Research Article

Riesz Transform Characterization of Weighted Hardy Spaces Associated with Schrödinger Operators

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We characterize the weighted local Hardy spaces $h^1_\rho(\omega)$ related to the critical radius function ρ and weights $\omega \in A_1^{\rho,\infty}(\mathbb{R}^n)$ by localized Riesz transforms $\tilde{R}_j$; in addition, we give a characterization of weighted Hardy spaces $H^1_\rho(\omega)$ via Riesz transforms associated with Schrödinger operator $\mathcal{L}$, where $\mathcal{L} = -\Delta + V$ is a Schrödinger operator on $\mathbb{R}^n$ ($n \geq 3$) and V is a nonnegative function satisfying the reverse Hölder inequality.

1. Introduction

Let $\mathcal{L} = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$, where $V \not\equiv 0$ is a fixed nonnegative potential. We assume that V belongs to the reverse Hölder class $RH_s(\mathbb{R}^n)$ for some $s \geq n/2$; that is, there exists $C = C(s, V) > 0$ such that

$$\left(\frac{1}{|B|} \int_B V(x)^s dx \right)^{1/s} \leq C \left(\frac{1}{|B|} \int_B V(x) dx \right), \quad (1)$$

for every ball $B \subset \mathbb{R}^n$. With regard to the Schrödinger operator $\mathcal{L}$, we know that the operators derived from $\mathcal{L}$ behave “locally” quite similar to those corresponding to the Laplacian (see [1, 2]). The notion of locality is given by the critical radius function

$$\rho(x) = \frac{1}{m_V(x)} = \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq 1 \right\}. \quad (2)$$

Throughout the paper we assume that $V \not\equiv 0$, so that $0 < \rho(x) < \infty$ (see [2]).

The study of Schrödinger operator $\mathcal{L} = -\Delta + V$ recently attracted much attention; see [2–14]. In particular, Dziubański and Zienkiewicz [5, 6] studied Hardy space H^1_ρ associated with Schrödinger operators $\mathcal{L}$. In [8], we introduce the weighted Hardy space $H^1_\rho(\omega)$ with $\omega \in A_1(\mathbb{R}^n)$ (for Muckenhoupt’s weights, see [15–19]), and in [14], we

obtain some weighted estimates for bilinear operators on $H^1_\rho(\omega)$.

Recently, Bongioanni et al. [3] introduced new classes of weights, related to Schrödinger operators $\mathcal{L}$, that is, $A_p^{\rho,\infty}(\mathbb{R}^n)$ weight, which are in general larger than Muckenhoupt’s (see Section 2 for notions of $A_p^{\rho,\infty}(\mathbb{R}^n)$ weight). In [9, 10, 20], Tang gave some results about weighted norm inequalities with weights $\omega \in A_p^{\rho,\infty}(\mathbb{R}^n)$. In [21], we have obtained an atomic characterization for weighted Hardy space $H^1_\rho(\omega)$ with $\omega \in A_1^{\rho,\infty}(\mathbb{R}^n)$, by introducing the weighted local Hardy spaces $h^p_\rho(\omega)$ with $A_q^{\rho,\infty}(\mathbb{R}^n)$ weights and establishing the atomic characterization of the weighted local Hardy spaces $h^p_\rho(\omega)$ with $\omega \in A_q^{\rho,\infty}(\mathbb{R}^n)$ weights.

An interesting question is whether we can give a characterization of weighted Hardy spaces $H^1_\rho(\omega)$ with $\omega \in A_1^{\rho,\infty}(\mathbb{R}^n)$ via Riesz transforms associated with Schrödinger operator $\mathcal{L}$. Since it is difficult to give the characterization by the same method in [5, 8], in this paper, we will establish the characterization by a new method, which will take advantage of weighted local Hardy spaces $h^p_\rho(\omega)$ and localized Riesz transforms. In fact, by this method, we can give the Riesz transforms characterization of $h^p_\rho(\omega)$ at the same time.

The paper is organized as follows. In Section 2, we review some notions and notations concerning the weight classes $A_p^{\rho,\theta}(\mathbb{R}^n)$ introduced in [3, 9, 10], and we also review some

results about $h_p^1(\omega)$ and $H_p^1(\omega)$. In Section 3, we establish the Riesz transforms characterization of weighted Hardy spaces $H_{\mathcal{L}}^1(\omega)$.

Throughout this paper, we let C denote constants that are independent of the main parameters involved but whose value may differ from line to line. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $1/C \leq A/B \leq C$. The symbol $A \lesssim B$ means that $A \leq CB$. The symbol $[s]$ for $s \in \mathbb{R}$ denotes the maximal integer not more than s . We also set $\mathbb{N} \equiv \{1, 2, \dots\}$ and $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$.

2. Preliminaries

In this section, we first review some notions and notations concerning the weight classes $A_p^{\rho, \theta}(\mathbb{R}^n)$ introduced in [3, 9, 10]. Given $B = B(x, r)$ and $\lambda > 0$, we will write λB for the λ -dilate ball, which is the ball with the same center x and with radius λr . Similarly, $Q(x, r)$ denotes the cube centered at x with side length r (here and below only cubes with sides parallel to the axes are considered), and $\lambda Q(x, r) = Q(x, \lambda r)$. Particularly, we will denote $2B$ by B^* and $2Q$ by Q^* . The following lemma is a basic property of the critical radius function $\rho(x)$.

Lemma 1 (see [2]). *There exist $C_0 \geq 1$ and $k_0 \geq 1$ so that for all $x, y \in \mathbb{R}^n$*

$$\begin{aligned} C_0^{-1} \rho(x) \left(1 + \frac{|x - y|}{\rho(x)}\right)^{-k_0} &\leq \rho(y) \\ &\leq C_0 \rho(x) \left(1 + \frac{|x - y|}{\rho(x)}\right)^{k_0/(k_0+1)}. \end{aligned} \quad (3)$$

In particular, $\rho(x) \sim \rho(y)$ when $y \in B(x, r)$ and $r \leq C\rho(x)$, where C is a positive constant.

Now we recall the covering of $\mathbb{R}^n$ and the partition of unity related to ρ from [5]. For $m \in \mathbb{Z}$ we defined the sets $\mathcal{B}_m$ by $\mathcal{B}_m = \{x : 2^{-(m+1)/2} < \rho(x) \leq 2^{-m/2}\}$, where critical radii $\rho(x)$ have been defined in (3). Since $0 < \rho(x) < \infty$, we have $\mathbb{R}^n = \bigcup_{m \in \mathbb{Z}} \mathcal{B}_m$.

Lemma 2 (see [5]). *There is a positive constant C and a collection of balls $B_{(m,k)} = B(x_{(m,k)}, 2^{2-m/2})$, $m \in \mathbb{Z}$, $k = 1, 2, \dots$, such that $x_{(m,k)} \in \mathcal{B}_m$, $\mathcal{B}_m \subset \bigcup_k B(x_{(m,k)}, 2^{-m/2})$, and for every (m, k) and $R \geq 2$*

$$\begin{aligned} \#\left\{(m', k') : B(x_{(m,k)}, R2^{-m/2}) \cap B(x_{(m',k')}, R2^{-m'/2}) \right. \\ \left. \neq \emptyset\right\} \leq R^C; \end{aligned} \quad (4)$$

here and in what follows, for any set E , $\#E$ denotes its cardinality.

Lemma 3 (see [5]). *There are nonnegative functions $\psi_{(m,k)}$ such that*

$$(i) \text{ for all } m \in \mathbb{Z} \text{ and } k \in \mathbb{N}, \psi_{(m,k)} \in C_c^\infty(B(x_{(m,k)}), 2^{1-m/2});$$

- (ii) $\sum_{(m,k)} \psi_{(m,k)}(x) = 1$ for all $x \in \mathbb{R}^n$;
- (iii) *there exists a positive constant C s.t. for all $m \in \mathbb{Z}$, $k \in \mathbb{N}$, and $\|\nabla \psi_{(m,k)}\|_{L^\infty(\mathbb{R}^n)} \leq C2^m$.*

In this paper, we write $\Psi_\theta(B) = (1+r/\rho(x_0))^\theta$, where $\theta \geq 0$ and x_0 and r denote the center and radius of B , respectively. As in [3], we say that a weight ω belongs to the class $A_p^{\rho, \theta}(\mathbb{R}^n)$ for $1 < p < \infty$, if there is a constant C such that for all balls B

$$\begin{aligned} &\left(\frac{1}{\Psi_\theta(B)|B|} \int_B \omega(y) dy\right) \\ &\cdot \left(\frac{1}{\Psi_\theta(B)|B|} \int_B \omega^{-1/(p-1)}(y) dy\right)^{p-1} \leq C. \end{aligned} \quad (5)$$

We also say that a nonnegative function ω satisfies the $A_1^{\rho, \theta}(\mathbb{R}^n)$ condition if there exists a constant C such that $M_{V, \theta}(\omega)(x) \leq C\omega(x)$, a.e. $x \in \mathbb{R}^n$, where

$$M_{V, \theta} f(x) \equiv \sup_{x \in B} \frac{1}{\Psi_\theta(B)|B|} \int_B |f(y)| dy. \quad (6)$$

When $V = 0$, we denote $M_0 f(x)$ by $Mf(x)$ (the standard Hardy-Littlewood maximal function). It is easy to see that $|f(x)| \leq M_{V, \theta} f(x) \leq Mf(x)$ for a.e. $x \in \mathbb{R}^n$ and any $\theta \geq 0$. Clearly, the classes $A_p^{\rho, \theta}$ are increasing with θ , and we denote $A_p^{\rho, \infty} = \bigcup_{\theta \geq 0} A_p^{\rho, \theta}$ and $A_\infty^{\rho, \infty} = \bigcup_{p \geq 1} A_p^{\rho, \infty}$. In addition, for $1 \leq p \leq \infty$, denote by p' the adjoint number of p ; that is, $1/p + 1/p' = 1$.

We remark that balls can be replaced by cubes in definition of $A_p^{\rho, \theta}$ and $M_{V, \theta}$, since $\Psi(B) \leq \Psi(2B) \leq 2^\theta \Psi(B)$. In fact, for the cube $Q = Q(x_0, r)$, we can also define $\Psi_\theta(Q) = (1+r/\rho(x_0))^\theta$. Now, we give some properties of weights class $A_p^{\rho, \theta}$ for $p \geq 1$.

Lemma 4 (see [3, 10, 21]). *Let $\omega \in A_p^{\rho, \infty} = \bigcup_{\theta \geq 0} A_p^{\rho, \theta}$ for $p \geq 1$. Then*

- (i) *if $1 \leq p_1 < p_2 < \infty$, then $A_{p_1}^{\rho, \theta} \subset A_{p_2}^{\rho, \theta}$;*
- (ii) *if $\omega \in A_p^{\rho, \infty}$, $1 < p < \infty$, then there exists $\epsilon > 0$ such that $\omega \in A_{p-\epsilon}^{\rho, \infty}$;*
- (iii) *for $\omega \in A_p^{\rho, \theta}$, $Q = Q(x, r)$, and $\lambda > 1$, there exists a positive constant C such that $\omega(\lambda Q) \leq C(\Psi_\theta(\lambda Q))^p \lambda^{np} \omega(Q)$.*

For any $\omega \in A_\infty^{\rho, \infty}(\mathbb{R}^n)$, define the critical index of ω by

$$q_\omega \equiv \inf \{p \in [1, \infty) : \omega \in A_p^{\rho, \infty}(\mathbb{R}^n)\}. \quad (7)$$

For any $\varphi \in \mathcal{D}(\mathbb{R}^n)$, let $\varphi_t(x) = t^{-n} \varphi(x/t)$ for $t > 0$ and $\varphi_j(x) = 2^{jn} \varphi(2^j x)$ for $j \in \mathbb{Z}$. Now let us introduce the space $H_{\mathcal{L}}^1(\omega)$. Let $\{T_t\}_{t>0}$ be the semigroup of linear operators generated by $\mathcal{L}$ and let $T_t(x, y)$ be their kernels; that is, $T_t f(x) = e^{-t\mathcal{L}} f(x) = \int_{\mathbb{R}^n} T_t(x, y) f(y) dy$, for $t > 0$ and $f \in L^2(\mathbb{R}^n)$. A weighted Hardy-type space related to $\mathcal{L}$ with

$A_1^{\rho,\theta}(\mathbb{R}^n)$ weights is naturally defined by $H_{\mathcal{L}}^1(\omega) \equiv \{f \in L^1_\omega(\mathbb{R}^n) : \mathcal{T}^* f(x) \in L^1_\omega(\mathbb{R}^n)\}$, with $\|f\|_{H_{\mathcal{L}}^1(\omega)} \equiv \|\mathcal{T}^* f\|_{L^1_\omega(\mathbb{R}^n)}$. $H_{\mathcal{L}}^1(\omega)$ with $\omega \in A_1(\mathbb{R}^n)$ has been studied in [8, 14], and we have the following theorem.

Theorem A (see [8]). *If $\omega \in A_1(\mathbb{R}^n)$ and $V \in RH_n(\mathbb{R}^n)$ is a nonnegative potential, then, for all $f \in H_{\mathcal{L}}^1(\omega)$, one has $\|f\|_{H_{\mathcal{L}}^1(\omega)} \sim \|f\|_{L^1_\omega(\mathbb{R}^n)} + \sum_{j=1}^n \|R_j f\|_{L^1_\omega(\mathbb{R}^n)}$.*

Then let us recall some results about weighted local Hardy spaces $h_{\rho}^p(\omega)$. We first introduce some local maximal functions. For $N \in \mathbb{Z}_+$ and $R \in (0, \infty)$, let

$$\begin{aligned} \mathcal{D}_{N,R}(\mathbb{R}^n) &\equiv \left\{ \varphi \in \mathcal{D}(\mathbb{R}^n) : \text{supp}(\varphi) \right. \\ &\quad \left. \subset B(0, R), \|\varphi\|_{\mathcal{D}_{N,R}(\mathbb{R}^n)} \equiv \sup_{x \in \mathbb{R}^n, \alpha \in \mathbb{Z}_+^n, |\alpha| \leq N} |\partial^\alpha \varphi(x)| \right. \\ &\quad \left. \leq 1 \right\}. \end{aligned} \quad (8)$$

For any $f \in \mathcal{D}'(\mathbb{R}^n)$, the local nontangential grand maximal function $\widetilde{\mathcal{M}}_{N,R}(f)$ of f is defined as

$$\begin{aligned} \widetilde{\mathcal{M}}_{N,R}(f)(x) &\equiv \sup \left\{ |\varphi_l * f(z)| : |x - z| < 2^{-l} \right. \\ &\quad \left. < \rho(x), \varphi \in \mathcal{D}_{N,R}(\mathbb{R}^n) \right\}, \end{aligned} \quad (9)$$

and the local vertical grand maximal function $\mathcal{M}_{N,R}(f)$ of f is defined as

$$\begin{aligned} \mathcal{M}_{N,R}(f)(x) &\equiv \sup \left\{ |\varphi_l * f(x)| : 0 < 2^{-l} < \rho(x), \varphi \right. \\ &\quad \left. \in \mathcal{D}_{N,R}(\mathbb{R}^n) \right\}. \end{aligned} \quad (10)$$

For the sake of convenience, we denote $\mathcal{D}_{N,R}(\mathbb{R}^n)$, $\widetilde{\mathcal{M}}_{N,R}(f)$, and $\mathcal{M}_{N,R}(f)$ as $\mathcal{D}_N^0(\mathbb{R}^n)$, $\widetilde{\mathcal{M}}_N^0(f)$, and $\mathcal{M}_N^0(f)$, when $R = 1$, and as $\mathcal{D}_N(\mathbb{R}^n)$, $\widetilde{\mathcal{M}}_N(f)$, and $\mathcal{M}_N(f)$, when $R = \max\{R_1, R_2, R_3\} > 1$ (where R_1, R_2 , and R_3 are defined as in Lemmas 4.2, 4.4, and 4.8 in [21]). For any $N \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$, obviously, $\mathcal{M}_N^0(f)(x) \leq \mathcal{M}_N(f)(x) \leq \widetilde{\mathcal{M}}_N(f)(x)$.

Definition 5. Let $\omega \in A_{\infty}^{\rho,\infty}(\mathbb{R}^n)$ and let q_ω be as in (7), $p \in (0, 1]$, and $\widetilde{N}_{p,\omega} \equiv [n(q_\omega/p - 1)] + 2$. For each $N \in \mathbb{N}$ with $N \geq \widetilde{N}_{p,\omega}$, the weighted local Hardy space is defined by $h_{\rho,N}^p(\omega) \equiv \{f \in \mathcal{D}'(\mathbb{R}^n) : \mathcal{M}_N(f) \in L_\omega^p(\mathbb{R}^n)\}$. Moreover, let $\|f\|_{h_{\rho,N}^p(\omega)} \equiv \|\mathcal{M}_N(f)\|_{L_\omega^p(\mathbb{R}^n)}$.

Here and in what follows, we define

$$N_{p,\omega} \equiv \max\{\widetilde{N}_{p,\omega}, N_0\}, \quad (11)$$

where $\widetilde{N}_{p,\omega}$ and N_0 are, respectively, as in Definition 2.2 and Theorem 3.1 of [21].

Theorem B (see [21]). *Let $\omega \in A_{\infty}^{\rho,\infty}(\mathbb{R}^n)$ and let $N_{p,\omega}$ be as in (11). Then for any integer $N \geq N_{p,\omega}$ and $f \in h_{\rho,N}^p(\omega)$, one has*

$$\begin{aligned} \|f\|_{h_{\rho,N}^p(\omega)} &\sim \|\widetilde{\mathcal{M}}_N(f)\|_{L_\omega^p(\mathbb{R}^n)} \sim \|\widetilde{\mathcal{M}}_N^0(f)\|_{L_\omega^p(\mathbb{R}^n)} \\ &\sim \|\mathcal{M}_N^0(f)\|_{L_\omega^p(\mathbb{R}^n)}, \end{aligned} \quad (12)$$

where the implicit constants are independent of f .

Next, we introduce the weighted local atoms and weighted atomic local Hardy space.

Definition 6. Let $\omega \in A_{\infty}^{\rho,\infty}(\mathbb{R}^n)$ and let q_ω be as in (7). A triplet $(p, q, s)_\omega$ is called admissible, if $p \in (0, 1]$, $q \in (q_\omega, \infty]$, and $s \in \mathbb{N}$ with $s \geq [n(q_\omega/p - 1)]$. A function a on $\mathbb{R}^n$ is said to be a $(p, q, s)_\omega$ -atom if

- (i) $\text{supp } a \subset Q(x, r)$ and $r \leq L_1 \rho(x)$,
- (ii) $\|a\|_{L_\omega^q(\mathbb{R}^n)} \leq [\omega(Q)]^{1/q-1/p}$,
- (iii) $\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0$ for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq s$, when $Q = Q(x, r)$, $r < L_2 \rho(x)$,

where $L_1 = 4C_0(3\sqrt{n})^{k_0}$, $L_2 = 1/C_0^2(3\sqrt{n})^{k_0+1}$, and C_0 and k_0 are constants given in Lemma 1. Moreover, a function $a(x)$ on $\mathbb{R}^n$ is called a $(p, q)_\omega$ -single-atom with $q \in (q_\omega, \infty]$, if $\|a\|_{L_\omega^q(\mathbb{R}^n)} \leq [\omega(\mathbb{R}^n)]^{1/q-1/p}$.

Lemma 7 (see [21]). *Let $\omega \in A_{\infty}^{\rho,\theta}(\mathbb{R}^n)$ and a be a $(p, q, s)_\omega$ -atom, which satisfies $\text{supp } a \subset Q(x_0, r)$; then there exists a constant C such that $\|a\|_{L^1(\mathbb{R}^n)} \leq C|Q|\omega(Q)^{-1/p}\Psi_\theta(Q)$.*

Definition 8. Let $\omega \in A_{\infty}^{\rho,\infty}(\mathbb{R}^n)$ and let q_ω be as in (7), and let $(p, q, s)_\omega$ be admissible. The weighted atomic local Hardy space $h_{\rho}^{p,q,s}(\omega)$ is defined as the set of all $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfying that $f = \sum_{i=0}^{\infty} \lambda_i a_i$ in $\mathcal{D}'(\mathbb{R}^n)$, where $\{a_i\}_{i \in \mathbb{N}}$ are $(p, q, s)_\omega$ -atoms, a_0 is a $(p, q)_\omega$ -single-atom, and $\{\lambda_i\}_{i \in \mathbb{Z}_+} \subset \mathbb{C}$. Moreover, the quasi-norm of $f \in h_{\rho}^{p,q,s}(\omega)$ is defined by $\|f\|_{h_{\rho}^{p,q,s}(\omega)} \equiv \inf\{[\sum_{i=0}^{\infty} |\lambda_i|^p]^{1/p}\}$, where the infimum is taken over all the decompositions of f as above.

Theorem C (see [21]). *Let $\omega \in A_{\infty}^{\rho,\infty}(\mathbb{R}^n)$, and let q_ω and $N_{p,\omega}$ be, respectively, as in (7) and (11). If $q \in (q_\omega, \infty]$, $p \in (0, 1]$ and integers s and N satisfy $N \geq N_{p,\omega}$ and $N > s \geq [n(q_\omega/p - 1)]$, then $h_{\rho}^{p,q,s}(\omega) = h_{\rho,N}^p(\omega) = h_{\rho,N_{p,\omega}}^p(\omega)$ with equivalent norms.*

In virtue of the above theorem, for notational simplicity, we denote by $h_{\rho}^p(\omega)$ the weighted local Hardy space $h_{\rho,N}^p(\omega)$ when $N \geq N_{p,\omega}$.

Definition 9. Let $\omega \in A_{\infty}^{\rho,\infty}(\mathbb{R}^n)$ and let $(p, q, s)_\omega$ be admissible; then $h_{\rho,\text{fin}}^{p,q,s}(\omega)$ is defined as the vector space of all finite linear combinations of $(p, q, s)_\omega$ -atoms and a $(p, q)_\omega$ -single-atom, and the norm of f in $h_{\rho,\text{fin}}^{p,q,s}(\omega)$ is defined by $\|f\|_{h_{\rho,\text{fin}}^{p,q,s}(\omega)} \equiv \inf\{[\sum_{i=0}^k |\lambda_i|^p]^{1/p}\}$, where $f = \sum_{i=0}^k \lambda_i a_i$, $k \in \mathbb{Z}_+$, $\{\lambda_i\}_{i=0}^k \subset \mathbb{C}$, $\{a_i\}_{i=1}^k$ are $(p, q, s)_\omega$ -atoms, and a_0 is a $(p, q)_\omega$ -single-atom.

Let $\beta \in (0, 1]$. A quasi-Banach space $\mathcal{B}_\beta$ with the quasi-norm $\|\cdot\|_{\mathcal{B}_\beta}$ is called a β -quasi-Banach space if $\|f + g\|_{\mathcal{B}_\beta}^\beta \leq \|f\|_{\mathcal{B}_\beta}^\beta + \|g\|_{\mathcal{B}_\beta}^\beta$ for all $f, g \in \mathcal{B}_\beta$. For any given β -quasi-Banach space $\mathcal{B}_\beta$ with $\beta \in (0, 1]$ and a linear space $\mathcal{Y}$, an operator T from $\mathcal{Y}$ to $\mathcal{B}_\beta$ is said to be $\mathcal{B}_\beta$ -sublinear if, for any $f, g \in \mathcal{B}_\beta$ and $\lambda, \nu \in \mathbb{C}$, $\|T(\lambda f + \nu g)\|_{\mathcal{B}_\beta} \leq (|\lambda|^\beta \|T(f)\|_{\mathcal{B}_\beta}^\beta + |\nu|^\beta \|T(g)\|_{\mathcal{B}_\beta}^\beta)^{1/\beta}$ and $\|T(f) - T(g)\|_{\mathcal{B}_\beta} \leq \|T(f - g)\|_{\mathcal{B}_\beta}$.

Theorem D (see [21]). Let $\omega \in A_\infty^{p,\infty}(\mathbb{R}^n)$, $0 < p \leq \beta \leq 1$, and $\mathcal{B}_\beta$ be a β -quasi-Banach space. Suppose $q \in (q_\omega, \infty)$ and $T : h_{p,fin}^{p,q,s}(\omega) \rightarrow \mathcal{B}_\beta$ is a $\mathcal{B}_\beta$ -sublinear operator such that $S \equiv \sup\{\|T(a)\|_{\mathcal{B}_\beta} : a \text{ is a } (p, q, s)_\omega\text{-atom or } (p, q)_\omega \text{ single-atom}\} < \infty$. Then there exists a unique bounded $\mathcal{B}_\beta$ -sublinear operator $\tilde{T}$ from $h_p^p(\omega)$ to $\mathcal{B}_\beta$ which extends T .

Theorem E (see [21]). Let $0 \neq V \in RH_{n/2}$ and $\omega \in A_1^{p,\infty}(\mathbb{R}^n)$; then $h_p^1(\omega) = H_{\mathcal{L}}^1(\omega)$ with equivalent norms; that is, $\|f\|_{h_p^1(\omega)} \sim \|f\|_{H_{\mathcal{L}}^1(\omega)}$.

3. Riesz Transform Characterization of Weighted Hardy Spaces

The main purpose of this section is to give a characterization of weighted Hardy spaces $H_{\mathcal{L}}^1(\omega)$ with $\omega \in A_1^{p,\infty}(\mathbb{R}^n)$ via Riesz transforms associated with Schrödinger operator $\mathcal{L}$. We begin with some useful lemmas.

Lemma 10 (see [8]). If $\omega \in A_1(\mathbb{R}^n)$, then there is a constant $C > 0$ such that

$$\sum_{(m,k)} \|R_j(\psi_{(m,k)} f) - \psi_{(m,k)} R_j(f)\|_{L^1(\omega)} \leq C \|f\|_{L^1(\omega)}. \quad (13)$$

Lemma 11 (see [5]). There exists a sequence of points $x_j \in \mathbb{R}^n$, $j \geq 1$, such that the family $B_j = B(x_j, \rho(x_j))$, $j \geq 1$, satisfies the following:

- (a) $\bigcup_j B_j = \mathbb{R}^n$.
- (b) For every $\sigma \geq 1$ there exist constants C and N_1 such that $\sum_j \chi_{\sigma B_j} \leq C \sigma^{N_1}$.

For $j = 1, 2, \dots, n$, let us define the Riesz transforms R_j associated with Schrödinger operator $\mathcal{L}$ by $R_j = (\partial/\partial x_j) \mathcal{L}^{-1/2}$. In addition, as in [22], for all $j \in \{1, 2, \dots, n\}$ and $x \in \mathbb{R}^n$, we define localized Riesz transforms as

$$\begin{aligned} R_j^l(f)(x) \\ \equiv \text{p.v. } c_n \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} \eta\left(\frac{|x - y|}{\rho(x)}\right) f(y) dy, \end{aligned} \quad (14)$$

where and in what follows $c_n \equiv \Gamma((n+1)/2)/[\pi^{(n+1)/2}]$, $\eta \in C^1(\mathbb{R}^n)$ supported in $(-1, 1)$, and $\eta(t) = 1$ if $|t| \leq 1/2$. For all $j \in \{1, 2, \dots, n\}$ and $x \in \mathbb{R}^n$, we define

$$R_{j,*}(f)(x) \equiv c_n \sup_{\varepsilon > 0} \left| \int_{|x-y| \geq \varepsilon} \frac{x_j - y_j}{|x - y|^{n+1}} f(y) dy \right|. \quad (15)$$

With these operators, we can get the following boundedness.

Lemma 12. Let R_j^l be as in (14); then

- (i) $\|R_j^l(f)\|_{L_\omega^p(\mathbb{R}^n)} \leq C_{p,\omega} \|f\|_{L_\omega^p(\mathbb{R}^n)}$, for $1 < p < \infty$ and $\omega \in A_p(\mathbb{R}^n)$,
- (ii) $\|R_j^l(f)\|_{L_\omega^{1,\infty}(\mathbb{R}^n)} \leq C_\omega \|f\|_{L_\omega^{1,\infty}(\mathbb{R}^n)}$, for $\omega \in A_1(\mathbb{R}^n)$.

Proof. Let $j \in \{1, 2, \dots, n\}$; then by [15, 18], we know that $R_{j,*}$ is bounded on $L_\omega^p(\mathbb{R}^n)$ for $1 < p < \infty$ and bounded from $L_\omega^1(\mathbb{R}^n)$ to $L_\omega^{1,\infty}(\mathbb{R}^n)$. Let $\{\tilde{R}_j\}_{j=1}^n$ be the classical Riesz transforms; then for all $x \in \mathbb{R}^n$, we have

$$\begin{aligned} & |R_j^l(f)(x) - \tilde{R}_j(f)(x)| \\ & \leq \left| \int_{(1/2)\rho(x) \leq |x-y| < \rho(x)} \frac{x_j - y_j}{|x - y|^{n+1}} \left[1 - \eta\left(\frac{|x - y|}{\rho(x)}\right) \right] f(y) dy \right| \\ & + \left| \int_{|x-y| \geq \rho(x)} \frac{x_j - y_j}{|x - y|^{n+1}} f(y) dy \right| \leq M(f)(x) \\ & + R_{j,*}(f)(x), \end{aligned} \quad (16)$$

where and in what follows M denotes the Hardy-Littlewood maximal operator. By the above estimate, combining with the boundedness on $L_\omega^p(\mathbb{R}^n)$ for $1 < p < \infty$ and boundedness from $L_\omega^1(\mathbb{R}^n)$ to $L_\omega^{1,\infty}(\mathbb{R}^n)$ of M , $\tilde{R}_j$, and $R_{j,*}$, we can obtain the conclusions of the lemma. $\square$

Then for $\omega \in A_\infty^{p,\infty}(\mathbb{R}^n)$, we can obtain the following lemma.

Lemma 13. Let R_j^l be as in (14); then

- (i) $\|R_j^l(f)\|_{L_\omega^p(\mathbb{R}^n)} \leq C_{p,\omega} \|f\|_{L_\omega^p(\mathbb{R}^n)}$, for $1 < p < \infty$ and $\omega \in A_p^{p,\infty}(\mathbb{R}^n)$,
- (ii) $\|R_j^l(f)\|_{L_\omega^{1,\infty}(\mathbb{R}^n)} \leq C_\omega \|f\|_{L_\omega^{1,\infty}(\mathbb{R}^n)}$, for $\omega \in A_1^{p,\infty}(\mathbb{R}^n)$.

Proof. By Lemma 11, there exists a sequence of points $x_j \in \mathbb{R}^n$, $j \geq 1$, such that $\bigcup_j B_j = \bigcup_j B(x_j, \rho(x_j)) = \mathbb{R}^n$. Let $\sigma_1 = C_0 2^{k_0/(k_0+1)} + 1$, with C_0, k_0 as in Lemma 1, and set $\tilde{B}_j = \sigma_1 B_j$. Then by Lemma 1 of [3], for $\omega \in A_p^{p,\infty}(\mathbb{R}^n)$ and any B_j , there

exists $\bar{\omega} \in A_p(\mathbb{R}^n)$ such that $\bar{\omega} = \omega$ on $\tilde{B}_j$. Thus, by Lemma 12, we have

$$\begin{aligned} \|R_j^l(f)\|_{L_\omega^p(B_j)} &= \|R_j^l(\chi_{\tilde{B}_j} f)\|_{L_\omega^p(B_j)} \\ &\leq \|R_j^l(\chi_{\tilde{B}_j} f)\|_{L_\omega^p(B_j)} \leq \|\chi_{\tilde{B}_j} f\|_{L_\omega^p(\mathbb{R}^n)} \\ &\leq \|f\|_{L_\omega^p(\tilde{B}_j)}. \end{aligned} \quad (17)$$

Summing up all balls of $\{B_j\}_j$ gives (i).

For (ii), as in (i), for $\omega \in A_1^{\rho, \infty}(\mathbb{R}^n)$ and any B_j , there exists $\bar{\omega} \in A_1(\mathbb{R}^n)$ such that $\bar{\omega} = \omega$ on $\tilde{B}_j$. Then, by Lemma 12, we have

$$\begin{aligned} \omega(\{x \in B_j : |R_j^l(f)(x)| > \lambda\}) &\leq \omega(\{x \in B_j : |R_j^l(\chi_{\tilde{B}_j} f)(x)| > \lambda\}) \\ &= \bar{\omega}(\{x \in B_j : |R_j^l(\chi_{\tilde{B}_j} f)(x)| > \lambda\}) \\ &\leq \lambda^{-1} \|\chi_{\tilde{B}_j} f\|_{L_{\bar{\omega}}^1(\mathbb{R}^n)} \leq \lambda^{-1} \|f\|_{L_{\bar{\omega}}^1(\tilde{B}_j)}. \end{aligned} \quad (18)$$

Summing up all balls of $\{B_j\}_j$ gives (ii).

The proof is complete. $\square$

Now we can establish the Riesz transform characterization as follows.

Theorem 14. Let $\omega \in A_1^{\rho, \infty}(\mathbb{R}^n)$ and $V \in RH_n(\mathbb{R}^n)$; then one has

$$\begin{aligned} \|f\|_{H_{\mathcal{L}}^1(\omega)} &\sim \|f\|_{h_\rho^1(\omega)} \sim \|f\|_{L_\omega^1(\mathbb{R}^n)} + \sum_{j=1}^n \|R_j^l(f)\|_{L_\omega^1(\mathbb{R}^n)} \\ &\sim \|f\|_{L_\omega^1(\mathbb{R}^n)} + \sum_{j=1}^n \|R_j(f)\|_{L_\omega^1(\mathbb{R}^n)}. \end{aligned} \quad (19)$$

Proof. $\|f\|_{H_{\mathcal{L}}^1(\omega)} \sim \|f\|_{h_\rho^1(\omega)}$ is obvious by Theorem E. We now prove

$$\|f\|_{h_\rho^1(\omega)} \sim \|f\|_{L_\omega^1(\mathbb{R}^n)} + \sum_{j=1}^n \|R_j^l(f)\|_{L_\omega^1(\mathbb{R}^n)}. \quad (20)$$

We first assume that

$$\|f\|_{L_\omega^1(\mathbb{R}^n)} + \sum_{j=1}^n \|R_j^l f\|_{L_\omega^1(\mathbb{R}^n)} < \infty, \quad (21)$$

and here we will borrow some idea from [23]. Let $B_{(m,k)}$ be a ball as in Lemma 2, and let $\zeta_{a_1 B_{(m,k)}}$ be a C_0^∞ nonnegative function supported in $2a_1 B_{(m,k)}$ and $\zeta_{a_1 B_{(m,k)}} = 1$ on $a_1 B_{(m,k)}$, where $a_1 = 3C_0 + 1$. By properties of weights, we can set $\bar{\omega} \in A_1(\mathbb{R}^n)$ so that $\bar{\omega} = \omega$ on $a_5 B_{(m,k)}$, where a_5 is a constant

independent of $B_{(m,k)}$ and will be given in the following proof. Taking $\varphi \in \mathcal{D}_N^0(\mathbb{R}^n)$, by Theorem A, we have

$$\begin{aligned} &\left\| \sup_{0 < 2^{-l} < \rho(x)} |\varphi_l * f| \right\|_{L_\omega^1(B_{(m,k)})} \\ &\leq \left\| \sup_{0 < 2^{-l} < \rho(x)} |\varphi_l * (f \zeta_{a_1 B_{(m,k)}})| \right\|_{L_\omega^1(\mathbb{R}^n)} \\ &\leq \|f \zeta_{a_1 B_{(m,k)}}\|_{H_{\mathcal{L}}^1(\bar{\omega})} \\ &\leq \|f \zeta_{a_1 B_{(m,k)}}\|_{L_{\bar{\omega}}^1(\mathbb{R}^n)} + \sum_{j=1}^n \|R_j(f \zeta_{a_1 B_{(m,k)}})\|_{L_{\bar{\omega}}^1(\mathbb{R}^n)}. \end{aligned} \quad (22)$$

Let us denote the integral kernel of the operator $(\partial/\partial x_j)\mathcal{L}^{-1/2}$ by $R_j(x, y)$, and let $\tilde{R}_j(x, y)$ be the kernel of the classical Riesz transform $\tilde{R}_j f = (\partial/\partial x_j)\Delta^{-1/2} f$. Then we can define

$$\bar{R}_j(f)(x) \equiv \int_{\mathbb{R}^n} R_j(x, y) \eta\left(\frac{|x-y|}{\rho(x)}\right) f(y) dy, \quad (23)$$

and we also have

$$R_j^l(f)(x) = \int_{\mathbb{R}^n} \tilde{R}_j(x, y) \eta\left(\frac{|x-y|}{\rho(x)}\right) f(y) dy. \quad (24)$$

Let $a_2 = C_0(1 + 16a_1) + 2a_1 < a_5$; then we have

$$\begin{aligned} &\|R_j(f \zeta_{a_1 B_{(m,k)}}) - \zeta_{a_1 B_{(m,k)}} R_j^l(f)\|_{L_\omega^1(\mathbb{R}^n)} \\ &\leq \|R_j(f \zeta_{a_1 B_{(m,k)}} \chi_{a_2 B_{(m,k)}})\|_{L_\omega^1(\mathbb{R}^n)} \\ &\quad - \zeta_{a_1 B_{(m,k)}} R_j(f \chi_{a_2 B_{(m,k)}})\|_{L_\omega^1(\mathbb{R}^n)} \\ &\quad + \|\zeta_{a_1 B_{(m,k)}} R_j(f \chi_{a_2 B_{(m,k)}})\|_{L_\omega^1(\mathbb{R}^n)} \\ &\quad - \zeta_{a_1 B_{(m,k)}} \bar{R}_j(f \chi_{a_2 B_{(m,k)}})\|_{L_\omega^1(\mathbb{R}^n)} \\ &\quad + \|\zeta_{a_1 B_{(m,k)}} \bar{R}_j(f \chi_{a_2 B_{(m,k)}})\|_{L_\omega^1(\mathbb{R}^n)} \\ &\quad - \zeta_{a_1 B_{(m,k)}} R_j^l(f \chi_{a_2 B_{(m,k)}})\|_{L_\omega^1(\mathbb{R}^n)} \equiv I_1 + I_2 + I_3. \end{aligned} \quad (25)$$

For I_1 , by the same method of the proof of Lemma 10 (Lemma 3.14 of [8]), we get

$$I_1 \leq \|f\|_{L^1(a_2 B_{(m,k)}, \bar{\omega})} \leq \|f\|_{L_\omega^1(a_2 B_{(m,k)})}. \quad (26)$$

For I_2 , by Lemma 3 of [24], we have

$$\begin{aligned}
 I_2 &\leq \left\| \zeta_{a_1 B(m,k)} R_j (f \chi_{a_2 B(m,k)}) - \zeta_{a_1 B(m,k)} \bar{R}_j (f \chi_{a_2 B(m,k)}) \right\|_{L^1_{\bar{\omega}}(\mathbb{R}^n)} \\
 &= \left\| R_j (f \chi_{a_2 B(m,k)}) - \bar{R}_j (f \chi_{a_2 B(m,k)}) \right\|_{L^1(2a_1 B(m,k), \bar{\omega})} \\
 &\leq \int_{2a_1 B(m,k)} \int_{|x-y| > \rho(x)/2} |R_j(x, y)| \left| 1 - \eta \left(\frac{|x-y|}{\rho(x)} \right) \right| \\
 &\quad \cdot |f \chi_{a_2 B(m,k)}(y)| dy \bar{\omega}(x) dx \\
 &\leq \int_{2a_1 B(m,k)} \int_{|x-y| > \rho(x)/2} \frac{|f \chi_{a_2 B(m,k)}(y)|}{|x-y|^n} dy \bar{\omega}(x) dx \\
 &\leq \int_{2a_1 B(m,k)} \int_{|x-y| > \rho(x)/2} \frac{|f \chi_{a_2 B(m,k)}(y)|}{(\rho(x))^n} dy \bar{\omega}(x) dx.
 \end{aligned} \tag{27}$$

Since $x \in 2a_1 B(m,k)$ and $y \in a_2 B(m,k)$, we have $\rho(x) \sim \rho(x_{(m,k)}) \sim \rho(y)$, and there is a constant $a_3 > 1$ independent of $B(m,k)$ such that

$$\begin{aligned}
 I_2 &\leq \int_{a_2 B(m,k)} \left(\int_{|x-y| < a_3 \rho(y)} \frac{\bar{\omega}(x)}{(\rho(y))^n} dx \right) \\
 &\quad \cdot |f \chi_{a_2 B(m,k)}(y)| dy \leq \int_{a_2 B(m,k)} |f(y)| \bar{\omega}(y) dy \\
 &\leq \|f\|_{L^1_{\bar{\omega}}(a_2 B(m,k))}.
 \end{aligned} \tag{28}$$

We now estimate I_3 . For $x \in 2a_1 B(m,k)$ and $y \in a_2 B(m,k)$, we have $\rho(x) \sim \rho(x_{(m,k)}) \sim \rho(y)$, and there exists a constant $a_4 > 1$ independent of $B(m,k)$ such that $\rho(x) \leq a_4 \rho(y)$. On the other hand, if $V \in RH_n(\mathbb{R}^n)$, then there exists $\varepsilon > 0$, which depends only on n and the constant C in (3), such that $V \in RH_{n+\varepsilon}(\mathbb{R}^n)$ (see [25]). Thus, according to Lemma 3 of [24], we get

$$\begin{aligned}
 I_3 &\leq \int_{2a_1 B(m,k)} \int_{\mathbb{R}^n} |R_j(x, y) - \bar{R}_j(x, y)| \left| \eta \left(\frac{|x-y|}{\rho(x)} \right) \right| |f \chi_{a_2 B(m,k)}(y)| dy \bar{\omega}(x) dx \\
 &\leq \int_{2a_1 B(m,k)} \int_{|x-y| < \rho(x)} \frac{|f \chi_{a_2 B(m,k)}(y)|}{|x-y|^n} \left(\frac{|x-y|}{\rho(x)} \right)^{2-n/(n+\varepsilon)} dy \bar{\omega}(x) dx \\
 &\leq \int_{a_2 B(m,k)} \left(\int_{|x-y| < a_4 \rho(y)} \frac{1}{|x-y|^n} \left(\frac{|x-y|}{\rho(y)} \right)^{2-n/(n+\varepsilon)} \bar{\omega}(x) dx \right) |f(y)| dy.
 \end{aligned} \tag{29}$$

For the inner integral, there exists a constant a_5 independent of $B(m,k)$ such that $a_5 > a_2$ and $B(y, \max\{a_3, 2a_4\}\rho(y)) \subset a_5 B(m,k)$ for all $y \in a_2 B(m,k)$; then by the properties of A_1 weights, we have

$$\begin{aligned}
 &\int_{|x-y| < a_4 \rho(y)} \frac{1}{|x-y|^n} \left(\frac{|x-y|}{\rho(y)} \right)^{2-n/(n+\varepsilon)} \bar{\omega}(x) dx \\
 &\leq \sum_{k=k_0}^{\infty} \int_{|x-y| \sim 2^{-k} \rho(y)} \frac{1}{|x-y|^n} \left(\frac{|x-y|}{\rho(y)} \right)^{2-n/(n+\varepsilon)} \\
 &\quad \cdot \bar{\omega}(x) dx \\
 &\leq \sum_{k=k_0}^{\infty} \int_{|x-y| \sim 2^{-k} \rho(y)} \frac{(2^{-k})^{2-n/(n+\varepsilon)}}{|2^{-k} \rho(y)|^n} \bar{\omega}(x) dx \\
 &\leq \sum_{k=k_0}^{\infty} (2^{-k})^{2-n/(n+\varepsilon)} \int_{|x-y| \leq 2^{-k} \rho(y)} \frac{\bar{\omega}(x)}{|2^{-k} \rho(y)|^n} dx \\
 &\leq \bar{\omega}(y),
 \end{aligned} \tag{30}$$

where k_0 satisfies $2^{-k_0-1} < a_5 \leq 2^{-k_0}$. Hence, we obtain

$$I_3 \leq \int_{a_2 B(m,k)} |f(y)| \bar{\omega}(y) dy \leq \|f\|_{L^1_{\bar{\omega}}(a_2 B(m,k))}. \tag{31}$$

Combining (22), (26), (28), and (31), we get

$$\left\| \sup_{0 < 2^{-l} < \rho(x)} |\varphi_l * f| \right\|_{L^1_{\bar{\omega}}(B(m,k))} \tag{32}$$

$$\leq \|f\|_{L^1_{\bar{\omega}}(a_2 B(m,k))} + \sum_{j=1}^n \|R_j^l(f)\|_{L^1_{\bar{\omega}}(2a_1 B(m,k))}.$$

Therefore, by Theorem B and Lemma 2, we obtain

$$\begin{aligned}
 \|f\|_{h^1_{\rho}(\omega)} &\leq \sum_{(m,k)} \left\| \sup_{0 < 2^{-l} < \rho(x)} |\varphi_l * f| \right\|_{L^1_{\bar{\omega}}(B(m,k))} \\
 &\leq \sum_{(m,k)} \left(\|f\|_{L^1_{\bar{\omega}}(a_2 B(m,k))} + \sum_{j=1}^n \|R_j^l(f)\|_{L^1_{\bar{\omega}}(2a_1 B(m,k))} \right) \\
 &\leq \|f\|_{L^1_{\bar{\omega}}(\mathbb{R}^n)} + \sum_{j=1}^n \|R_j^l(f)\|_{L^1_{\bar{\omega}}(\mathbb{R}^n)}.
 \end{aligned} \tag{33}$$

In order to prove the converse inequality, by Theorem D, it suffices to show that, for all j and any $(1, 2, s)_{\omega}$ -atom or $(1, 2)_{\omega}$ -single-atom a ,

$$\|R_j^l(a)\|_{L^1_{\bar{\omega}}(\mathbb{R}^n)} \leq 1. \tag{34}$$

If a is a single atom, by Hölder inequality and $L^2_\omega(\mathbb{R}^n)$ boundedness of R_j^l , we have

$$\begin{aligned} \|R_j^l(a)\|_{L^1_\omega(\mathbb{R}^n)} &\leq \|R_j^l(a)\|_{L^2_\omega(\mathbb{R}^n)} \omega(\mathbb{R}^n)^{1/2} \\ &\leq \|a\|_{L^2_\omega(\mathbb{R}^n)} \omega(\mathbb{R}^n)^{1/2} \lesssim 1. \end{aligned} \quad (35)$$

Next we assume that a is a $(1, 2, s)_\omega$ -atom and $\text{supp } a \subset Q(x_0, r)$ with $r \leq L_1 \rho(x_0)$; then $\rho(x) \leq C_0^2 2^{k_0} (1 + L_1) \rho(x_0)$ for any x satisfying $|x - y| < \rho(x)$ and $y \in Q(x_0, r)$.

We first consider the atom a with $L_2 \rho(x_0) \leq r \leq L_1 \rho(x_0)$. Taking $b_1 = 1 + C_0^2 2^{k_0} (1 + L_1) / L_2$, then $\text{supp } (R_j^l(a)) \subset Q(x_0, b_1 r)$; thus, by Hölder inequality, Lemma 4, and $L^2_\omega(\mathbb{R}^n)$ boundedness of R_j^l , we get

$$\begin{aligned} \|R_j^l(a)\|_{L^1_\omega(\mathbb{R}^n)} &= \|R_j^l(a)\|_{L^1_\omega(Q(x_0, b_1 r))} \\ &\leq \|a\|_{L^2_\omega(\mathbb{R}^n)} \omega(Q(x_0, b_1 r))^{1/2} \\ &\leq \|a\|_{L^2_\omega(\mathbb{R}^n)} \omega(Q(x_0, r))^{1/2} \lesssim 1. \end{aligned} \quad (36)$$

If $r < L_2 \rho(x_0)$, then $\text{supp } (R_j^l(a)) \subset Q(x_0, b_2 \rho(x_0))$, where $b_2 = C_0^2 2^{k_0} (1 + L_1) + L_2$. Applying the Hölder inequality, Lemma 4 and $L^2_\omega(\mathbb{R}^n)$ boundedness of R_j^l give us that

$$\|R_j^l(a)\|_{L^1_\omega(Q(x_0, 2r))} \leq \|a\|_{L^2_\omega(\mathbb{R}^n)} \omega(Q(x_0, r))^{1/2} \lesssim 1. \quad (37)$$

Moreover, for $x \in Q(x_0, b_2 \rho(x_0)) \setminus Q(x_0, 2r)$, there is a constant $b_3 > 1$ such that $b_3^{-1} \rho(x_0) \leq \rho(x) \leq b_3 \rho(x_0)$, and by the vanishing moment of a , we get

$$\begin{aligned} &|R_j^l(a)(x)| \\ &\leq \left| \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} \left[\eta\left(\frac{|x - y|}{\rho(x)}\right) - 1 \right] a(y) dy \right| \\ &\quad + \left| \int_{\mathbb{R}^n} \left[\frac{x_j - y_j}{|x - y|^{n+1}} - \frac{x_j - (x_0)_j}{|x - x_0|^{n+1}} \right] a(y) dy \right| \\ &\leq \int_{|x - y| \geq \rho(x_0)/2b_3} \frac{|a(y)|}{|x - y|^n} dy \\ &\quad + \int_{\mathbb{R}^n} \frac{|x_0 - y|}{|x - x_0|^{n+1}} |a(y)| dy \\ &\leq \left[\frac{1}{(\rho(x_0))^n} + \frac{r}{|x - x_0|^{n+1}} \right] \|a\|_{L^1(\mathbb{R}^n)}, \end{aligned} \quad (38)$$

where, in the penultimate inequality, we use the fact that if $|x - y| < \rho(x_0)/2b_2$, then $\eta(|x - y|/\rho(x)) = 1$. This implies that

$$\begin{aligned} &\int_{Q(x_0, b_2 \rho(x_0)) \setminus Q(x_0, 2r)} |R_j^l(a)(x)| \omega(x) dx \\ &\leq \|a\|_{L^1(\mathbb{R}^n)} \int_{Q(x_0, b_2 \rho(x_0)) \setminus Q(x_0, 2r)} \frac{\omega(x)}{(\rho(x_0))^n} dx \\ &\quad + \|a\|_{L^1(\mathbb{R}^n)} \int_{Q(x_0, b_2 \rho(x_0)) \setminus Q(x_0, 2r)} \frac{r \omega(x)}{|x - x_0|^{n+1}} dx \\ &\equiv I + II. \end{aligned} \quad (39)$$

For I , since $\text{supp } a \subset Q_0 \equiv Q(x_0, r) \subset Q(x_0, \rho(x_0)) \equiv Q$, by Lemmas 4 and 7, we have

$$\begin{aligned} I &\leq \|a\|_{L^1(\mathbb{R}^n)} \frac{\omega(Q(x_0, b_2 \rho(x_0)))}{(\rho(x_0))^n} \\ &\leq \frac{|Q|}{\omega(Q)} \frac{\omega(Q(x_0, b_2 \rho(x_0)))}{(\rho(x_0))^n} \lesssim 1. \end{aligned} \quad (40)$$

For II , by Lemmas 4 and 7, we get

$$\begin{aligned} II &\leq \|a\|_{L^1(\mathbb{R}^n)} \int_{2r \leq |x - x_0| \leq b_2 \rho(x_0)} \frac{r \omega(x)}{|x - x_0|^{n+1}} dx \\ &\leq \|a\|_{L^1(\mathbb{R}^n)} \sum_{j=2}^{j_0} \int_{|x - x_0| \sim 2^j r} \frac{r \omega(x)}{|x - x_0|^{n+1}} dx \\ &\leq \frac{|Q_0|}{\omega(Q_0)} \sum_{j=2}^{j_0} 2^{-k} \frac{\omega(2^j Q_0)}{|2^k Q_0|} \lesssim 1, \end{aligned} \quad (41)$$

where j_0 is an integer such that $2^{j_0-1} r < b_2 \rho(x_0) \leq 2^{j_0} r$. Thus, for $r < L_2 \rho(x_0)$ we obtain

$$\|R_j^l(a)\|_{L^1_\omega(\mathbb{R}^n)} \lesssim 1. \quad (42)$$

Finally, we prove $\|f\|_{L^1_\omega(\mathbb{R}^n)} + \sum_{j=1}^n \|R_j(f)\|_{L^1_\omega(\mathbb{R}^n)} \sim \|f\|_{L^1_\omega(\mathbb{R}^n)} + \sum_{j=1}^n \|R_j^l(f)\|_{L^1_\omega(\mathbb{R}^n)}$. Indeed, we only need to show that

$$\|R_j^l(f) - R_j(f)\|_{L^1_\omega(\mathbb{R}^n)} \leq \|f\|_{L^1_\omega(\mathbb{R}^n)}. \quad (43)$$

In fact, note that $|x - y| \leq \rho(x)$; then there exists a constant $\beta > 0$ such that $\rho(y)/\beta \leq \rho(x) \leq \beta \rho(y)$ by Lemma 1. Set $\omega \in A_1^{\rho, \theta}(\mathbb{R}^n)$ with $\theta > 0$. Hence, by the properties of $A_1^{\rho, \theta}(\mathbb{R}^n)$, we get

$$\begin{aligned} &\|R_j^l(f) - R_j(f)\|_{L^1_\omega(\mathbb{R}^n)} \leq \iint_{\mathbb{R}^n} |R_j(x, y) - \tilde{R}_j(x, y)| \\ &\quad \cdot \eta\left(\frac{|x - y|}{\rho(x)}\right) |f(y)| dy \omega(x) dx \\ &\leq \int_{\mathbb{R}^n} \int_{|x - y| < \rho(y)/\beta} |R_j(x, y) - \tilde{R}_j(x, y)| \\ &\quad \cdot \omega(x) dx |f(y)| dy \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{R}^n} \int_{\rho(y)/\beta \leq |x-y| < \beta\rho(y)} |\tilde{R}_j(x, y)| \\
& \cdot \omega(x) dx |f(y)| dy \\
& + \int_{\mathbb{R}^n} \int_{\rho(y)/\beta \leq |x-y|} |R_j(x, y)| \omega(x) dx |f(y)| dy \\
& \leq \|f\|_{L^1_\omega(\mathbb{R}^n)} + \int_{\mathbb{R}^n} \frac{\omega(Q(y, \rho(y)))}{(\rho(y))^n} |f(y)| dy \\
& + \int_{\mathbb{R}^n} \sum_{k=1}^{\infty} 2^{-kN} |2^k \rho(y)|^{-n} \\
& \cdot \int_{|x-y| \leq 2^k \rho(y)/\beta} \omega(x) dx |f(y)| dy \\
& \leq \|f\|_{L^1_\omega(\mathbb{R}^n)} + \int_{\mathbb{R}^n} \sum_{k=1}^{\infty} 2^{-k(N-(k_0+1)\theta)} \\
& \cdot \frac{\omega(Q(y, \rho(y)))}{(\rho(y))^n} |f(y)| dy \lesssim \|f\|_{L^1_\omega(\mathbb{R}^n)},
\end{aligned} \tag{44}$$

if taking $N > (k_0 + 1)\theta + 1$.

Therefore, the proof is complete. $\square$

Competing Interests

The author declares that there are no competing interests regarding the publication of this paper.

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References

- [1] J. Dziubański, G. Garrigós, T. Martínez, J. L. Torrea, and J. Zienkiewicz, "BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality," *Mathematische Zeitschrift*, vol. 249, no. 2, pp. 329–356, 2005.
- [2] Z. Shen, " L^p estimates for Schrödinger operators with certain potentials," *Annales de l'Institut Fourier*, vol. 45, no. 2, pp. 513–546, 1995.
- [3] B. Bongioanni, E. Harboure, and O. Salinas, "Classes of weights related to Schrödinger operators," *Journal of Mathematical Analysis and Applications*, vol. 373, no. 2, pp. 563–579, 2011.
- [4] B. Bongioanni, E. Harboure, and O. Salinas, "Commutators of Riesz transforms related to Schrödinger operators," *The Journal of Fourier Analysis and Applications*, vol. 17, no. 1, pp. 115–134, 2011.
- [5] J. Dziubański and J. Zienkiewicz, "Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality," *Revista Matemática Iberoamericana*, vol. 15, no. 2, pp. 279–296, 1999.
- [6] J. Dziubański and J. Zienkiewicz, " H^p spaces for Schrödinger operators, Fourier Analysis and Related Topics," *Banach Center Publications*, vol. 56, pp. 45–53, 2002.
- [7] J. Dziubański and J. Zienkiewicz, " H^p spaces associated with Schrödinger operators with potentials from reverse Hölder classes," *Colloquium Mathematicum*, vol. 98, no. 1, pp. 5–38, 2003.
- [8] H. Liu, L. Tang, and H. Zhu, "Weighted Hardy spaces and BMO spaces associated with Schrödinger operators," *Mathematische Nachrichten*, vol. 285, no. 17–18, pp. 2173–2207, 2012.
- [9] L. Tang, "Weighted norm inequalities for Schrödinger type operators," *Forum Mathematicum*, vol. 27, no. 4, pp. 2491–2532, 2015.
- [10] L. Tang, "Extrapolation from $A_{\infty}^{\rho, \infty}$, vector-valued inequalities and applications in the Schrödinger settings," *Arkiv för Matematik*, vol. 52, no. 1, pp. 175–202, 2014.
- [11] D. Yang and Y. Zhou, "Boundedness of sublinear operators in Hardy spaces on RD-spaces via atoms," *Journal of Mathematical Analysis and Applications*, vol. 339, no. 1, pp. 622–635, 2008.
- [12] D. Yang and Y. Zhou, "Localized Hardy spaces H^1 related to admissible functions on RD-spaces and applications to Schrödinger operators," *Transactions of the American Mathematical Society*, vol. 363, no. 3, pp. 1197–1239, 2011.
- [13] J. Zhong, *Harmonic analysis for some Schrödinger type operators [Ph.D. thesis]*, Princeton University, 1993.
- [14] H. Zhu and H. Liu, "Weighted estimates for bilinear operators," *Journal of Function Spaces*, vol. 2014, Article ID 797956, 10 pages, 2014.
- [15] J. Duoandikoetxea, *Fourier Analysis*, vol. 29 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, USA, 2000.
- [16] J. García-Cuerva and J. Rubio de Francia, *Weighted norm inequalities and related topics*, vol. 116 of *North-Holland Mathematics Studies*, North-Holland, Amsterdam, The Netherlands, 1985.
- [17] J. García-Cuerva, "Weighted H^p spaces," *Dissertationes Mathematicae*, vol. 162, pp. 1–63, 1979.
- [18] E. M. Stein, *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals*, vol. 43 of *Princeton Mathematical Series*, Princeton University Press, Princeton, NJ, USA, 1993.
- [19] J.-O. Stromberg and A. Torchinsky, *Weighted Hardy Spaces*, vol. 1381 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1989.
- [20] L. Tang, "Weighted norm inequalities for pseudo-differential operators with smooth symbols and their commutators," *Journal of Functional Analysis*, vol. 262, no. 4, pp. 1603–1629, 2012.
- [21] H. Zhu and L. Tang, "Weighted local Hardy spaces associated to Schrödinger operators," <http://arxiv.org/abs/1403.7641>.
- [22] D. Yang, D. Yang, and Y. Zhou, "Endpoint properties of localized Riesz transforms and fractional integrals associated to Schrödinger operators," *Potential Analysis*, vol. 30, no. 3, pp. 271–300, 2009.
- [23] G. Mauceri, M. A. Picardello, and F. Ricci, "A Hardy space associated with twisted convolution," *Advances in Mathematics*, vol. 39, no. 3, pp. 270–288, 1981.
- [24] B. Bongioanni, E. Harboure, and O. Salinas, "Riesz transforms related to Schrödinger operators acting on BMO type spaces," *Journal of Mathematical Analysis and Applications*, vol. 357, no. 1, pp. 115–131, 2009.
- [25] F. W. Gehring, "The L^p -integrability of the partial derivatives of a quasi-conformal mapping," *Acta Mathematica*, vol. 130, pp. 265–277, 1973.

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