Research Article

Explicit Bounds and Sharp Results for the Composition Operators Preserving the Exponential Class

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Let $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a *quasiconformal mapping* whose Jacobian is denoted by J_f and let $\text{EXP}(\Omega)$ be the space of exponentially integrable functions on Ω . We give an explicit bound for the norm of the composition operator $T_f : u \in \text{EXP}(\Omega) \mapsto u \circ f^{-1} \in \text{EXP}(f(\Omega))$ and, as a related question, we study the behaviour of the norm of $\log J_f$ in the exponential class. The A_∞ property of J_f is the counterpart in higher dimensions of the area distortion formula due to Astala in the plane and it is the key tool to prove the sharpness of our results.

1. Introduction and Statement of the Main Result

This paper is concerned with the interplay between quasiconformal mappings and the space of exponentially integrable functions. Let us recall that a homeomorphism $f : \Omega \rightarrow \mathbb{R}^n$ defined on an open subset Ω of $\mathbb{R}^n$ (with $n \geq 2$) is a *K-quasiconformal mapping* if $f \in W_{\text{loc}}^{1,n}(\Omega, \mathbb{R}^n)$ and

$$|Df(x)|^n \leq K J_f(x) \quad \text{for a.e. } x \in \Omega, \quad (1)$$

for some constant $K \geq 1$. Here $Df(x)$ stands for the *differential matrix* of f and $J_f(x) = \det Df(x)$ denotes the *Jacobian determinant* of f . The norm $|Df(x)|$ of $Df(x)$ in (1) is defined as $|Df(x)| = \sup\{|Df(x)\xi| : \xi \in \mathbb{R}^n, |\xi| = 1\}$.

If Ω is a bounded domain of $\mathbb{R}^n$ with measure $|\Omega|$, the space of exponentially integrable functions $\text{EXP}(\Omega)$ is the set of measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that there exists $\lambda > 0$ for which

$$\int_{\Omega} \exp \frac{|u(x)|}{\lambda} dx < \infty, \quad (2)$$

where the mean value notation $\int_{\Omega} = (1/|\Omega|) \int_{\Omega}$ is used. One of the interesting properties of functions in $\text{EXP}(\Omega)$ consists

in the fact that they may be characterized as BMO-majorized functions. Indeed, in [1] it is proved that $u \in \text{EXP}(\Omega)$ if and only if there exists $v \in \text{BMO}(\mathbb{R}^n)$ such that

$$|u(x)| \leq v(x) \quad \text{a.e. } x \in \Omega. \quad (3)$$

For the definition of the space $\text{BMO}(\mathbb{R}^n)$ of functions of bounded mean oscillation see Section 2 below.

Quasiconformal mappings $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and BMO-functions $u : \mathbb{R}^n \rightarrow \mathbb{R}$ are related by the fact that the composition operator $u \mapsto u \circ f^{-1}$ maps $\text{BMO}(\mathbb{R}^n)$ into itself continuously, as stated by a result of Reimann [2]: there exists $C = C(n, K) \geq 1$ such that, for every $u \in \text{BMO}(\mathbb{R}^n)$, one has

$$\begin{aligned} \frac{1}{C(n, K)} \|u\|_{\text{BMO}(\mathbb{R}^n)} &\leq \|u \circ f^{-1}\|_{\text{BMO}(\mathbb{R}^n)} \\ &\leq C(n, K) \|u\|_{\text{BMO}(\mathbb{R}^n)}. \end{aligned} \quad (4)$$

In light of the connection between exponentially integrable functions and functions of bounded mean oscillation, composition operators acting continuously on EXP have been considered in [3, 4], where it is proved that, given a K -quasiconformal mapping $f : \Omega \rightarrow \mathbb{R}^n$, there exists

$C = C(n, K) \geq 1$ such that, for every $u \in \text{EXP}(\Omega)$ and for every ball $B \subset \subset \Omega$, one has

$$\begin{aligned} \frac{1}{C(n, K)} \|u\|_{\text{EXP}(B)} &\leq \|u \circ f^{-1}\|_{\text{EXP}(f(B))} \\ &\leq C(n, K) \|u\|_{\text{EXP}(B)}. \end{aligned} \quad (5)$$

The estimates above may be seen as the analogy of (4) in the framework of the space of exponentially integrable functions.

It is worth pointing out that spaces of functions of bounded mean oscillation and exponentially integrable functions are not the only ones which are stable under quasiconformal changes of variables. We recall that quasiconformal mappings and their generalizations provided by homeomorphisms of finite distortion or bi-Sobolev mappings (see, e.g., [5, 6]), turn to be the class of homeomorphisms for which the composition operator acts continuously between Sobolev, Lorentz-Sobolev, and Zygmund-Sobolev spaces (see [7–9] and the references therein).

Explicit estimates of the constants appearing in (4) and (5) have been considered a problem of its own interest (see, e.g., [10] for an application). In the planar case, sharp estimates for the constant $C(2, K)$ appearing in (5) are given in [4]. As a refinement of the result of Reimann, explicit estimates of the constant $C(n, K)$ appearing in (4) are provided in [11]. More precisely, a constant $\widehat{G}_1(J_f)$ depending on the Jacobian of f may be defined (see Section 2.4 below) in such a way that the results of [11] may be stated as follows

$$\|u \circ f^{-1}\|_{\text{BMO}(\mathbb{R}^n)} \leq C(n) \widehat{G}_1(J_f) \|u\|_{\text{BMO}(\mathbb{R}^n)}, \quad (6)$$

for every $u \in \text{BMO}(\mathbb{R}^n)$ and for some suitable constant $C = C(n) \geq 1$ depending only on the dimension n . The definition of the constant $\widehat{G}_1(J_f)$ strongly relies on the fact that the Jacobian of a quasiconformal mapping is a weight in A_∞ or equivalently in G_1 (see Section 2.4 for the definitions of such classes of weights). In particular, it is well known (see [12]) that $v \in G_1$ if and only if for every ball $B \subset \mathbb{R}^n$ and every measurable set $F \subset B$ it holds

$$\frac{\int_F v(x) dx}{\int_B v(x) dx} \leq L \left(\frac{|F|}{|B|} \right)^\beta, \quad (7)$$

for some $0 < \beta \leq 1 \leq L$ independent of F and B . Equivalently (see again [12]) a weight $w \in A_\infty$ if and only if for every ball $B \subset \mathbb{R}^n$ and every measurable set $E \subset B$ it holds

$$\frac{|E|}{|B|} \leq M \left(\frac{\int_E w(x) dx}{\int_B w(x) dx} \right)^\alpha, \quad (8)$$

for some $0 < \alpha \leq 1 \leq M$ independent of E and B .

The goal of this paper is to seek a precise estimate as in (6) in the framework of the class of exponentially integrable functions. To this aim, we define for a weight v in G_1 the constant $\widehat{G}_1(v)$ as

$$\begin{aligned} \widehat{G}_1(v) &= \inf \left\{ \max \left\{ \frac{1}{\beta}, 1 + \frac{\log L}{\beta} \right\} : 0 < \beta \leq 1 \right. \\ &\quad \left. \leq L \text{ and (7) holds} \right\}, \end{aligned} \quad (9)$$

and similarly, we define for a weight w in A_∞ the constant $\widehat{A}_\infty(w)$ as

$$\begin{aligned} \widehat{A}_\infty(w) &= \inf \left\{ \max \left\{ \frac{1}{\alpha}, 1 + \frac{\log M}{\alpha} \right\} : 0 < \alpha \leq 1 \right. \\ &\quad \left. \leq M \text{ and (8) holds} \right\}. \end{aligned} \quad (10)$$

Our main result reads as follows.

Theorem 1. *Let Ω, Ω' be bounded domains of $\mathbb{R}^n$, $n \geq 2$. Let $f : \Omega \rightarrow \Omega'$ be a K -quasiconformal mapping. Then*

$$\begin{aligned} \frac{1}{\widehat{A}_\infty(J_f)} \|u\|_{\text{EXP}(B)} &\leq \|u \circ f^{-1}\|_{\text{EXP}(f(B))} \\ &\leq \widehat{G}_1(J_f) \|u\|_{\text{EXP}(B)}, \end{aligned} \quad (11)$$

for each ball $B \subset \subset \Omega$ and for every $u \in \text{EXP}(\Omega)$.

The norm $\|\cdot\|_{\text{EXP}(B)}$ is defined in (16) of Section 2.2 below.

Our result is sharp, in the sense that equalities are attained in (11) for special choices of the mapping f (e.g., when f is the identity map). We are able to obtain such optimal result since we give a characterization of constant weights in terms of the constants in (9) and (10) in Section 4. In particular, a G_1 weight v is constant a.e. in $\mathbb{R}^n$ if and only if $\widehat{G}_1(v) = 1$ and $\widehat{A}_\infty(v) = 1$. Moreover, the result of Theorem 1 extends the one of [3], where the case of planar principal quasiconformal mappings has been considered. We call *principal* any quasiconformal mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which is conformal outside the unit disk $\mathbb{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$ and which satisfies the following normalization

$$|f(x) - x| = \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{if } |x| > 1. \quad (12)$$

For this peculiar class of quasiconformal mappings, the following result has been previously established.

Theorem 2 (see [3]). *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a K -quasiconformal principal mapping which maps $\mathbb{D}$ onto itself. Then, the following estimates hold true*

$$\begin{aligned} \frac{1}{1 + K \log K} \|u\|_{\text{EXP}(\mathbb{D})} &\leq \|u \circ f^{-1}\|_{\text{EXP}(\mathbb{D})} \\ &\leq (1 + K \log K) \|u\|_{\text{EXP}(\mathbb{D})}, \end{aligned} \quad (13)$$

for every $u \in \text{EXP}(\mathbb{D})$.

We will provide an alternative proof of the previous result, which is based on the fact that estimate (11) reduces to (13) for a principal quasiconformal mapping which maps the unit disk onto itself. In general, a principal quasiconformal map in the plane does not necessarily send the unit disk onto itself. However, there exist nontrivial examples of principal

quasiconformal mappings sending the unit disk onto itself, such as the radial stretching $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of the form

$$\begin{aligned} f(x) &= |x|^K \frac{x}{|x|} \quad \text{for } x \in \mathbb{D}, \\ f(x) &= x \quad \text{for } x \in \mathbb{R}^2 \setminus \mathbb{D}. \end{aligned} \quad (14)$$

The paper is organized as follows. Section 2 is devoted to the definitions of the function spaces object of our studies. In particular, the connection between BMO-functions and A_p -weights is treated. In Section 3 we give the proof of Theorem 1. In Section 4 we give the aforementioned characterization of the constant weights, which allows us to conclude the sharpness of Theorem 1. Finally, in Section 5 applications of Theorem 1 are given; in particular we provide a precise estimate which relates $\|\log J_{f^{-1}}\|_{\text{EXP}(f(B))}$ and $\|\log J_f\|_{\text{EXP}(B)}$ for any ball $B \subset \subset \Omega$.

2. Preliminaries

2.1. Properties of Quasiconformal Mappings. We report here some well-known facts about quasiconformal mappings. We recall that the change of variables formula holds for a quasiconformal mapping $f: \Omega \rightarrow \Omega'$. More precisely, if $\varphi \in L^1_{\text{loc}}(\Omega')$ then $(\varphi \circ f)J_f \in L^1_{\text{loc}}(\Omega)$ and

$$\int_E \varphi(f(x))J_f(x)dx = \int_{f(E)} \varphi(y)dy, \quad (15)$$

for every measurable $E \subset \subset \Omega$ (see, e.g., [13]). More generally, for an arbitrary Sobolev homeomorphism, the validity of the change of variables formula depends on the set of the points where the homeomorphism f is approximately differentiable (see, e.g., [14]); more generally the condition of being absolutely continuous on lines plays an important role, especially for planar mappings (see, e.g., [15]), since very often properties of this type of mappings are sufficient to prove statements which one would assign to general Sobolev homeomorphisms.

2.2. Exponentially Integrable Functions. We recall (see, e.g., [16]) that $\text{EXP}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{\text{EXP}(\Omega)} = \sup_{0 < t < |\Omega|} \frac{u^*(t)}{(1 + \log(|\Omega|/t))}, \quad (16)$$

where u^* is the *nonincreasing rearrangement* of u

$$u^*(t) = \sup \{ \tau \geq 0 : \mu_u(\tau) > t \} \quad \text{for every } t \in (0, |\Omega|), \quad (17)$$

and μ_u is the *distribution function* of u

$$\mu_u(\tau) = |\{x \in \Omega : |u(x)| > \tau\}| \quad \text{for every } \tau \geq 0. \quad (18)$$

On the other hand, $\text{EXP}(\Omega)$ may be also equipped with the *Luxemburg norm* defined as

$$\|u\|_{\text{EXP}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \exp \frac{|u(x)|}{\lambda} dx \leq 2 \right\}. \quad (19)$$

This norm is equivalent to the one in (16). As observed in [17], $L^\infty(\Omega)$ is not a dense subspace of $\text{EXP}(\Omega)$. The distance to $L^\infty(\Omega)$ in the space $\text{EXP}(\Omega)$ is defined as

$$\text{dist}_{\text{EXP}(\Omega)}(u, L^\infty(\Omega)) = \inf_{\varphi \in L^\infty(\Omega)} \|u - \varphi\|_{\text{EXP}(\Omega)}. \quad (20)$$

Appealing to the results in [17] the distance to $L^\infty(\Omega)$ in $\text{EXP}(\Omega)$ evaluated with respect to the Luxemburg norm (19) is given by

$$\begin{aligned} \text{dist}_{\text{EXP}(\Omega)}(u, L^\infty(\Omega)) \\ = \inf \left\{ \lambda > 0 : \int_{\Omega} \exp \frac{|u(x)|}{\lambda} dx < \infty \right\}, \end{aligned} \quad (21)$$

for every $u \in \text{EXP}(\Omega)$.

2.3. Functions of Bounded Mean Oscillation. A locally integrable function $u: \Omega \rightarrow \mathbb{R}$ has bounded mean oscillation, $u \in \text{BMO}(\Omega)$, if

$$\|u\|_{\text{BMO}(\Omega)} = \sup_B \int_B |u(x) - u_B| dx < \infty. \quad (22)$$

The supremum in (22) is taken over all open balls $B \subset \Omega$ and the notation

$$u_B = \int_B u(x) dx, \quad (23)$$

is used for averages.

2.4. A_p and G_q Classes. For our purposes, it is fundamental to introduce the *Muckenhoupt class* A_∞ and the *Gehring class* G_1 . First of all, we say that a measurable function $w: \mathbb{R}^n \rightarrow \mathbb{R}$ is a *weight* if w is positive a.e. and locally integrable in $\mathbb{R}^n$. A weight w belongs to the *Muckenhoupt class* A_∞ if

$$A_\infty(w) = \sup_B \left(\int_B w dx \right) \left(\exp \int_B \log \frac{1}{w} dx \right) < \infty. \quad (24)$$

Similarly, a weight v belongs to the *Gehring class* G_1 if

$$G_1(v) = \sup_B \left(\exp \int_B \frac{v}{v_B} \log \frac{v}{v_B} dx \right) < \infty. \quad (25)$$

The suprema in (24) and (25) are taken over all balls $B \subset \mathbb{R}^n$.

The link between Muckenhoupt and Gehring classes is given in [18, 19] where it is proved that

$$A_\infty = G_1. \quad (26)$$

As a corollary of Gehring's Lemma [20], Jacobians of quasiconformal mappings are weights in the A_∞ (or equivalently G_1) class. By virtue of the change of variables formula for quasiconformal mappings (see Section 2.1 above) and (7), for every ball $B \subset \mathbb{R}^n$ and every measurable set $F \subset B$ it holds

$$\frac{|f(F)|}{|f(B)|} \leq L \left(\frac{|F|}{|B|} \right)^\beta, \quad (27)$$

for some $0 < \beta \leq 1 \leq L$ independent of F and B .

For a weight ν in G_1 , in [21] (and also studied in [22]) the constant $\widetilde{G}_1(\nu)$ is defined as

$$\widetilde{G}_1(\nu) = \inf \left\{ \frac{L}{\beta} : 0 < \beta \leq 1 \leq L \text{ and (7) holds} \right\}. \quad (28)$$

We briefly refer to $\widetilde{G}_1(\nu)$ as the $\widetilde{G}_1$ -constant of ν .

As for (27), the change of variables formula for quasiconformal mappings and (8) immediately give us, for every ball $B \subset \mathbb{R}^n$ and every measurable set $E \subset B$

$$\frac{|E|}{|B|} \leq M \left(\frac{|f(E)|}{|f(B)|} \right)^\alpha, \quad (29)$$

for some $0 < \alpha \leq 1 \leq M$ independent of E and B .

As done for the $\widetilde{G}_1$ -constant, in [21] a second auxiliary constant is defined as

$$\begin{aligned} \widetilde{A}_\infty(w) \\ = \inf \left\{ \frac{M}{\alpha} : 0 < \alpha \leq 1 \leq M \text{ and (8) holds} \right\}. \end{aligned} \quad (30)$$

We briefly refer to $\widetilde{A}_\infty(w)$ as the $\widetilde{A}_\infty$ -constant of w .

It is worth pointing out that to each function $u \in \text{BMO}(\mathbb{R}^n)$ there corresponds a weight in the G_1 class of Gehring given by $e^{\delta u}$, for some $\delta > 0$ depending on n and $\|u\|_{\text{BMO}}$ (see [23]). Conversely (see again [23]) if $\nu \in G_1$ (or equivalently if $\nu \in A_\infty$) then $\log \nu \in \text{BMO}(\mathbb{R}^n)$. In particular, $\log J_f$ is a BMO-function, whenever $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a quasiconformal mapping.

For the sake of completeness, we also recall the definition of the Muckenhoupt class A_p for $1 \leq p < \infty$. A weight w belongs to the Muckenhoupt class A_p for $1 < p < \infty$ if

$$A_p(w) = \sup_B \left(\int_B w \, dx \right) \left(\int_B w^{-1/(p-1)} \, dx \right)^{p-1} < \infty. \quad (31)$$

As a natural extension of the above definition, one can consider the Muckenhoupt class A_1 which covers the limit case $p = 1$. A weight w belongs to the Muckenhoupt class A_1 if

$$A_1(w) = \sup_B \frac{\int_B w \, dx}{\text{ess inf}_{x \in B} w(x)} < \infty. \quad (32)$$

The suprema in (31) and (32) are taken over all balls $B \subset \mathbb{R}^n$. For each $1 \leq p < \infty$ we call $A_p(w)$ the A_p -constant of the weight w .

We recall here the definition of the Gehring class G_q for $1 < q \leq \infty$. A weight ν belongs to the Gehring class G_q for $1 < q < \infty$ if

$$G_q(\nu) = \sup_B \left[\frac{(\int_B \nu^q \, dx)^{1/q}}{\int_B \nu \, dx} \right]^{q/(q-1)} < \infty. \quad (33)$$

As a natural extension of the above definition, one can consider the class G_∞ which cover the limit case $q = \infty$. A weight ν belongs to the Gehring class G_∞ if

$$G_\infty(\nu) = \sup_B \frac{\text{ess sup}_{x \in B} \nu(x)}{\int_B \nu \, dx} < \infty. \quad (34)$$

The suprema in (33) and (34) are taken over all balls $B \subset \mathbb{R}^n$. For each $1 < q \leq \infty$ we call $G_q(\nu)$ the G_q -constant of the weight ν . Each weight in the G_q class satisfies a reverse Hölder inequality. This is a key fact in order to study the regularity of the Jacobian of quasiconformal mappings (see [20]).

For more details related to the Muckenhoupt and Gehring classes we refer to [19, 23, 24].

3. Explicit Bounds

We start by proving Theorem 1.

Proof of Theorem 1. Let us assume that L and β are arbitrary constants for which (27) holds, with $0 < \beta \leq 1 \leq L$. We notice that for every $t > 0$

$$\begin{aligned} \{y \in f(B) : |u(f^{-1}(y))| > t\} \\ = f(\{x \in B : |u(x)| > t\}). \end{aligned} \quad (35)$$

We compare the distribution functions of u and $u \circ f^{-1}$ by means of the estimate (27) and we obtain

$$\begin{aligned} \mu_{u \circ f^{-1}}(t) &= |\{y \in f(B) : |u(f^{-1}(y))| > t\}| \\ &= |f(\{x \in B : |u(x)| > t\})| \\ &\leq L |f(B)| \left(\frac{\mu_u(t)}{|B|} \right)^\beta. \end{aligned} \quad (36)$$

Let $s \in (0, |f(B)|)$ and let $t \geq 0$ be such that

$$\mu_{u \circ f^{-1}}(t) > s. \quad (37)$$

From (36) we get

$$\mu_u(t) > |B| \left(\frac{s}{L |f(B)|} \right)^{1/\beta}, \quad (38)$$

and therefore, from the definition of nonincreasing rearrangement (17), we obtain that

$$(u \circ f^{-1})^*(s) \leq u^* \left(|B| \left(\frac{s}{L |f(B)|} \right)^{1/\beta} \right). \quad (39)$$

We deduce directly from the definition of the norm (16) that

$$\begin{aligned} u^* \left(|B| \left(\frac{s}{L |f(B)|} \right)^{1/\beta} \right) \\ \leq \|u\|_{\text{EXP}(B)} \left\{ 1 + \frac{1}{\beta} \left[\log \frac{|f(B)|}{s} + \log L \right] \right\}. \end{aligned} \quad (40)$$

Combining (39) and (40) we obtain

$$\begin{aligned} (u \circ f^{-1})^*(s) \\ \leq \|u\|_{\text{EXP}(B)} \left\{ 1 + \frac{1}{\beta} \left[\log \frac{|f(B)|}{s} + \log L \right] \right\}. \end{aligned} \quad (41)$$

Let us introduce the function

$$\phi_{L,\beta}(s) = 1 + \frac{1}{\beta} \left[\log \frac{|f(B)|}{s} + \log L \right] \quad (42)$$

so that (41) may be rewritten as

$$(u \circ f^{-1})^*(s) \leq \|u\|_{\text{EXP}(B)} \phi_{L,\beta}(s), \quad (43)$$

and let us define

$$\begin{aligned} \psi_{L,\beta}(s) &= \frac{\phi_{L,\beta}(s)}{1 + \log(|f(B)|/s)} \\ &= \frac{1 + (1/\beta) [\log(|f(B)|/s) + \log L]}{1 + \log(|f(B)|/s)}. \end{aligned} \quad (44)$$

Our aim is to show that $\psi_{L,\beta}$ is bounded in $(0, |f(B)|)$ and we want to compute explicitly its upper bound. To this aim, we compute the derivative of $\psi_{L,\beta}$

$$\frac{d\psi_{L,\beta}}{ds}(s) = -\frac{1 - \beta - \log L}{\beta s (1 + \log(|f(B)|/s))^2} \quad (45)$$

so, the monotonicity of the function $\psi_{L,\beta}$ depends only on the sign of the constant $M(L, \beta) := 1 - \beta - \log L$ and for every $s \in (0, |f(B)|)$ we deduce

$$\begin{aligned} \psi_{L,\beta}(s) &\leq \frac{1}{\beta} \quad \text{if } 1 - \beta - \log L \geq 0 \\ \psi_{L,\beta}(s) &\leq 1 + \frac{\log L}{\beta} \quad \text{if } 1 - \beta - \log L \leq 0. \end{aligned} \quad (46)$$

The definition of $\psi_{L,\beta}$ (see (44)) and (46) imply

$$\phi_{L,\beta}(s) \leq \max \left\{ \frac{1}{\beta}, 1 + \frac{\log L}{\beta} \right\} \left(1 + \log \frac{|f(B)|}{s} \right). \quad (47)$$

Combining the latter estimate with (43) we conclude that

$$\begin{aligned} (u \circ f^{-1})^*(s) &\leq \max \left\{ \frac{1}{\beta}, 1 + \frac{\log L}{\beta} \right\} \|u\|_{\text{EXP}(B)} \left(1 + \log \frac{|f(B)|}{s} \right) \\ &\leq \max \left\{ \frac{1}{\beta}, 1 + \frac{\log L}{\beta} \right\} \|u\|_{\text{EXP}(B)} \left(1 + \log \frac{|f(B)|}{s} \right) \end{aligned} \quad (48)$$

and, by virtue of the definition of the norm in (16), we conclude that

$$\|u \circ f^{-1}\|_{\text{EXP}(f(B))} \leq \max \left\{ \frac{1}{\beta}, 1 + \frac{\log L}{\beta} \right\} \|u\|_{\text{EXP}(B)}. \quad (49)$$

Due to the definition of the constant $\widehat{G}_1(J_f)$ and to the fact that L and β are arbitrary constants for which (27) holds, we obtain

$$\|u \circ f^{-1}\|_{\text{EXP}(f(B))} \leq \widehat{G}_1(J_f) \|u\|_{\text{EXP}(B)}, \quad (50)$$

for each ball $B \subset \subset \Omega$ and for every $u \in \text{EXP}(\Omega)$.

It remains to prove the first inequality in (11). Let us assume that M and α are arbitrary constants for which (29) holds, with $0 < \alpha \leq 1 \leq M$. As before, we compare the distribution functions of u and $u \circ f^{-1}$. This time we make use of the estimate (29) instead of (27) and we argue as in the proof of estimate (36). Indeed, if we pick $E = \{x \in B : |u(x)| > t\}$ we obtain

$$\begin{aligned} \mu_u(t) &= |\{x \in B : |u(x)| > t\}| \\ &\leq M |B| \left(\frac{|\{x \in B : |u(x)| > t\}|}{|f(B)|} \right)^\alpha \\ &= M |B| \left(\frac{\mu_{u \circ f^{-1}}(t)}{|f(B)|} \right)^\alpha. \end{aligned} \quad (51)$$

Let $s \in (0, |B|)$ and let $t \geq 0$ be such that

$$\mu_u(t) > s. \quad (52)$$

From (51) we get

$$s < M |B| \left(\frac{\mu_{u \circ f^{-1}}(t)}{|f(B)|} \right)^\alpha, \quad (53)$$

which in turn implies

$$\mu_{u \circ f^{-1}}(t) > |f(B)| \left(\frac{s}{M |B|} \right)^{1/\alpha}. \quad (54)$$

It follows from the definition of nonincreasing rearrangement (17) that

$$u^*(s) \leq (u \circ f^{-1})^* \left(|f(B)| \left(\frac{s}{M |B|} \right)^{1/\alpha} \right). \quad (55)$$

The argument which leads to (41) and the definition of the norm in (16), allows us to conclude that

$$\begin{aligned} u^*(s) &\leq \|u \circ f^{-1}\|_{\text{EXP}(f(B))} \left\{ 1 + \frac{1}{\alpha} \left[\log \frac{|B|}{s} + \log M \right] \right\}. \end{aligned} \quad (56)$$

Arguing as in the proof of estimate (49) we have

$$\|u\|_{\text{EXP}(B)} \leq \max \left\{ \frac{1}{\alpha}, 1 + \frac{\log M}{\alpha} \right\} \|u \circ f^{-1}\|_{\text{EXP}(f(B))}. \quad (57)$$

Due to the definition of the constant $\widehat{A}_\infty(J_f)$ and to the fact that M and α are arbitrary constants for which (29) holds, we obtain

$$\|u\|_{\text{EXP}(B)} \leq \widehat{A}_\infty(J_f) \|u \circ f^{-1}\|_{\text{EXP}(f(B))}, \quad (58)$$

for each ball $B \subset \subset \Omega$ and for every $u \in \text{EXP}(\Omega)$. The proof is complete. $\square$

We are in a position to prove Theorem 2.

Proof of Theorem 2. Any principal quasiconformal mapping in the plane, which maps the unit disk onto itself, satisfies

$$\frac{|f(E)|}{|f(\mathbb{D})|} \leq K \left(\frac{|E|}{|\mathbb{D}|} \right)^{1/K}, \quad (59)$$

for every measurable subset $E \subset \mathbb{D}$ (see [25, Theorem 13.1.4]). Appealing to the definition (9) we deduce that

$$\begin{aligned} \widehat{G}_1(J_f) &= \inf \left\{ \max \left\{ \frac{1}{\beta}, 1 + \frac{\log L}{\beta} \right\} : 0 < \beta \leq 1 \right. \\ &\quad \left. \leq L \text{ and (27) holds} \right\}, \end{aligned} \quad (60)$$

therefore

$$\widehat{G}_1(J_f) \leq \max \{K, 1 + K \log K\}, \quad (61)$$

directly follows from (59).

We introduce the auxiliary function

$$\psi(K) = 1 + K \log K - K. \quad (62)$$

Since $K \geq 1$ we have $\psi'(K) = \log K \geq 0$ and in particular $\psi(K) \geq \psi(1) = 0$, that is

$$\max \{K, 1 + K \log K\} = 1 + K \log K. \quad (63)$$

Combining the latter identity with (61) we conclude that

$$\widehat{G}_1(J_f) \leq 1 + K \log K. \quad (64)$$

The proof is complete, since (11) infers (13) in case of any K -quasiconformal principal mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which maps $\mathbb{D}$ onto itself. $\square$

4. A Characterization of Constant Weights

We explicitly remark that, for a weight v in A_∞ , one always has

$$\widetilde{A}_\infty(v) = 1 \quad \text{iff } v \text{ is constant a.e. in } \mathbb{R}^n. \quad (65)$$

This result is proved in [21]. With a similar proof, one can show the same property for the $\widetilde{G}_1$ -constant, that is

$$\widetilde{G}_1(v) = 1 \quad \text{iff } v \text{ is constant a.e. in } \mathbb{R}^n. \quad (66)$$

Our next result gives a similar characterization of constant weights in terms of the constant appearing in (9) and (10). It is crucial to prove sharpness of Theorem 1.

Proposition 3. *Let w be an A_∞ weight such that $\widehat{A}_\infty(w) = 1$. Then w is constant a.e. in $\mathbb{R}^n$.*

Proof of Proposition 3. Let us fix some integer $j \geq 1$. By the definition (10) of $\widehat{A}_\infty(w)$, we may find two sequences $\{\alpha_j\}$ and $\{M_j\}$ fulfilling

$$0 < \alpha_j \leq 1 \leq M_j, \quad (67)$$

$$1 \leq \max \left\{ \frac{1}{\alpha_j}, 1 + \frac{\log M_j}{\alpha_j} \right\} < 1 + \frac{1}{j} \quad (68)$$

and for every ball $B \subset \mathbb{R}^n$ and every measurable set $E \subset B$ it holds

$$M_j \left(\frac{|E|}{|B|} \right)^{\alpha_j} \leq \frac{\int_E w(x) dx}{\int_B w(x) dx}. \quad (69)$$

Moreover, α_j and M_j are independent of E and B . Because of condition (67), we may assume (up to a subsequence) that α_j converges to some $\alpha_0 \in [0, 1]$. Since one clearly has $1/\alpha_j \geq 1$ for every $j \geq 1$, condition (67) implies $\alpha_0 = 1$. Conditions (67) and (68) also imply

$$0 \leq \log M_j < \frac{\alpha_j}{j}. \quad (70)$$

Appealing to the fact that $\alpha_j \rightarrow 1$ as $j \rightarrow \infty$, the latter relation implies $M_j \rightarrow 1$ as $j \rightarrow \infty$. Therefore, we may pass to the limit as $j \rightarrow \infty$ in (69) in order to get

$$\frac{|E|}{|B|} \leq \frac{\int_E w(x) dx}{\int_B w(x) dx}, \quad (71)$$

for every ball $B \subset \mathbb{R}^n$ and every measurable set $E \subset B$. In particular, for every ball $B \subset \mathbb{R}^n$ and for a.e. $x_0 \in B$ we have

$$\int_B w dx \leq w(x_0) \quad (72)$$

which in turn implies that w is a constant function. $\square$

Adapting suitably the proof of Proposition 3 one can prove next result featuring the $\widehat{G}_1(\cdot)$ -constant.

Proposition 4. *Let v be an G_1 weight such that $\widehat{G}_1(v) = 1$. Then v is constant a.e. in $\mathbb{R}^n$.*

5. The Logarithm of the Jacobian of a Quasiconformal Mapping

Let $f : \Omega \rightarrow \mathbb{R}^n$ be a quasiconformal mapping. In Section 2.4 we mentioned that $\log J_f$ is a BMO-function. Similarly (see, e.g., [4]) $\log J_f \in \text{EXP}(\Omega)$. Our next results are consequences of Theorem 1 and relate the logarithm of the Jacobian of a quasiconformal mapping with the one of its inverse taking into account the constants $\widehat{G}_1(J_f)$ and $\widehat{A}_\infty(J_f)$. Moreover, these results can be seen as the counterpart in the space of exponentially integrable functions of the results of Reimann [10] in the setting of the BMO-space.

Corollary 5. *Let Ω, Ω' be bounded domains of $\mathbb{R}^n$, $n \geq 2$. Let $f : \Omega \rightarrow \Omega'$ be a K -quasiconformal mapping and let $\widehat{G}_1(J_f)$ be the constant defined in (9). Then*

$$\begin{aligned} \frac{1}{\widehat{A}_\infty(J_f)} \|\log J_f\|_{\text{EXP}(B)} &\leq \|\log J_{f^{-1}}\|_{\text{EXP}(f(B))} \\ &\leq \widehat{G}_1(J_f) \|\log J_f\|_{\text{EXP}(B)} \end{aligned} \quad (73)$$

for each ball $B \subset \subset \Omega$ and for every $u \in \text{EXP}(\Omega)$.

Corollary 6. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a quasiconformal mapping with $n \geq 2$. Then, there exists a constant $C = C(n) \geq 1$ depending only on n , such that

$$\begin{aligned} \frac{1}{C(n) \widehat{G}_1(J_{f^{-1}})} \|\log J_f\|_{\text{BMO}} &\leq \|\log J_{f^{-1}}\|_{\text{BMO}} \\ &\leq C(n) \widehat{G}_1(J_f) \|\log J_f\|_{\text{BMO}}. \end{aligned} \quad (74)$$

Proof of Corollary 5. We start by observing (see, e.g., [13]) that

$$J_{f^{-1}}(f(x)) = \frac{1}{J_f(x)} \quad \text{for a.e. } x \in \Omega. \quad (75)$$

Therefore, we may write the following identity for the Jacobian of f

$$-(\log J_f) \circ f^{-1} = \log J_{f^{-1}} \quad (76)$$

then, from (11), we have

$$\|\log J_{f^{-1}}\|_{\text{EXP}(f(B))} \leq \widehat{G}_1(J_f) \|\log J_f\|_{\text{EXP}(B)}. \quad (77)$$

It remains to prove the first inequality in (73). Consider the first inequality in (11), namely

$$\|u\|_{\text{EXP}(B)} \leq \widehat{A}_\infty(J_f) \|u \circ f^{-1}\|_{\text{EXP}(f(B))}, \quad (78)$$

We recall that f and f^{-1} are both quasiconformal. So, we pick $u = v \circ f$ where v is any function in $\text{EXP}(\Omega')$, then (78) gives us

$$\|v \circ f\|_{\text{EXP}(B)} \leq \widehat{A}_\infty(J_f) \|v\|_{\text{EXP}(f(B))}. \quad (79)$$

In (76) we may replace f by f^{-1} , so that

$$-\log J_f = (\log J_{f^{-1}}) \circ f. \quad (80)$$

Thus, (79) and (80) infer

$$\|\log J_f\|_{\text{EXP}(B)} \leq \widehat{A}_\infty(J_f) \|\log J_{f^{-1}}\|_{\text{EXP}(f(B))}. \quad (81)$$

The desired inequality (73) is proved. $\square$

Proof of Corollary 6. As in the proof of Corollary 5, we know that (76) holds. Then from (6), we have

$$\|\log J_{f^{-1}}\|_{\text{BMO}} \leq C(n) \widehat{G}_1(J_f) \|\log J_f\|_{\text{BMO}}. \quad (82)$$

It remains to prove the first inequality in (74). In (82) we may replace f by f^{-1} . Hence

$$\|\log J_f\|_{\text{BMO}} \leq C(n) \widehat{G}_1(J_{f^{-1}}) \|\log J_{f^{-1}}\|_{\text{BMO}}. \quad (83)$$

The desired inequality (74) is proved. $\square$

Next results provide quantitative estimates as in (73) and (74) where the norms are replaced by the distances to L^∞ .

Corollary 7. Let Ω, Ω' be bounded domains of $\mathbb{R}^n$, $n \geq 2$. Let $f : \Omega \rightarrow \Omega'$ be a K -quasiconformal mapping and let $\widehat{G}_1(J_f)$ be the constant defined in (9). Then

$$\begin{aligned} \text{dist}_{\text{EXP}(f(B))}(\log J_{f^{-1}}, L^\infty(f(B))) \\ \leq \widehat{G}_1(J_f) \text{dist}_{\text{EXP}(B)}(\log J_f, L^\infty(B)), \end{aligned} \quad (84)$$

$$\begin{aligned} \frac{1}{\widehat{A}_\infty(J_f)} \text{dist}_{\text{EXP}(B)}(\log J_f, L^\infty(B)) \\ \leq \text{dist}_{\text{EXP}(f(B))}(\log J_{f^{-1}}, L^\infty(f(B))) \end{aligned} \quad (85)$$

for each ball $B \subset \subset \Omega$.

Corollary 8. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a quasiconformal mapping with $n \geq 2$. Then, there exists a constant $C = C(n)$ depending only on n , such that

$$\begin{aligned} \text{dist}_{\text{BMO}}(\log J_{f^{-1}}, L^\infty) \\ \leq C(n) \widehat{G}_1(J_f) \text{dist}_{\text{BMO}}(\log J_f, L^\infty), \end{aligned} \quad (86)$$

$$\begin{aligned} \frac{1}{C(n) \widehat{G}_1(J_{f^{-1}})} \text{dist}_{\text{BMO}}(\log J_f, L^\infty) \\ \leq \text{dist}_{\text{BMO}}(\log J_{f^{-1}}, L^\infty). \end{aligned} \quad (87)$$

Proof of Corollary 7. Let $\varphi \in L^\infty(B)$ and let $\psi = -\varphi \circ f^{-1}$. Then, from (11), we have

$$\begin{aligned} \|\log J_{f^{-1}} - \psi\|_{\text{EXP}(f(B))} &= \|(\log J_f - \varphi) \circ f^{-1}\|_{\text{EXP}(f(B))} \\ &\leq \widehat{G}_1(J_f) \|\log J_f - \varphi\|_{\text{EXP}(B)}. \end{aligned} \quad (88)$$

It is clear that there exists a measurable set N_0 of zero measure such that φ is bounded in $B \setminus N_0$. Thus, ψ is bounded in $f(B) \setminus f(N_0)$; on the other hand, one has $|f(N_0)| = 0$, since f satisfies the Lusin condition (see, e.g., [13]). We proved that $\psi \in L^\infty(f(B))$. Thus, from (88) we know that

$$\begin{aligned} \text{dist}_{\text{EXP}(f(B))}(\log J_{f^{-1}}, L^\infty) \\ \leq \widehat{G}_1(J_f) \|\log J_f \varphi\|_{\text{EXP}(B)}. \end{aligned} \quad (89)$$

Since φ is an arbitrary function in $L^\infty(B)$, we conclude that

$$\begin{aligned} \text{dist}_{\text{EXP}(f(B))}(\log J_{f^{-1}}, L^\infty) \\ \leq \widehat{G}_1(J_f) \text{dist}_{\text{EXP}(B)}(\log J_f, L^\infty). \end{aligned} \quad (90)$$

Now we prove the inequality in (85). Thanks to (79) and arguing as before, we may conclude that

$$\begin{aligned} \text{dist}_{\text{EXP}(B)}(v \circ f, L^\infty(B)) \\ \leq \widehat{A}_\infty(J_f) \text{dist}_{\text{EXP}(f(B))}(v, L^\infty(f(B))). \end{aligned} \quad (91)$$

Thus

$$\begin{aligned} \text{dist}_{\text{EXP}(B)}(\log J_f, L^\infty(B)) \\ = \text{dist}_{\text{EXP}(B)}((\log J_{f^{-1}}) \circ f, L^\infty(B)) \\ \leq \widehat{A}_\infty(J_f) \text{dist}_{\text{EXP}(f(B))}(\log J_{f^{-1}}, L^\infty(f(B))). \end{aligned} \quad (92)$$

The inequality in (85) is proved. $\square$

Proof of Corollary 8. Let $\varphi \in L^\infty(\mathbb{R}^n)$ and let $\psi = -\varphi \circ f^{-1}$. Then, from (6), we have

$$\begin{aligned} \|\log J_{f^{-1}} - \psi\|_{\text{BMO}} &= \|(\log J_f - \varphi) \circ f^{-1}\|_{\text{BMO}} \\ &\leq C(n) \widetilde{G}_1(J_f) \|\log J_f - \varphi\|_{\text{BMO}}. \end{aligned} \quad (93)$$

It is clear that there exists a measurable set N_0 of zero measure such that φ is bounded in $\mathbb{R}^n \setminus N_0$. Thus, ψ is bounded in $\mathbb{R}^n \setminus f(N_0)$; on the other hand, one has $|f(N_0)| = 0$, since f satisfies the Lusin condition. We proved that $\psi \in L^\infty(\mathbb{R}^n)$. Thus, from (93) we know that

$$\begin{aligned} \text{dist}_{\text{BMO}}(\log J_{f^{-1}}, L^\infty) \\ \leq C(n) \widetilde{G}_1(J_f) \|\log J_f - \varphi\|_{\text{BMO}}. \end{aligned} \quad (94)$$

Since φ is an arbitrary function in $L^\infty(B)$, we conclude that

$$\begin{aligned} \text{dist}_{\text{BMO}}(\log J_{f^{-1}}, L^\infty) \\ \leq C(n) \widetilde{G}_1(J_f) \text{dist}_{\text{BMO}}(\log J_f, L^\infty). \end{aligned} \quad (95)$$

The inequality (87) follows by replacing f by f^{-1} in (95), namely

$$\begin{aligned} \text{dist}_{\text{BMO}}(\log J_f, L^\infty) \\ \leq C(n) \widetilde{G}_1(J_{f^{-1}}) \text{dist}_{\text{BMO}}(\log J_{f^{-1}}, L^\infty). \end{aligned} \quad (96)$$

$\square$

Given a function $u \in \text{BMO}(\mathbb{R}^n)$, we consider the quantity

$$\varepsilon(u) = \inf \left\{ \lambda > 0 : \sup_B \int_B \exp \frac{|u - u_B|}{\lambda} dx < \infty \right\}, \quad (97)$$

introduced in [26] by Garnett and Jones. The supremum in (97) is taken over all balls $B \subset \mathbb{R}^n$. The main result of [26] states that $\varepsilon(\cdot)$ is equivalent to the distance to $L^\infty(\mathbb{R}^n)$ in the space $\text{BMO}(\mathbb{R}^n)$ defined as

$$\text{dist}_{\text{BMO}(\mathbb{R}^n)}(u, L^\infty(\mathbb{R}^n)) = \inf_{\varphi \in L^\infty(\Omega)} \|u - \varphi\|_{\text{BMO}(\mathbb{R}^n)}. \quad (98)$$

We also refer to [17, 27] for the problem of finding a formula for the distance to L^∞ in grand Sobolev and grand Orlicz spaces. The next result is an immediate consequence of the previous result.

Corollary 9. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a quasiconformal mapping with $n \geq 2$. Then, there exists a constant $C = C(n)$ depending only on n , such that

$$\begin{aligned} \frac{1}{C(n) \widetilde{G}_1(J_{f^{-1}})} \varepsilon(\log J_f) &\leq \varepsilon(\log J_{f^{-1}}) \\ &\leq C(n) \widetilde{G}_1(J_f) \varepsilon(\log J_f). \end{aligned} \quad (99)$$

Competing Interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

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