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Research Article

Some Fixed Points Results of Quadratic Functions in Split Quaternions

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We attempt to find fixed points of a general quadratic polynomial in the algebra of split quaternion. In some cases, we characterize fixed points in terms of the coefficients of these polynomials and also give the cardinality of these points. As a consequence, we give some simple examples to strengthen the infinitude of these points in these cases. We also find the roots of quadratic polynomials as simple consequences.

1. Introduction

Because of the ever increasing interests in higher dimension spaces and algebras, mathematicians and physicists are in search for developing an easy understanding towards them so that they better understand the physical phenomenons. In an attempt to satisfy this curiosity, they consider different functions to analyze them and also discuss some fixed points of these functions see [1, 2]. These algebras may present nice understanding towards general rotations and describe some easy ways to consider geometric problems and also problems in mechanics and dynamical systems [3–7].

This paper mainly covers split quaternion algebra or coquaternion over real numbers. It was in fact introduced by James Cockle in 1849 on already established quaternions by Hamilton in 1843. Both of these algebras are actually associative but noncommutative and non-division ring generated by four basic elements. Like quaternion, split quaternion also forms a four-dimensional real vector space equipped with a multiplicative operation. But, unlike the quaternion algebra, the split quaternion algebra contains lots of zero divisors, nilpotents, and nontrivial idempotents. For a detailed description of quaternion and its generalization (octonion), see [3, 8–10]. As mathematical structures, both are algebras over the real numbers which are isomorphic to the algebra of 2×2 real matrices. The name split quaternion is used due to the division into positive and negative terms in the modulus function. The set $(1, i \cdot j, k)$ forms a basis. The products of these elements are

$$\begin{aligned} i^2 &= -1, \\ j^2 &= 1 = k^2, \\ ij &= k = -ji, \\ jk &= -i = -kj, \\ ki &= j = -ik, \\ ijk &= 1. \end{aligned} \tag{1}$$

It follows from the defining relations that the set $(\pm 1, \pm i, \pm j, \pm k)$ is a group under split quaternion multiplication; it is

isomorphic to the dihedral group of a square. Equation (2) encodes the multiplication of basis split quaternions:

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	1	$-i$
k	k	j	i	1

(2)

Both of these algebras have a long history and a great deal of literature is at hand on them. Brand in [4] gave the roots of a quaternion. Strictly speaking, he proved mainly De Moivre's theorem and then used it to find nth roots of a quaternion. His approach paved the way for finding roots of a quaternion in an efficient and intelligent way. Recently, authors in [1] considered split quaternionic matrices and discussed their interesting properties. Erdoğdu and Özdemir computed the eigenvalues for these matrices [5]. Ozdemir also discussed the roots of split quaternions in [2]. He also gave some geometrical applications of split quaternions in [6]. Rotations in Minkowski 3-space can be explained easily with split quaternions, such as Euclidean rotations used with quaternions [7]. So, it is very much possible to perform many things with split quaternions which can ordinarily be done using vectors analysis with Lorentzian metric and vector products. It is important to mention that these two algebras can also be constructed for $\mathbb{Z}_p$ over prime finite fields of characteristic P. Then, we obtain finite families of algebras with entirely different properties. For these finite algebras, see [11].

In the present paper, we first obtain the roots of a general quadratic polynomial in algebra of split quaternion. As simple implications, we state that there are infinitely many roots for some special cases. We also associate some general functions with these polynomials and explore their fixed points. In the end, we give some nice examples for each case. First, we make a preliminary introduction which will be used for the rest of this paper.

Definition 1. Let $x \in \mathbb{H}_s$; then, x has the form $x = a_0 + a_1 i + a_2 j + a_3 k$, where $a_i \in \mathbb{R}$. The *conjugate of* x is defined as $x = a_0 - a_1 i - a_2 j - a_3 k$. The norm of x is given by

$$N(x) = a_0^2 + a_1^2 - a_3^2 - a_4^2. \tag{3}$$

We are first interested in explicit formulas for computing the roots of a quadratic polynomial of the form $x^2 + bx + c$, where $b, c \in H_s$. Let us fix $x = x_0 + x_1 i + x_2 j + x_3 k$, $b = b_0 + b_1 i + b_2 j + b_3 k$, and $c = c_0 + c_1 i + c_2 j + c_3 k$. So, $x^2 + bx + c = 0$ becomes the real system of nonlinear equations

$$\begin{aligned}
&(x_0 + x_1 i + x_2 j + x_3 k)^2 \\
&\quad + (b_0 + b_1 i + b_2 j + b_3 k)(x_0 + x_1 i + x_2 j + x_3 k) + c_0 \\
&\quad + c_1 i + c_2 j + c_3 k = 0,
\end{aligned}$$

$$\begin{aligned}
&x_0^2 - x_1^2 + x_2^2 + x_3^2 + b_0 x_0 - b_1 x_1 + b_2 x_2 + b_3 x_3 \\
&\quad + c_0 = 0, \\
&2x_0 x_1 + b_1 x_0 + b_0 x_1 + b_3 x_2 - b_2 x_3 + c_1 = 0, \\
&2x_0 x_2 + b_2 x_0 + b_3 x_1 + b_0 x_2 - b_1 x_3 + c_2 = 0, \\
&2x_0 x_3 + b_3 x_0 + b_1 x_2 + b_0 x_3 - b_2 x_1 + c_3 = 0.
\end{aligned} \tag{4}$$

We obtain five equations, and by no means is it obvious that this nonlinear system will have an explicit solution. There have been several attempts in the literature for the case of quaternion. Historically speaking, split quaternion preceded Cayley's matrix algebra; split quaternion (along with quaternion and tessarines) evokes the broader Linear Algebra.

2. Main Results

We first solve a general quadratic polynomial in the algebra of split quaternion. As a consequence, we find fixed points of associated functions in this algebra.

Proposition 2. *$x = (-b/2 + \beta i + \gamma j + \delta k)$ is a root of $x^2 + bx + c = 0$ with $b^2 < 4c$ and $b, c \in \mathbb{R}$, where $\beta^2 - \gamma^2 - \delta^2 = (4c - b^2)/4$ and $\beta, \gamma, \delta \in \mathbb{R}$.*

Proof. Take $x = \alpha + \beta i + \gamma j + \delta k$ in $x^2 + bx + c = 0$ given that $b, c \in \mathbb{R}$ and $b^2 < 4c$.

We get $(\alpha + \beta i + \gamma j + \delta k)^2 + b(\alpha + \beta i + \gamma j + \delta k) + c = 0$. Consider

$$\begin{aligned}
&\alpha^2 - \beta^2 + \gamma^2 + \delta^2 + 2\alpha\beta i + 2\alpha\gamma j + 2\alpha\delta k + b\alpha + b\beta i \\
&\quad + b\gamma j + b\delta k + c = 0 \\
&\alpha^2 - \beta^2 + \gamma^2 + \delta^2 + b\alpha + c = 0 \\
&2\alpha\beta + b\beta = 0 \\
&2\alpha\gamma + b\gamma = 0 \\
&2\alpha\delta + b\delta = 0 \\
&\beta \neq 0 \\
&\text{so } 2\alpha + b = 0 \Longrightarrow \alpha = -\frac{b}{2};
\end{aligned} \tag{5}$$

put $\alpha = -b/2$; consider

$$\begin{aligned}
\left(-\frac{b}{2}\right)^2 - \beta^2 + \gamma^2 + \delta^2 + b\left(-\frac{b}{2}\right) + c = 0 \\
\frac{b^2}{4} - \beta^2 + \gamma^2 + \delta^2 - \frac{b^2}{2} + c = 0 \\
\beta^2 - \gamma^2 - \delta^2 = \frac{4c - b^2}{4},
\end{aligned} \tag{6}$$

where

$$\beta, \gamma, \delta \in \mathbb{R}; \tag{7}$$

hence, $x = (-b/2+\beta i+\gamma j+\delta k)$, where $\beta^2-\gamma^2-\delta^2 = (4c-b^2)/4$ and $\beta, \gamma, \delta \in \mathbb{R}$.

Example. Consider $x^2 - x + 1 = 0, b = -1,\ c = 1, 4c - b^2 = 3,$

$$x = \frac{1}{2}(-b + \beta i + \gamma j + \delta k),$$
$$\beta^2 - \gamma^2 - \delta^2 = \frac{4c - b^2}{4} = \frac{3}{4} \tag{8}$$
$$\text{this implies } \beta^2 = \frac{3}{4} + \gamma^2 + \delta^2.$$

It is satisfied for all choices of $\gamma, \delta \in \mathbb{R}$; let $\gamma = 1, \delta = 1$; then $\beta = \sqrt{11}/2$, and

$$x = \left(\frac{1}{2} + \frac{\sqrt{11}}{2} i + j + k\right). \tag{9}$$

□

Corollary 3. *The quadratic equation $x^2 + bx + c = 0$ has infinitely many solutions for $b^2 < 4c$ and $\beta \neq 0$, in any case. If $\beta = 0$, then $4c - b^2 = -4(\gamma^2 + \delta^2)$ is absurd.*

Theorem 4. *If $b, c \in \mathbb{R}$ and $b^2 < 4c$, then $x = (-b/2+\beta i+\gamma j+\delta k)$ is a fixed point of a class of functions of the form $f(x) = x^2+(b+1)x+c$, where $\beta^2-\gamma^2-\delta^2 = (4c-b^2)/4$ and $\beta, \gamma, \delta \in \mathbb{R}$.*

Proof. It is enough to give a new relation $f(x) = g(x) + x$, where $g(x) = x^2+bx+c$; then, the existence of fixed points for $f(x)$ is equivalent to the solutions of $g(x)$. Then, the required result is immediate from Proposition 2. □

Proposition 5. *If $b, c \in \mathbb{R}$ and $b^2 \geq 4c$, then $x = (-b/2 + \beta i + \gamma j+\delta k)$ is a root of $x^2+bx+c = 0$, where $\beta^2-\gamma^2-\delta^2 = (4c-b^2)/4$ and $\beta, \gamma, \delta \in \mathbb{R}$.*

Proof. Consider $b, c \in \mathbb{R}$ and $b^2 \geq 4c$.

Put $x = \alpha + \beta i + \gamma j + \delta k$ in $x^2 + bx + c = 0$:

$$\begin{aligned}
&(\alpha + \beta i + \gamma j + \delta k)^2 + b(\alpha + \beta i + \gamma j + \delta k) + c = 0\\
&\alpha^2 - \beta^2 + \gamma^2 + \delta^2 + 2\alpha\beta i + 2\alpha\gamma j + 2\alpha\delta k + b\alpha + b\beta i\\
&\quad + b\gamma j + b\delta k + c = 0\\
&\alpha^2 - \beta^2 + \gamma^2 + \delta^2 + b\alpha + c = 0\\
&2\alpha\beta + b\beta = 0\\
&2\alpha\gamma + b\gamma = 0\\
&2\alpha\delta + b\delta = 0\\
&\beta \neq 0\\
&\text{so } 2\alpha + b = 0 \Longrightarrow \alpha = -\frac{b}{2};
\end{aligned} \tag{10}$$

put $\alpha = -b/2$; hence,

$$\begin{aligned}
\left(-\frac{b}{2}\right)^2 - \beta^2 + \gamma^2 + \delta^2 + b\left(-\frac{b}{2}\right) + c &= 0\\
\frac{b^2}{4} - \beta^2 + \gamma^2 + \delta^2 - \frac{b^2}{2} + c &= 0\\
-\beta^2 + \gamma^2 + \delta^2 &= \frac{b^2 - 4c}{4},
\end{aligned} \tag{11}$$

where

$$\beta, \gamma, \delta \in \mathbb{R}; \tag{12}$$

hence, $x = (-b/2 + \beta i + \gamma j + \delta k)$, where $b^2 \geq 4c$ and $-\beta^2 + \gamma^2 + \delta^2 = b^2 - 4c$. □

Example 6. Consider $x^2 - 2x + 1 = 0$; here, $b^2 - 4c = 4 - 4 = 0$ and $\beta^2 = \gamma^2 + \delta^2$; we can choose infinitely many values of $\beta, \gamma, \delta \in \mathbb{R}$; let $\gamma = 1 = \delta$; then, $\beta = \sqrt{2}$; then $x = (-b/2 + \beta i + \gamma j + \delta k) = (1/2 + \sqrt{2}i + j + k)$.

Corollary 7. *It has infinitely many solutions for $b^2 - 4c \geq 0$.*

Theorem 8. *$x = (-b/2+\beta i+\gamma j+\delta k)$ with $b, c \in \mathbb{R}$ and $b^2 \geq 4c$, where $\beta^2 - \gamma^2 - \delta^2 = (4c - b^2)/4$ and $\beta, \gamma, \delta \in \mathbb{R}$, is a fixed point of a class of functions of the form $f(x) = x^2 + (b+1)x + c$, where $\beta^2 - \gamma^2 - \delta^2 = (4c - b^2)/4$ and $\beta, \gamma, \delta \in \mathbb{R}$.*

Proof. Again, it is enough to use relation $f(x) = g(x) + x$, where $g(x) = x^2+bx+c$; then, the existence of fixed points for $f(x)$ is equivalent to the solutions of $g(x)$. Then, the required result is immediate from Proposition 5. □

Proposition 9. *If $b \in \mathbb{R}$ and $c \notin \mathbb{R}$, then $x = (-b/2 \pm \rho/2) \mp (c_1/\rho)i \mp (c_2/\rho)j \mp (c_3/\rho)k$ is a root of equation $x^2 + bx + c = 0$, where $c = c_0 + c_1 i + c_2 j + c_3 k$ and $\beta = -c_1/(2\alpha + b)$, $\gamma = -c_2/(2\alpha + b)$, $\delta = -c_3/(2\alpha + b)$, $\sqrt{(b^2 - 4c_0 \pm \sqrt{(4c_0 - b^2)^2 + 16(c_1^2 - c_2^2 - c_3^2)})/2} = \rho$, and $\operatorname{Re} x = \alpha$.*

Proof. If $b \in \mathbb{R}$ and $c \notin \mathbb{R}$,

$$\begin{aligned}
&x = \alpha + \beta i + \gamma j + \delta k,\\
&c = c_0 + c_1 i + c_2 j + c_3 k,\\
&(\alpha + \beta i + \gamma j + \delta k)^2 + b(\alpha + \beta i + \gamma j + \delta k) + c_0 + c_1 i\\
&\quad + c_2 j + c_3 k = 0,\\
&\alpha^2 - \beta^2 + \gamma^2 + \delta^2 + b\alpha = -c_0,\\
&2\alpha\beta + b\beta = -c_1,
\end{aligned}$$

$$\begin{aligned}
&2\alpha\gamma + b\gamma = -c_2,\\
&2\alpha\delta + b\delta = -c_3,\\
&\qquad \beta = -\frac{c_1}{2\alpha+b},\ \gamma = -\frac{c_2}{2\alpha+b},\ \delta = -\frac{c_3}{2\alpha+b},
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\alpha^2 - \left(-\frac{c_1}{2\alpha+b}\right)^2 + \left(-\frac{c_2}{2\alpha+b}\right)^2 + \left(-\frac{c_3}{2\alpha+b}\right)^2\\
&\quad + b\alpha = -c_0,\\
&\alpha^2 + b\alpha - \frac{c_1^2}{(2\alpha+b)^2} + \frac{c_2^2}{(2\alpha+b)^2} + \frac{c_3^2}{(2\alpha+b)^2} = -c_0,\\
&\left(\alpha + \frac{b}{2}\right)^2 - \frac{c_1^2}{(2\alpha+b)^2} + \frac{c_2^2}{(2\alpha+b)^2}\\
&\quad + \frac{c_3^2}{(2\alpha+b)^2} = -c_0 + \frac{b^2}{4},\\
&\frac{(2\alpha+b)^2}{4}(2\alpha+b)^2 - c_1^2 + c_2^2\\
&\quad + c_3^2 = (2\alpha+b)^2\left(\frac{b^2}{4} - c_0\right),\\
&(2\alpha+b)^4 - 4\left(c_1^2 - c_2^2 - c_3^2\right) = (2\alpha+b)^2\left(b^2 - 4c_0\right),\\
&(2\alpha+b)^4 + \left(4c_0 - b^2\right)(2\alpha+b)^2\\
&\quad - 4\left(c_1^2 - c_2^2 - c_3^2\right) = 0.
\end{aligned} \tag{14}$$

It is quadratic in $(2\alpha+b)^2$; hence,

$$\begin{aligned}
&(2\alpha+b)^2\\
&\quad = \frac{-\left(4c_0 - b^2\right) \pm \sqrt{\left(4c_0 - b^2\right)^2 + 16\left(c_1^2 - c_2^2 - c_3^2\right)}}{2},\\
&2\alpha + b\\
&\quad = \pm\sqrt{\frac{b^2 - 4c_0 \pm \sqrt{\left(4c_0 - b^2\right)^2 + 16\left(c_1^2 - c_2^2 - c_3^2\right)}}{2}};
\end{aligned} \tag{15}$$

let

$$\begin{aligned}
&\sqrt{\frac{b^2 - 4c_0 \pm \sqrt{\left(4c_0 - b^2\right)^2 + 16\left(c_1^2 - c_2^2 - c_3^2\right)}}{2}} = \rho\\
&2\alpha + b = \pm\rho,\\
&2\alpha = \pm\rho - b,\\
&\alpha = \pm\frac{\rho}{2} - \frac{b}{2},\\
&x = \alpha + \beta i + \gamma j + \delta k,\\
&x = \alpha - \frac{c_1}{2\alpha+b}i - \frac{c_2}{2\alpha+b}j - \frac{c_3}{2\alpha+b}k,\\
&x = \left(-\frac{b}{2} \pm \frac{\rho}{2}\right) - \frac{c_1}{\pm\rho}i - \frac{c_2}{\pm\rho}j - \frac{c_3}{\pm\rho}k,\\
&x = \left(-\frac{b}{2} \pm \frac{\rho}{2}\right) \mp \frac{c_1}{\rho}i \mp \frac{c_2}{\rho}j \mp \frac{c_3}{\rho}k.
\end{aligned} \tag{16}$$

□

Example 10. One has $x^2 + 3x + (2 + \sqrt{2}i + j + k) = 0$ with $b^2 - 4c_0 = 1 > 0$ and $\rho = 1/\sqrt{2}$; then $x = (-3/2 \pm 1/2\sqrt{2}) \mp 2i \mp \sqrt{2}j \mp \sqrt{2}k$.

Theorem 11. *If $x = (-b/2 \pm \rho/2) \mp (c_1/\rho)i \mp (c_2/\rho)j \mp c_3/\rho$ is a fixed point of functions $x^2 + (b+1)x + c$, where $c = c_0 + c_1 i + c_2 j + c_3 k$, $b \in \mathbb{R}$, and $c \notin \mathbb{R}$ and $\beta = -c_1/(2\alpha+b)$, $\gamma = -c_2/(2\alpha+b)$, $\delta = -c_3/(2\alpha+b)$, $\sqrt{(b^2 - 4c_0 \pm \sqrt{(4c_0 - b^2)^2 + 16(c_1^2 - c_2^2 - c_3^2)})/2} = \rho$, and $\operatorname{Re} x = \alpha$.*

Proof. The proof follows as a simple result from Proposition 9 using the same argument as given in previous theorems. □

2.1. Open Question and Discussion. We obtained some fixed point results relating to a general quadratic polynomial in algebra of split quaternions. We however remark that if we take $x^2 + bx + c = 0$ with $b, c \notin \mathbb{R}$ and $y^2 + \acute{b}y + \acute{c} = 0$ with

$$\begin{aligned}
&y = x + \frac{\operatorname{Re} b}{2} = x + \frac{b_0}{2},\\
&\acute{b} = b - \operatorname{Re} b = b_1 i + b_2 j + b_3 k,\\
&\acute{c} = c - \frac{\operatorname{Re} b}{2}\left(b - \frac{\operatorname{Re} b}{2}\right) = c - \frac{b_0}{2}\left(b - \frac{b_0}{2}\right),\\
&\acute{c} = c - \frac{b_0}{4}\left(b_0 + 2b_1 i + 2b_2 j + 2b_3 k\right),
\end{aligned} \tag{17}$$

after putting all these values in $y^2 + \acute{b}y + \acute{c} = 0$, we get $x^2 + bx + c = 0$. Solution of the quadratic equation $y^2 + \acute{b}y + \acute{c} = 0$ also satisfies $y^2 - Jy + K = 0$ with $K = \overline{y}y \geq 0$ for $\beta^2 > \gamma^2 + \delta^2$ and $J = y + \overline{y} \in \mathbb{R}$. Hence,

$$\begin{aligned}
&\left(\acute{b} + J\right)y + (\acute{c} - K) = 0,\\
&\qquad y = -\left(\acute{b} + J\right)^{-1}(\acute{c} - K)\\
&\qquad J \in \mathbb{R},\ \acute{b} = b - \operatorname{Re} b \notin \mathbb{R},\ \acute{b} + J \neq 0;
\end{aligned} \tag{18}$$

$K = y\overline{y}$ implies $K^2 - K(B + J^2) + E = 0$, where $B = \acute{b}\overline{\acute{b}} + \acute{c} + \overline{\acute{c}}$, $|\acute{c}|^2 = E \in \mathbb{R}$.

$J = y + \overline{y}$ implies $J^3 + J(\acute{B} - 2K) + D = 0$, where $\acute{B} = -\acute{b}\overline{\acute{b}} + \acute{c} + \acute{c}$, $D = \acute{b}\overline{\acute{c}}, + \acute{b}\acute{c}$. Taking the above remark into consideration, we leave this case as an open problem, where we have $b, c \notin \mathbb{R}$.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final paper.

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