Research Article

Solutions for Impulsive Fractional Differential Equations via Variational Methods

Peiluan Li,¹ Hui Wang,² and Zheqing Li³

¹School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang 471023, China

²College of Information Engineering, Henan University of Science and Technology, Luoyang 471023, China

³Network and Information Center, Henan University of Science and Technology, Luoyang 471023, China

Correspondence should be addressed to Peiluan Li; lpplpl.lpl@163.com

Received 20 January 2016; Accepted 16 March 2016

Academic Editor: Jozef Banas

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We investigate the boundary value problems of impulsive fractional order differential equations. First, we obtain the existence of at least one solution by the minimization result of Mawhin and Willem. Then by the variational methods and a very recent critical points theorem of Bonanno and Marano, the existence results of at least triple solutions are established. At last, two examples are offered to demonstrate the application of our main results.

1. Introduction

Fractional differential equations have been an area of great interest recently. This is because of both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various scientific fields such as physics, mechanics, chemistry, and engineering; see [1–3] and the references therein. Fractional differential equations involving the Riemann-Liouville fractional derivative or the Caputo fractional derivative have been paid more and more attentions. But for almost all the papers above, the main methods are some fixed theorems, the coincidence degree theory, and the monotone iterative methods.

On the other hand, critical point theory and the variational methods have been very useful in dealing with the existence and multiplicity of solutions for integer order differential equations with some boundary conditions. We refer the readers to the books (or surveys) of Mawhin and Willem [4] and Rabinowitz [5] and [6, 7] and the references therein. But until now, there are few works that deal with the fractional differential equations via the variational methods; [8, 9] are the pioneer in the use of the variational methods to fractional models. By using the variational methods, Jiao and

Zhou [8] first considered the following fractional boundary value problems:

$$\begin{aligned} {}_t D_T^\alpha ({}_0 D_t^\alpha u(t)) &= \nabla F(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ u(0) &= u(T) = 0, \end{aligned} \quad (1)$$

where $\alpha \in (0, 1)$ and ${}_0 D_t^\alpha$ and ${}_t D_T^\alpha$ are the left and right Riemann-Liouville fractional derivatives, respectively. $F : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ (with $N \geq 1$) is a suitable given function and $\nabla F(t, x)$ is the gradient of F with respect to x . The problem in [8] and the related problems were further considered by critical point theory and the variational methods in [10–12].

By means of critical point theory, Jiao and Zhou [9] considered the following fractional boundary value problems:

$$\begin{aligned} -\frac{1}{2} \frac{d}{dt} ({}_0 D_t^{-\beta} + {}_t D_T^{-\beta}) u'(t) &= \nabla F(t, u(t)), \\ \text{a.e. } t \in [0, T], \end{aligned} \quad (2)$$

$$u(0) = u(T) = 0,$$

where $\beta \in [0, 1)$ and ${}_0 D_t^{-\beta}$ and ${}_t D_T^{-\beta}$ are the left and right Riemann-Liouville fractional derivatives gradient of F with respect to x .

The above problem arises from the phenomena of advection dispersion and was first investigated by Ervin and Roop in [13]. The authors in [14–17] further studied the existence and multiplicity of solutions for the above problem or related problems by critical point theory.

The impulsive differential equations arise from the real world problems to describe the dynamics of processes in which sudden, discontinuous jumps occur. Such processes are naturally seen in biology, physics, engineering, and so forth. Due to their significance, many authors have established the solvability of impulsive differential equations. There have been many different approaches to the study of the existence of solutions to impulsive fractional differential equations, such as topological degree theory, fixed point theory, upper and lower solutions method, and monotone iterative technique.

To the best of our knowledge, the fractional boundary value problems with impulses using variational methods and critical point theory have received considerably less attention [18–20].

In [20], the authors investigated the following fractional differential equations with impulses:

$$\begin{aligned} {}_t D_T^\alpha ({}_0^c D_t^\alpha u(t)) + a(t)u(t) &= f(t, u(t)), \\ t &\neq t_j, \text{ a.e. } t \in [0, T], \\ \Delta({}_t D_T^{\alpha-1} ({}_0^c D_t^\alpha u))(t_j) &= I_j(u(t_j)), \\ j &= 1, 2, \dots, n, \\ u(0) &= u(T) = 0. \end{aligned} \quad (3)$$

By using critical point theory and variational methods, the authors give some criteria of the existence of solutions.

Motivated by the work above, we consider the following problem (4) of impulsive fractional differential equations:

$$\begin{aligned} -\frac{1}{2} \frac{d}{dt} ({}_0 D_t^{-\beta} + {}_t D_T^{-\beta}) u'(t) &= \lambda \nabla F(t, u(t)), \\ t &\neq t_k, \text{ a.e. } t \in [0, T], \\ \Delta(D_t^\alpha u)(t_k) &= \lambda I_k(u(t_k)), \\ t_k &\in (0, T), \quad k = 1, 2, \dots, l, \\ u(0) &= u(T) = 0, \end{aligned} \quad (4)$$

where $\lambda \in (0, +\infty)$, $\beta \in [0, 1]$, $\alpha = 1 - \beta/2 \in (1/2, 1]$, ${}_0 D_t^{-\beta}$, ${}_t D_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order β , ${}_0^c D_t^\alpha$ and ${}_t^c D_T^\alpha$ are the left and right Caputo fractional derivative of order α , respectively, $0 = t_0 < t_1 < t_2 < \dots < t_l < t_{l+1} = 1$, $F \in [0, T] \times R \rightarrow R$ a given function

satisfying some assumptions and $\nabla F(t, x)$ is the gradient of F at x , $I_k \in C([0, T], R)$, and

$$\begin{aligned} (D_t^\alpha u)(t) &= \frac{1}{2} \{ {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u) \}(t), \\ \Delta(D_t^\alpha u)(t_k) &= \frac{1}{2} \{ {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u) \}(t_k^+) \\ &\quad - \frac{1}{2} \{ {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u) \}(t_k^-), \\ \frac{1}{2} \{ {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u) \}(t_k^+) &= \lim_{t \rightarrow t_k^+} \frac{1}{2} \{ {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u) \}(t), \\ \frac{1}{2} \{ {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u) \}(t_k^-) &= \lim_{t \rightarrow t_k^-} \frac{1}{2} \{ {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u) \}(t), \\ &\text{for } k = 1, \dots, l. \end{aligned} \quad (5)$$

In this paper, the existence results of at least one solution or triple solutions of problem (4) are established. The rest of this paper is organized as follows. In Section 2, some definitions and lemmas which are essential to prove our main results are stated. In Section 3, we give the main results. At last, two examples are offered to demonstrate the application of our main results.

2. Preliminaries

At first, we present the necessary definitions for the fractional calculus theory and several lemmas which are used further in this paper.

Definition 1 (see [3]). Let f be a function defined by $[a, b]$. The left and right Riemann-Liouville fractional integrals of order α for function f denoted by ${}_a D_t^{-\alpha} f(t)$ and ${}_t D_b^{-\alpha} f(t)$ function, respectively, are defined by

$$\begin{aligned} {}_a D_t^{-\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \\ {}_t D_b^{-\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds, \end{aligned} \quad (6)$$

for $t \in [a, b]$, $\alpha > 0$.

The left and right Riemann-Liouville fractional derivatives of order α for function f denoted by ${}_a D_t^\alpha f(t)$ and ${}_t D_b^\alpha f(t)$ function, respectively, are defined by

$$\begin{aligned} {}_a D_t^\alpha f(t) &= \frac{d^n}{dt^n} {}_a D_t^{\alpha-n} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s) ds, \\ {}_t D_b^\alpha f(t) &= (-1)^n \frac{d^n}{dt^n} {}_t D_b^{\alpha-n} f(t) \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (s-t)^{n-\alpha-1} f(s) ds, \end{aligned} \quad (7)$$

for $t \in [a, b]$, $n-1 \leq \alpha < n$, $n \in \mathbb{N}$,

provided that the right-hand side integral is pointwise defined on $[a, b]$.

Lemma 2 (see [3]). *The left and right Riemann-Liouville fractional integral operators have the property of a semigroup; that is,*

$$\begin{aligned} {}_a D_t^{-\gamma_1} ({}_a D_t^{-\gamma_2} f(t)) &= {}_a D_t^{-\gamma_1-\gamma_2} f(t), \\ {}_t D_b^{-\gamma_1} ({}_t D_b^{-\gamma_2} f(t)) &= {}_t D_b^{-\gamma_1-\gamma_2} f(t), \end{aligned} \quad (8)$$

$\forall \gamma_1, \gamma_2 > 0$

in any point $t \in [a, b]$ for continuous function f and for almost every point in $[a, b]$ if the function $f \in L^1([a, b], \mathbb{R}^N)$.

Definition 3 (see [3]). If $\alpha \in (n-1, n)$ and $f \in AC^n([a, b], \mathbb{R})$, then the left and right Caputo fractional derivatives of order α of a function f denoted by ${}_a^c D_t^\alpha f(t)$ and ${}_t^c D_b^\alpha f(t)$ function, respectively, are defined by

$$\begin{aligned} {}_a^c D_t^\alpha f(t) &= {}_a D_t^{\alpha-n} \frac{d^n}{dt^n} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \\ {}_t^c D_b^\alpha f(t) &= (-1)^n {}_t D_b^{\alpha-n} \frac{d^n}{dt^n} f(t) \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} f^{(n)}(s) ds, \end{aligned} \quad (9)$$

respectively, where $t \in [a, b]$.

In view of Definition 1 and Lemma 2, we can easily transfer problem (4) to the following problem:

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u(t)) - \frac{1}{2} {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u(t)) \right) \\ + \lambda \nabla F(t, u(t)) = 0, \quad t \neq t_k, \text{ a.e., } t \in [0, T], \\ \Delta (D_t^\alpha u)(t_k) = \lambda I_k(u(t_k)), \\ t_k \in (0, T), \quad k = 1, 2, \dots, l, \\ u(0) = u(T) = 0, \end{aligned} \quad (10)$$

where $\beta \in [0, 1]$, $\alpha = 1 - \beta/2 \in (1/2, 1]$, $(D_t^\alpha u)(t) = (1/2)\{{}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u)\}(t)$.

Then problem (4) is equivalent to problem (10). Therefore, a solution of problem (10) corresponds to a solution of BVP (4).

In order to establish a variational structure which enables us to reduce the existence of solution of problem (10) to existence of the critical point of corresponding functional, we construct the following appropriate function spaces.

Let us recall that, for any fixed $t \in [0, T]$ and $1 \leq p \leq \infty$,

$$\begin{aligned} \|u\|_\infty &= \max_{t \in [0, T]} |u(t)|, \\ \|u\|_{L^p} &= \left(\int_0^T |u(s)|^p ds \right)^{1/p}. \end{aligned} \quad (11)$$

Let $0 < \alpha \leq 1$, and we define the fractional derivative spaces E_0^α by the closure of $C_0^\infty([0, T])$ with respect to the weighted norm $\|u\|_\alpha = (\int_0^T |{}_0^c D_t^\alpha u(t)|^2 dt + \int_0^T |u(t)|^2 dt)^{1/2}$, $\forall u \in E_0^\alpha$, where

$$C_0^\infty([0, T]) = \{u \in C^\infty([0, T]) : u(0) = u(T) = 0\}. \quad (12)$$

Clearly, the fractional derivative space E_0^α is the space of functions $u \in L^2([0, T] \setminus \{t_1, t_2, \dots, t_l\})$ having α -order Caputo left and right fractional derivatives and Riemann-Liouville left and right fractional derivatives, ${}_0^c D_t^\alpha$, ${}_t^c D_T^\alpha$, ${}_0 D_t^\alpha u(t)$, ${}_t D_T^\alpha u(t) \in L^2([0, T] \setminus \{t_1, t_2, \dots, t_l\})$ and $u(0) = u(T) = 0$.

Lemma 4 (see [9]). *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For all $u \in E_0^\alpha$, one has*

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0^c D_t^\alpha u\|_{L^p}. \quad (13)$$

Moreover, if $\alpha > 1/p$ and $1/p + 1/q = 1$, then

$$\|u\|_\infty \leq \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \|{}_0^c D_t^\alpha u\|_{L^p}. \quad (14)$$

Then we can conclude that $\|u\|_\alpha = (\int_0^T |{}_0^c D_t^\alpha u(t)|^2 dt + \int_0^T |u(t)|^2 dt)^{1/2}$ is equivalent to $\|u\|_\alpha = (\int_0^T |{}_0^c D_t^\alpha u(t)|^2 dt)^{1/2}$, $\forall u \in E_0^\alpha$. In the following, we will consider the fractional derivative spaces E_0^α with respect to the norm $\|u\|_\alpha = (\int_0^T |{}_0^c D_t^\alpha u(t)|^2 dt)^{1/2}$. Obviously, E_0^α is a reflexive and separable Banach space with the norm $\|u\|_\alpha = (\int_0^T |{}_0^c D_t^\alpha u(t)|^2 dt)^{1/2}$.

Definition 5. A function

$$u \in \left\{ u \in AC([0, T]) : \int_{t_j}^{t_{j+1}} \left(|{}_0^c D_t^\alpha u(t)|^2 + |u(t)|^2 \right)^{1/2} dt < \infty, j = 0, \dots, l \right\} \quad (15)$$

is called a classic solution of problem (10) if

(i)

$$\begin{aligned} & \frac{1}{2} \left\{ {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u) \right\} (t_k^+) \\ &= \lim_{t \rightarrow t_k^+} \frac{1}{2} \left\{ {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u) \right\} (t), \\ & \frac{1}{2} \left\{ {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u) \right\} (t_k^-) \\ &= \lim_{t \rightarrow t_k^-} \frac{1}{2} \left\{ {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u) \right\} (t) \end{aligned} \quad (16)$$

exist and satisfy the impulsive condition $\Delta(D_t^\alpha u)(t_k) = I_k(u(t_k))$ and the boundary condition $u(0) = u(T) = 0$ holds,

(ii) u satisfies (10) a.e. on $t \in [0, T] \setminus \{t_1, t_2, \dots, t_l\}$.

Definition 6. A function $u \in E_0^\alpha$ is called a weak solution of problem (10) if

$$\begin{aligned} & \int_0^T -\frac{1}{2} \left[{}_0^c D_t^\alpha u(t) {}_t^c D_T^\alpha v(t) + {}_t^c D_T^\alpha u(t) {}_0^c D_t^\alpha v(t) \right] dt \\ &+ \lambda \sum_{j=1}^l I_j(u(t_j)) v(t_j) \\ &- \lambda \int_0^T \nabla F(t, u(t)) v(t) dt = 0, \end{aligned} \quad (17)$$

for all $v(t) \in E_0^\alpha$.

Similar to the proof of Lemma 2.1 in [18], we have the following Lemma 7.

Lemma 7. The function $u \in E_0^\alpha$ is weak solution of (10), if and only if u is a classical solution of (10).

Lemma 8 (see [9]). Let $1/2 < \alpha \leq 1$ and the sequence $\{u_k\}$ converges weakly to u in E_0^α ; then $u_k \rightarrow u$ in $C([0, T], R)$; that is, $\|u - u_k\|_\infty \rightarrow 0$, as $k \rightarrow \infty$.

Lemma 9 (see [9]). Let $1/2 < \alpha \leq 1$. For any $u \in E_0^\alpha$, one has

$$\begin{aligned} |\cos(\pi\alpha)| \|u\|_\alpha^2 &\leq - \int_0^T ({}_0^c D_t^\alpha u(t), {}_t^c D_T^\alpha u(t)) dt \\ &\leq \frac{1}{|\cos(\pi\alpha)|} \|u\|_\alpha^2. \end{aligned} \quad (18)$$

The proofs of the main results in this paper are based on the following critical point theorems.

Theorem 10 (see [4, Theorem 1.1]). If Φ is weakly lower semicontinuous (WLSC) on a reflexive Banach space X and has a bounded minimizing sequence, then Φ has a minimum on X .

Theorem 11 (see [21]). Let X be a reflexive real Banach space, let $\Phi : X \rightarrow R$ be a sequentially weakly lower semicontinuous, coercive, and continuously Gateaux differentiable functional whose Gateaux derivative admits a continuous inverse on X^* , and let $\Psi : X \rightarrow R$ be a sequentially weakly upper semicontinuous and continuously Gateaux differentiable functional whose Gateaux derivative is compact. Assume that there exist $r \in R$ and $u_1 \in X$ with $0 < r < \Phi(u_1)$, such that

- (i) $\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u) < r(\Psi(u_1)/\Phi(u_1))$,
- (ii) for every $\lambda \in \Lambda_r :=]\Phi(u_1)/\Psi(u_1), r/ \sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)[$, the functional $\Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X .

3. Main Results

For convenience, we give the following hypothesis (H_1) .

(H_1) λ is a positive real parameter, $\beta \in [0, 1)$, $\alpha = 1 - \beta/2 \in (1/2, 1]$, $F : [0, T] \times R \rightarrow R$ is a function such that $F(\cdot, x)$ is continuous in $[0, T]$ for every $x \in R$, $F(t, \cdot)$ is a C^1 function in R for any $t \in [0, T]$, $F(t, 0) = 0$, and $\nabla F(t, x)$ is the gradient of F at x , $I_j \in C([0, T], R^+)$, $j = 1, \dots, l$.

Theorem 12. Let (H_1) hold. If $\lambda < 1$, the following assumption (H_2) is satisfied.

(H_2) There exists a positive constant $a_1 < \Gamma^2(\alpha + 1)|\cos(\pi\alpha)|/2T^{2\alpha}$ such that

$$\limsup_{|x| \rightarrow \infty} \frac{F(t, x)}{|x|^2} < a_1, \quad (19)$$

uniformly for $x \in R$, $t \in [0, T]$.

Then (4) possesses at least one solution.

Proof. We define

$$\begin{aligned} \varphi(u) &= \int_0^T -\frac{1}{2} {}_0^c D_t^\alpha u(t) {}_t^c D_T^\alpha u(t) dt \\ &- \lambda \int_0^T F(t, u(t)) dt + \lambda \sum_{j=1}^l \int_0^{u(t_j)} I_j(s) ds. \end{aligned} \quad (20)$$

Then, for all $v(t) \in E_0^\alpha$, we know

$$\begin{aligned} & \varphi'(u) v \\ &= \int_0^T -\frac{1}{2} [{}_0^c D_t^\alpha u(t) {}_t^c D_T^\alpha v(t) + {}_t^c D_T^\alpha u(t) {}_0^c D_t^\alpha v(t)] dt \\ &+ \lambda \sum_{j=1}^l I_j(u(t_j)) v(t_j) \\ &- \lambda \int_0^T \nabla F(t, u(t)) v(t) dt, \end{aligned} \quad (21)$$

which shows that a critical point of the functional φ is a weak solution of problem (10).

Our aim is to apply Theorem 10 to problem (10).

We begin by proving that φ is weakly lower semicontinuous. Since X is a separable and reflexive real Banach space, we assume that $\{u_k\} \subset X$ converges weakly to u in X . By Lemma 8, we can obtain that $u_k \rightarrow u$ uniformly in $C([0, T], R)$, as $k \rightarrow \infty$; that is,

$$\begin{aligned} \|u_k - u\|_\infty &\longrightarrow 0, \quad \text{as } k \longrightarrow \infty, \\ \liminf_{k \rightarrow \infty} \|u_k\|_\alpha &\geq \|u\|_\alpha. \end{aligned} \quad (22)$$

Together with (H_1) , one has

$$\begin{aligned} &\liminf_{k \rightarrow \infty} \varphi(u_k) \\ &= \liminf_{k \rightarrow \infty} \left\{ \int_0^T -\frac{1}{2} {}^c D_t^\alpha u_k(t) {}^c D_T^\alpha u_k(t) dt \right. \\ &\quad \left. - \lambda \int_0^T F(t, u_k(t)) dt + \lambda \sum_{j=1}^l \int_0^{u_k(t_j)} I_j(s) ds \right\} \\ &\geq \int_0^T -\frac{1}{2} {}^c D_t^\alpha u(t) {}^c D_T^\alpha u(t) dt \\ &\quad - \lambda \int_0^T F(t, u(t)) dt + \lambda \sum_{j=1}^l \int_0^{u(t_j)} I_j(s) ds \\ &= \varphi(u). \end{aligned} \quad (23)$$

Then it implies that φ is weakly lower semicontinuous.

Now, we are in the position of showing that the functional φ is coercive.

From (H_2) , we know that there exists a positive constant a_2 large enough such that

$$F(t, u) < a_1 |u|^2, \quad \text{for } |u| > a_2, \quad t \in [0, T]. \quad (24)$$

On the other hand, from the continuity of $F(t, u)$, we concluded that $F(t, u)$ is bounded for $|u| \leq a_2, t \in [0, T]$. Then there exists a constant $b_1 > 0$ such that

$$F(t, u) < b_1, \quad (25)$$

for $|u| \leq a_2, t \in [0, T]$.

Hence, for all $(t, u) \in [0, T] \times R$, we can get

$$F(t, u) < a_1 |u|^2 + b_1. \quad (26)$$

Then it follows from (H_1) and Lemmas 4 and 9 that

$$\begin{aligned} \varphi(u) &= \int_0^T -\frac{1}{2} {}^c D_t^\alpha u(t) {}^c D_T^\alpha u(t) dt \\ &\quad - \lambda \int_0^T F(t, u(t)) dt + \lambda \sum_{j=1}^l \int_0^{u(t_j)} I_j(s) ds \\ &\geq \frac{1}{2} |\cos(\pi\alpha)| \|u\|_\alpha^2 - \int_0^T (a_1 |u|^2 + b_1) dt \\ &\geq \left[\frac{1}{2} |\cos(\pi\alpha)| - a_1 \frac{T^{2\alpha}}{\Gamma^2(\alpha+1)} \right] \|u\|_\alpha^2 - b_1 T. \end{aligned} \quad (27)$$

In view of $a_1 < \Gamma^2(\alpha+1)|\cos(\pi\alpha)|/2T^{2\alpha}$, we have

$$\varphi(u) \longrightarrow \infty, \quad \text{as } \|u\|_\alpha \longrightarrow \infty. \quad (28)$$

Then we know that φ is coercive. Thus, by virtue of Theorem 10, the functional φ has a minimum, which is a critical point of φ . It follows that the boundary value problem (10) has one weak solution. By virtue of Lemma 7, we can deduce that the boundary value problem (10) has one solution which implies that the boundary value problem (4) possesses at least one solution. $\square$

Remark 13. If the asymptotically quadratic case in (H_2) becomes the subquadratic case, that is,

$$\limsup_{|x| \rightarrow \infty} F(t, x) < e |x|^{e_1}, \quad 1 < e_1 < 2, \quad e > 0, \quad (29)$$

then we can get the similar result.

Put

$$\begin{aligned} M &:= \frac{T^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|}, \\ L &:= \frac{2T^{2\alpha}}{\Gamma^2(\alpha+1)}, \end{aligned} \quad (30)$$

$$\Omega(c) = \left\{ \xi \in R : \frac{1}{2} |\xi|^2 \leq c \right\}.$$

For $u \in E_0^\alpha$, we define the functional $\Phi, \Psi : E_0^\alpha \rightarrow R$ as follows:

$$\Phi(u) = \int_0^T -\frac{1}{2} {}^c D_t^\alpha u(t) {}^c D_T^\alpha u(t) dt, \quad (31)$$

$$\Psi(u) = \int_0^T F(t, u(t)) dt - \sum_{j=1}^l \int_0^{u(t_j)} I_j(s) ds. \quad (32)$$

Theorem 14. Let (H_1) hold. Suppose that there exist a constant $r > 0$ and a function ω such that $\|\omega\|_\alpha^2 > 2r/|\cos(\pi\alpha)|$ and the following assumptions (H_3) - (H_4) are satisfied:

$$(H_3) \sup_{\xi \in \Omega(Mr)} \Psi(\xi)/r < 2|\cos(\pi\alpha)|\Psi(\omega)/\|\omega\|_\alpha^2;$$

$$(H_4) \lim_{|\xi| \rightarrow +\infty} F(t, \xi)/|\xi|^2 < |\cos(\pi\alpha)| \sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)/2Lr, \quad \text{where } \Omega(Mr) = \{\xi \in R : (1/2)|\xi|^2 \leq Mr\}.$$

Then, for each

$$\lambda \in \Lambda$$

$$:= \left[\frac{\|\omega\|_\alpha^2}{2 |\cos(\pi\alpha)| \Psi(\omega)}, \frac{r}{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)} \right], \quad (33)$$

the boundary value problem (4) has at least three distinct solutions in E_0^α .

Proof. Obviously E_0^α is compactly embedded in $C^0([0, T], R)$. From (31), (32), and (H_1) , it is well known that Φ, Ψ are well defined and are Gateaux differentiable functional whose Gateaux derivative at the point $u \in E_0^\alpha$ is the functionals $\Phi'(u), \Psi'(u) \in (E_0^\alpha)^*$, given by

$$\begin{aligned} \Phi'(u)(x) &= \int_0^T -\frac{1}{2} \left[{}^c D_t^\alpha u(t) {}^c D_T^\alpha x(t) + {}^c D_T^\alpha u(t) {}^c D_t^\alpha x(t) \right] dt, \\ \Psi'(u)(x) &= \int_0^T \nabla F(t, u(t)) x(t) dt - \sum_{j=1}^l I_j(u(t_j)) x(t_j) \end{aligned} \quad (34)$$

for every $x \in E_0^\alpha$.

For $\{u_n\}$, $u \in E_0^\alpha$, $u_n \rightarrow u$ in E_0^α as $n \rightarrow \infty$, by Lemma 8, we know that $\{u_n\}$ converges uniformly to u in $C([0, T], R)$. Hence

$$\begin{aligned} \limsup_{n \rightarrow \infty} \Psi(u_n) &\leq \int_0^T \limsup_{n \rightarrow \infty} F(t, u_n) dt \\ &\quad - \sum_{j=1}^l \int_0^{u(t_j)} \limsup_{n \rightarrow \infty} I_j(s) ds \\ &= \int_0^T F(t, u) dt - \sum_{j=1}^l \int_0^{u(t_j)} I_j(s) ds \\ &= \Psi(u), \end{aligned} \quad (35)$$

which implies that Ψ is sequentially weak upper semicontinuous.

By (H_1) , one can get $F(t, u_n) \rightarrow F(t, u)$ as $n \rightarrow \infty$. By the Lebesgue control convergence theorem, $\Psi'(u_n) \rightarrow \Psi'(u)$ strongly, which implies that $\Psi' : E_0^\alpha \rightarrow E_0^\alpha$ is strongly continuous on E_0^α ; that is, Ψ' is a compact operator.

From (H_1) , Lemma 9, and (31), it is also easy to verify that $\Phi : E_0^\alpha \rightarrow R$ is sequentially weakly lower semicontinuous, coercive, and its derivative Φ' admits a continuous inverse

on $(E_0^\alpha)^*$. By the condition $\|\omega\|_\alpha^2 > 2r/|\cos(\pi\alpha)|$, (31), and Lemma 9, we have

$$\begin{aligned} \Phi(\omega) &= -\frac{1}{2} \int_0^T {}^c D_t^\alpha \omega(t) {}^c D_T^\alpha \omega(t) dt \\ &> \frac{1}{2} |\cos(\pi\alpha)| \|\omega\|_\alpha^2 > r > 0 \end{aligned} \quad (36)$$

and $\Phi(0) = 0$.

Then, in order to apply Theorem 11, we only need to show that (i) and (ii) of Theorem 11 hold.

From Lemma 9, (14), and (31), one has

$$\begin{aligned} \Phi^{-1}((-\infty, r]) &= \{u \in E_0^\alpha : \Phi(u) \leq r\} \\ &= \left\{ u \in E_0^\alpha : -\frac{1}{2} \int_0^T {}^c D_t^\alpha u(t) {}^c D_T^\alpha u(t) dt \leq r \right\} \\ &\subseteq \left\{ u \in E_0^\alpha : \frac{1}{2} |\cos(\pi\alpha)| \|\omega\|_\alpha^2 \leq r \right\} \\ &\subseteq \left\{ u \in E_0^\alpha : \frac{|\cos(\pi\alpha)| \Gamma^2(\alpha) (2\alpha - 1)}{2T^{2\alpha-1}} \|u\|_\infty^2 \leq r \right\} \\ &\subseteq \left\{ u \in E_0^\alpha : \frac{|u(t)|^2}{2} \leq Mr, \forall t \in [0, T] \right\}, \end{aligned} \quad (37)$$

and it follows that

$$\begin{aligned} \sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u) &= \sup_{u \in \Phi^{-1}([-\infty, r])} \left(\int_0^T F(t, u(t)) dt \right. \\ &\quad \left. - \sum_{j=1}^l \int_0^{u(t_j)} I_j(s) ds \right) \leq \sup_{u \in \Omega(Mr)} \left(\int_0^T F(t, u(t)) dt \right. \\ &\quad \left. - \sum_{j=1}^l \int_0^{u(t_j)} I_j(s) ds \right), \end{aligned} \quad (38)$$

From (31), (32), Lemma 9, and (H_3) , we obtain

$$\begin{aligned} &\frac{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)}{r} \\ &\leq \frac{\sup_{u \in \Omega(Mr)} \left(\int_0^T F(t, u(t)) dt - \sum_{j=1}^l \int_0^{u(t_j)} I_j(s) ds \right)}{r} \\ &< \frac{2 |\cos(\pi\alpha)| \left(\int_0^T F(t, \omega(t)) dt - \sum_{j=1}^l \int_0^{\omega(t_j)} I_j(s) ds \right)}{\|\omega\|_\alpha^2} \\ &\leq \frac{\Psi(\omega)}{\Phi(\omega)}. \end{aligned} \quad (39)$$

Then (i) of Theorem 11 holds.

Furthermore, according to (H_4) there exist two constants $k, \varepsilon \in \mathbb{R}$ with

$$k < \frac{|\cos(\pi\alpha)| \sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)}{r}, \quad (40)$$

such that

$$F(t, \xi) \leq \frac{k}{2L} |\xi|^2 + \varepsilon, \quad \text{for } t \in [0, T], \xi \in \mathbb{R}. \quad (41)$$

Next we consider the functional $\Phi(u) - \lambda\Psi(u)$. From (41), one has

$$\begin{aligned} \Phi(u) - \lambda\Psi(u) &= -\frac{1}{2} \int_0^T {}^c D_t^\alpha u(t) {}^c D_T^\alpha u(t) dt \\ &\quad - \lambda \left(\int_0^T F(t, u(t)) dt - \sum_{j=1}^l \int_0^{u(t_j)} I_j(s) ds \right) \\ &\geq -\frac{1}{2} \int_0^T {}^c D_t^\alpha u(t) {}^c D_T^\alpha u(t) dt \\ &\quad - \lambda \int_0^T F(t, u(t)) dt \\ &\geq \frac{1}{2} |\cos(\pi\alpha)| \|u\|_\alpha^2 - \frac{\lambda k}{2L} \|u\|_{L^2}^2 - \lambda T \varepsilon. \end{aligned} \quad (42)$$

Obviously, if $k \leq 0$, it follows that $\Phi(u) - \lambda\Psi(u) \rightarrow +\infty$, as $\|u\|_\alpha \rightarrow +\infty$.

On the other hand, if $k > 0$, from (13) and (30), then we have

$$\begin{aligned} \Phi(u) - \lambda\Psi(u) &\geq \frac{1}{2} |\cos(\pi\alpha)| \|u\|_\alpha^2 - \frac{\lambda k}{2L} \|u\|_{L^2}^2 - \lambda T \varepsilon \\ &\geq \frac{1}{2} |\cos(\pi\alpha)| \|u\|_\alpha^2 \\ &\quad - \frac{\lambda k}{2L} \frac{T^{2\alpha}}{\Gamma^2(\alpha+1)} \|u\|_\alpha^2 - \lambda T \varepsilon \\ &= \frac{1}{2} \left(|\cos(\pi\alpha)| - \frac{\lambda k}{2} \right) \|u\|_\alpha^2 - \lambda T \varepsilon. \end{aligned} \quad (43)$$

For $\lambda \in]\|u\|_\alpha^2/2|\cos(\pi\alpha)|\Psi(u), r/\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)[\subseteq]\Phi(u)/\Psi(u), r/\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)[$, by (40), we can deduce

$$\begin{aligned} \Phi(u) - \lambda\Psi(u) &\geq \frac{1}{2} \left(|\cos(\pi\alpha)| - \frac{\lambda k}{2} \right) \|u\|_\alpha^2 - \lambda T \varepsilon \\ &\geq \frac{1}{2} |\cos(\pi\alpha)| \left(1 - \frac{\lambda}{2} \frac{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)}{r} \right) \|u\|_\alpha^2 \\ &\quad - \lambda T \varepsilon \geq \frac{1}{4} |\cos(\pi\alpha)| \|u\|_\alpha^2 - \lambda T \varepsilon \longrightarrow +\infty, \end{aligned} \quad (44)$$

$$\|u\|_\alpha \longrightarrow +\infty.$$

Hence the functional $\Phi(u) - \lambda\Psi(u)$ is coercive. So condition (ii) of Theorem 11 holds. Then, by virtue of Theorem 11, we can conclude that the equation $\Phi'(u) - \lambda\Psi'(u) = 0$ has

at least three distinct solutions. That is, the boundary value problem (10) has at least three distinct weak solutions. As a consequence of Lemma 7, we deduce that the boundary value problem (10) has at least three distinct solutions which implies that problem (4) possesses at least three distinct solutions. $\square$

Finally, we give two examples to illustrate the usefulness of our main result. Consider the following impulsive system of fractional differential equations.

Example 15. Consider

$$\begin{aligned} -\frac{1}{2} \frac{d}{dt} ({}_0 D_t^{-0.5} + {}_t D_1^{-0.5}) u'(t) &= \lambda \nabla F(t, u(t)), \\ t &\neq t_k, \text{ a.e., } t \in [0, 1], \\ \Delta(D_t^\alpha u)(t_k) &= \frac{\lambda}{2}, \quad k = 1, \dots, l, \\ u(0) &= u(1) = 0, \end{aligned} \quad (45)$$

where $0 = t_0 < t_1 < t_2 < \dots < t_l < t_{l+1} = 1$. From (45), we know $\beta = 0.5$, $\alpha = 0.75$, $I(u) = 1/2$, $T = 1$. Let $F(t, u(t)) = u^2(t)/18$; we can easily verify that all the conditions of hypothesis (H_1) are satisfied.

Choose $a_1 = 1/16$; it follows that

$$\begin{aligned} \frac{\Gamma^2(\alpha+1) |\cos(\pi\alpha)|}{2T^{2\alpha}} &= \frac{\sqrt{2}}{4} \Gamma^2\left(\frac{3}{2}\right) \approx 0.277621 > a_1 \\ &= \frac{1}{16}. \end{aligned} \quad (46)$$

It is also easy to see

$$\limsup_{|u| \rightarrow \infty} \frac{F(t, u)}{|u|^2} = \frac{1}{18} < a_1 = \frac{1}{16}, \quad (47)$$

which implies that condition (H_2) holds.

Then problem (45) satisfies all the conditions in Theorem 12. In view of Theorem 12, problem (45) has at least one solution for $0 < \lambda < 1$.

Example 16. Consider

$$\begin{aligned} -\frac{1}{2} \frac{d}{dt} ({}_0 D_t^{-0.5} + {}_t D_1^{-0.5}) u'(t) &= \lambda \nabla F(t, u(t)), \\ t &\neq t_k, \text{ a.e., } t \in [0, 1], \\ \Delta(D_t^\alpha u)(t_k) &= \lambda 10^{-6}, \\ k &= 1, \dots, l, \\ u(0) &= u(1) = 0, \end{aligned} \quad (48)$$

where $0 = t_0 < t_1 < t_2 < \dots < t_l < t_{l+1} = 1$. From (48), we can see that $\beta = 0.5$, $\alpha = 0.75$, $I(u) = 10^{-6}$, $T = 1$. Let

$$F(t, u(t)) = \begin{cases} (1+t) 2u^4, & u \leq 1, \\ (1+t) (3u - u^2), & u > 1. \end{cases} \quad (49)$$

Then it is easy to verify that assumption (H_1) holds.

We choose $\omega(t) = t(1-t)\Gamma(1.25)$, $r = 0.001$. By a direct calculation, we can obtain

$$\begin{aligned}
 M &= \frac{T^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|} \approx 1.50177, \\
 L &= \frac{2T^{2\alpha}}{\Gamma^2(\alpha+1)} \approx 2.3677, \\
 \|\omega\|_\alpha^2 &= \int_0^1 \left| {}^c D_t^{0.75} \omega(t) \right|^2 dt \approx 0.158 > \frac{0.004}{\sqrt{2}} = \frac{2r}{|\cos(\pi\alpha)|}, \\
 \frac{\sup_{\xi \in \Omega(Mr)} \left(\int_0^T F(t, \xi) dt - \sum_{j=1}^l \int_0^{\xi(t_j)} I_j(s) ds \right)}{r} &\approx 0.26 < 5.733 \approx \frac{2|\cos(\pi\alpha)|\Psi(\omega)}{\|\omega\|_\alpha^2}, \\
 \lim_{|\xi| \rightarrow +\infty} \frac{F(t, \xi)}{|\xi|^2} &= -1 < 0 < \frac{|\cos(\pi\alpha)| \sup_{u \in \Phi^{-1}([-\infty, r])} \left(\int_0^T F(t, u(t)) dt - \sum_{j=1}^l \int_0^{u(t_j)} I_j(s) ds \right)}{2Lr}.
 \end{aligned} \tag{50}$$

Hence all the assumptions of Theorem 14 are satisfied. Then our results can be applied to problem (48), which shows that problem (48) possesses at least three distinct solutions in E_0^α , for $\lambda \in [0.17443, 38.461538]$.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

Acknowledgments

This work is supported by National Natural Science Foundation of China (nos. 11001274, 11101126, and 11261010), China Postdoctoral Science Foundation (no. 20110491249), Youth Science Foundation of Henan University of Science and Technology (no. 2012QN010), and Innovative Natural Science Foundation of Henan University of Science and Technology (no. 2013ZCX020).

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