \).
\]

(8)

So

\[
\|x_i - y_{n_i}\|^p \leq (d(x, Y_i)^p + \|x - y\|^p - (d(x, Y))^p)
\]

\[
\rightarrow (d(x, Y))^p (n \rightarrow \infty).
\]

This implies that, for every \( i \), \( \{y_{n_i}\} \) is a minimizing sequence for \( x_i \) in \( Y_i \). Then \( \{y_{n_i}\} \) has a \( \tau \)-convergent subsequence by the approximative \( \tau \)-compactness of \( Y_i \). By employing the diagonal process, we can choose a subsequence \( \{y_{n}\} \) of \( \{y_{n_i}\} \) such that, for every \( i \), \( \{y_{n_i}\} \) has a \( \tau \)-convergent to some \( y_i \in Y_i \). Obviously, \( y_i \in P_{Y_i}(x_i) \) and \( y = (y_i) \in Y \), \( \|x - y\| = d(x, Y) \).

We still denote the subsequence \( \{y_{n}\} \) as \( \{y_n\} \). Next, to complete the proof, we will prove that \( \{y_{n}\} \) has a \( \tau \)-convergent to \( y \).

Case 1. \( \tau \) is the norm topology. For every \( \varepsilon > 0 \), by Remark 4, there exists some \( j \) such that \( \sum_{i>j} \|y_i\|^p < \varepsilon^p \) and for all \( n \), \( \sum_{i>j} \|y_{n_i} - y_i\|^p < \varepsilon^p \). Then we can choose some \( n_0 \) such that, for \( n > n_0 \), \( \sum_{i>j} \|y_{n_i} - y_i\|^p < \varepsilon^p \). Hence for all \( n > n_0 \),

\[
\|y_{n} - y\| = \left( \sum_{i>j} \|y_{n_i} - y_i\|^p + \sum_{i\leq j} \|y_{n_i} - y_i\|^p \right)^{1/p} \\
\leq \left( \sum_{i>j} \|y_{n_i} - y_i\|^p \right)^{1/p} + \left( \sum_{i\leq j} \|y_{n} - y\|^p \right)^{1/p} \\
< \varepsilon.
\]

(10)

Therefore, by the arbitrariness of \( \varepsilon \), we have that \( \{y_{n}\} \) converges to \( y \).

Case 2. \( \tau \) is the weak topology. Suppose \( f = (f_i) \in \Phi_y X_i^* = X^* \) with \( \|f\| = 1 \), where \( 1/p + 1/q = 1 \) when \( p > 1 \) and \( q = \infty \) when \( p = 1 \). For every \( \varepsilon > 0 \), again by Remark 4, we can choose some \( j \) such that \( \sum_{i>j} \|y_i\|^q < \varepsilon^q \) and for all \( n \), \( \sum_{i>j} \|y_{n_i} - y_i\|^q < \varepsilon^q \). Note that, for every \( 1 \leq i \leq j, \{y_{n_i}\} \) weakly converges to \( y_i \); hence there exists some \( n_0 \) such that, for \( n > n_0 \), \( \sum_{i\leq j} f_i(y_{n_i} - y_i) < \varepsilon \). Then for all \( n > n_0 \),

\[
| f(y_n - y_i) | = \left| \sum_{i \in \mathbb{N}} f_i(y_{n_i} - y_i) \right| \\
\leq \left| \sum_{i \leq j} f_i(y_{n_i} - y_i) + \sum_{i>j} f_i(y_{n_i} - y_i) \right| \\
\leq \varepsilon + \sum_{i\leq j} \|y_{n_i} - y_i\|^q + \left( \sum_{i \leq j} \|y_{n} - y\|^q \right)^{1/p} \\
< \varepsilon.
\]

(11)

Again by the arbitrariness of \( \varepsilon \), we have that \( \{y_{n}\} \) weakly converges to \( y \).
(2) Necessity: fix $\lambda_0 \in \Lambda$. Suppose that $x_{\lambda_0} \in X_{\lambda_0}$, and 
$\{y_{n,\lambda_0}\} \subset B_{Y_{\lambda_0}}$ is a minimizing sequence of $x_{\lambda_0}$ in $B_{Y_{\lambda_0}}$. Let 
$x = (x_1), y_\lambda = (y_{n,\lambda}),$ where $x_1 = 0$ and $y_{n,\lambda} = 0$ for $\lambda \neq \lambda_0$. Then $x \in X, \{y_\lambda\} \subset B_Y$, and

$$
\|x - y_\lambda\| = \|x_{\lambda_0} - y_{n,\lambda_0}\| \rightarrow d\left(x_{\lambda_0}, B_{Y_{\lambda_0}}\right),
$$

(12)

Notice that $d(x, B_Y) \geq d(x, B_{Y_{\lambda_0}})$. Hence

$$
\|x - y_\lambda\| \rightarrow d\left(x_{\lambda_0}, B_{Y_{\lambda_0}}\right) = d\left(x, B_Y\right),
$$

(13)

which implies that $\{y_n\}$ is a minimizing sequence of $x$ in $B_Y$. By approximative $r$-compactness of $B_Y$ in $X$, $\{y_n\}$ has a $r$-convergent subsequence $\{y_{n,\lambda_0}\}$. Therefore $\{y_{n,\lambda_0}\}$ is $r$-convergent.

Sufficiency: suppose that $x \in X \setminus B_Y$ and $\{y_n\} \subset B_Y$ is a minimizing sequence of $x$ in $B_Y$. Like the proof in (1), we will prove that $\{y_n\}$ has a $r$-convergent subsequence and we can assume $A = N$, $x = (x_1)$, and $y_n = (y_{n,\lambda})$. By employing the diagonal process, we can choose a subsequence $\{y_{n,\lambda}\}$ of $\{y_n\}$ (we still denote the subsequence as $\{y_{n,\lambda}\}$ such that $
lim_{n \rightarrow \infty}\|y_{n,\lambda}\| = r$, and for every $i \in N, \lim_{n \rightarrow \infty}\|y_{n,i}\| = r_i$. Then by Lemma 5, we have $r = \left(\sum_{i \in N} r_i^2\right)^{1/p}$, and

$$
d(x, B_Y) = d\left(x, B_{Y_{\lambda_0}}\right) = \left(\sum_{i \in N} d(x_i, B_{Y_i})\right)^{1/p} \quad (14)
$$

Next, for every $i \in N$, we will show that $\{y_{n,i}\}$ has a $r$-convergent subsequence. We can assume that, for all $n$ and $i$, $\|y_{n,i}\| \leq r_i$. Otherwise, we can replace $\{y_n\}$ with $\{z_n\}$ which we define in the proof of Lemma 5(2).

Note that, for every $i \in N$, 

$$
\|x_i - y_{n,i}\|^p + \sum_{j \neq i} \left[d\left(x_j, r_i B_{Y_i}\right)\right]^p \leq \|x - y_n\|^p
$$

$$
= \left[d\left(x, B_{Y_i}\right)\right]^p + \|x - y_n\|^p - \left[d\left(x, B_{Y_i}\right)\right]^p
$$

$$
= \sum_{j \in N} \left[d\left(x_j, r_j B_{Y_j}\right)\right]^p + \|x - y_n\|^p - \left[d\left(x, B_{Y_i}\right)\right]^p.
$$

(15)

Then

$$
\left[d\left(x_j, r_j B_{Y_j}\right)\right]^p \leq \|x_i - y_{n,i}\|^p
$$

$$
\leq \left[d\left(x_i, r_i B_{Y_i}\right)\right]^p + \|x - y_n\|^p - \left[d\left(x, B_{Y_i}\right)\right]^p.
$$

(16)

Hence, when $n \rightarrow \infty$,

$$
\|x_i - y_{n,i}\| \rightarrow d\left(x_i, r_i B_{Y_i}\right).
$$

(17)

This implies that $\{y_{n,i}\}$ is a minimizing sequence of $x_i$ in $r_i B_{Y_i}$. By approximative $r$-compactness of $r_i B_{Y_i}$ in $X_i, \{y_{n,i}\}$ has a $r$-convergent subsequence.

Employing the diagonal process again, we can choose a subsequence $\{y_{n,l}\}$ of $\{y_{n,i}\}$ such that, for every $i, \{y_{n,j}\}$ has a $r$-convergent to some $y_i \in r_i B_{Y_i}$. Let $y = (y_l)$; then $y \in \Phi r_i B_{Y_i}$. We still denote $\{y_{n,l}\}$ as $\{y_n\}$. Finally, just like the proof in (1), we can prove that $\{y_n\}$ has a $r$-convergent to $y$.

Remark 7. The above theorem does not hold for $p = \infty$. Indeed, suppose that $Z$ is an infinite-dimensional proper closed subspace of $L_1$. By Theorem 1.4 in [12], $B_Z$ is approximatively compact in $L_2$. Next, we show that $Z \Phi r_i Z$ is not approximatively compact in $L_2 Z_i$. Choose $x_0 \in L_1 \setminus Z$ and $y_0 \in Z$ such that $d(x_0, Z) = 1 = \|x_0 - y_0\|$. Furthermore, we take a sequence $\{z_n\} \subset Z$ with $\|z_n\| = 1$ satisfying that $\{z_n\}$ has no convergent subsequence. Note that

$$
\|\{0, x_0\} - (z_n, y_0)\|_\infty = \max\{\|z_n\|, \|x_0 - y_0\|\} = 1; \quad (18)
$$

and for any $(z, y) \in Z \Phi r_i Z$,

$$
\|\{0, x_0\} - (z, y)\|_\infty = \max\{\|z\|, \|x_0 - y\|\} \geq \|x_0 - y\| \geq 1.
$$

(19)

This means that $d((0, x_0), Z \Phi r_i Z) = 1$ and $\{(z_n, y_0)\}$ is a minimizing sequence of $(0, x_0)$ in $Z \Phi r_i Z$. But $\{z_n, y_0\}$ has no convergent subsequence.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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