Research Article

Multiple Positive Solutions of Third-Order BVP with Advanced Arguments and Stieltjes Integral Conditions

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We consider the following third-order boundary value problem with advanced arguments and Stieltjes integral boundary conditions:

\[ u'''(t) + f(t, u(\alpha(t))) = 0, \quad t \in (0, 1), \]
\[ u(0) = u''(0) = 0, \]
\[ u(1) = \alpha u(\eta) + \lambda [u], \]

where \( \alpha : [0, 1] \to [0, 1] \) is continuous and \( \alpha(t) \geq t \) for \( t \in [0, 1] \).

\[ u''(0) = 0, \]
\[ u(1) = \beta u(\eta) + \lambda [u], \]

where \( \lambda [u] \) was defined as in (2) and the condition (\( A_0 \)) was imposed on the advanced argument \( \alpha \). The main tools used were the Guo-Krasnoselskii fixed point theorem \([28, 29]\) and a fixed point theorem due to Avery and Peterson \([30]\).

1. Introduction

Third-order differential equations arise from a variety of different areas of applied mathematics and physics, for example, in the deflection of a curved beam having a constant or varying cross-section, a three-layer beam, electromagnetic waves or gravity driven flows, and so on \([1]\). Recently, third-order boundary value problems (BVPs for short) have received much attention from many authors; see \([2–20]\) and the references therein. However, it is necessary to point out that all the unknown functions in the above-mentioned papers do not depend on advanced arguments.

In 2012, Jankowski \([21]\) studied the existence of multiple positive solutions to the BVP

\[ u'''(t) + f(t, u(\alpha(t))) = 0, \quad t \in (0, 1), \]
\[ u(0) = u''(0) = 0, \]
\[ u(1) = \alpha u(\eta) + \lambda [u], \]

where the unknown function \( u \) depended on an advanced argument \( \alpha \) satisfying the following condition:

(\( A_0 \)) \( \alpha : [0, 1] \to [0, 1] \) was continuous and \( \alpha(t) \geq t \) for \( t \in [0, 1] \).

\[ \lambda [u] = \int_0^1 u(t) d\Lambda(t) \] (2)

involving a Stieltjes integral with a suitable function \( \Lambda \) of bounded variation. The measure \( d\Lambda \) could be a signed one. The situation with a signed measure \( d\Lambda \) was first discussed in \([22, 23]\) for second-order differential equations; it was also discussed in \([24, 25]\) for second-order impulsive differential equations.

Among the boundary conditions in (1), only \( u(1) \) was related to \( u(\eta) \) and a Stieltjes integral. When \( u(0) \) was also related to \( u(\eta) \) and the Stieltjes integral, the authors in \([26, 27]\) obtained the existence and multiplicity of positive solutions to the BVP

\[ u'''(t) + f(t, u(\alpha(t))) = 0, \quad t \in (0, 1), \]
\[ u(0) = \gamma u(\eta) + \lambda [u], \]
\[ u''(0) = 0, \]
\[ u(1) = \beta u(\eta) + \lambda [u], \]

where \( \lambda [u] \) was defined as in (2) and the condition (\( A_0 \)) was imposed on the advanced argument \( \alpha \). The main tools used were the Guo-Krasnoselskii fixed point theorem \([28, 29]\) and a fixed point theorem due to Avery and Peterson \([30]\).
In this paper, we are concerned with the following third-order BVP with advanced arguments and Stieltjes integral boundary conditions:

\[ u'''(t) + f(t, u(a(t))) = 0, \quad t \in (0, 1), \]
\[ u(0) = \gamma u(\eta_1) + \lambda_1 u, \]
\[ u''(0) = 0, \]
\[ u(1) = \beta u(\eta_2) + \lambda_2 u, \]  \( (4) \)

where

\[ \lambda_i[u] = \int_0^1 u(t) d\Lambda_i(t), \quad i = 1, 2, \]  \( (5) \)

are suitable functions of bounded variation. It is important to indicate that it is not assumed that \( \lambda_i[u] (i = 1, 2) \) is positive for all positive \( u \). Throughout this paper, we always assume that \( 0 < \eta_1 < \eta_2 < 1, 0 \leq \gamma, \beta \leq 1, f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty) \) is continuous, and the advanced argument \( \alpha \) satisfies the following condition:

\[ \alpha : [0, 1] \rightarrow [0, 1] \text{ is continuous, } \alpha(t) \geq t \text{ for } t \in [0, 1] \text{ and } \alpha(t) \leq \eta_2 \text{ for } t \in [\eta_1, \eta_2]. \]

In order to obtain our main results, we need the following concepts and fixed point theorem [30].

Let \( E \) be a real Banach space and let \( K \) be a cone in \( E \). A map \( \Theta \) is said to be a nonnegative continuous convex functional on \( K \) if \( \Theta : K \rightarrow [0, +\infty) \) is continuous and \( \Theta(u + v) \leq \Theta(u) + \Theta(v) \) for all \( u, v \in K \) and \( t \in [0, 1] \).

Similarly, a map \( \Phi \) is said to be a nonnegative continuous concave functional on \( K \) if \( \Phi : K \rightarrow [0, +\infty) \) is continuous and \( \Phi(\eta u + (1 - \eta) v) \geq \eta \Phi(u) + (1 - \eta) \Phi(v) \) for all \( u, v \in K \) and \( \eta \in [0, 1] \).

Let \( \Psi \) be a nonnegative continuous concave functional on \( K \), let \( \Phi \) be a nonnegative continuous concave functional on \( K \), and let \( \Psi \) be a nonnegative continuous functional on \( K \). For positive numbers \( a, b, c, d \), we define the following sets:

\[ K(\phi, d) = \{ u \in K : \phi(u) < d \}, \]
\[ K(\phi, \Theta, \Phi, b, c, d) = \{ u \in K : b \leq \Phi(u), \phi(u) \leq d \}, \]
\[ K(\phi, \Theta, \Phi, b, c, d) = \{ u \in K : b \leq \Phi(u), \Theta(u) \leq c, \phi(u) \leq d \}, \]
\[ R(\phi, \Psi, a, d) = \{ u \in K : a \leq \Psi(u), \phi(u) \leq d \}. \]  \( (6) \)

**Theorem 1** (Avery and Peterson fixed point theorem). Let \( E \) be a real Banach space and let \( K \) be a cone in \( E \). Let \( \phi \) and \( \Theta \) be nonnegative continuous convex functionals on \( K \), let \( \Phi \) be a nonnegative continuous concave functional on \( K \), and let \( \Psi \) be a nonnegative continuous functional on \( K \) satisfying \( \Psi(ku) \leq k\Psi(u) \) for \( 0 \leq k \leq 1 \), such that, for some positive numbers \( M \) and \( d \),

\[ \Phi(u) \leq \Psi(u), \]
\[ \|u\| \leq M\phi(u). \]  \( (7) \)

Then \( S \) has at least three fixed points \( u_1, u_2, u_3 \in K(\phi, d) \), such that

\[ b < \Phi(u_1), \]
\[ a < \Psi(u_2) \text{ with } \Phi(u_2) < b, \]
\[ \Psi(u_3) < a. \]  \( (8) \)

**2. Main Results**

For convenience, we denote

\[ \Delta = (1 - \gamma)(1 - \eta_2\beta) + (1 - \beta)\eta_1\gamma, \]
\[ k(t, s) = \frac{1}{2}(1 - t)(t - s^2), \quad 0 \leq s \leq t \leq 1, \]
\[ \rho_i = (1 - \eta_2\beta) \int_0^1 d\Lambda_i(t) - (1 - \beta) \int_0^1 t d\Lambda_i(t), \]
\[ i = 1, 2, \]
\[ \tau_i = \eta_1\gamma \int_0^1 d\Lambda_i(t) + (1 - \gamma) \int_0^1 t d\Lambda_i(t), \]
\[ i = 1, 2. \]

In the remainder of this paper, we always assume that \( \Delta - \rho_1 > 0, \Delta - \tau_2 > 0, \) and \( (\Delta - \rho_1)(\Delta - \tau_2) > \rho_2\tau_1 \), and for \( \Lambda_i (i = 1, 2) \), the following conditions are fulfilled:

\[ \int_0^1 d\Lambda_i(t) \geq \int_0^1 t d\Lambda_i(t) \geq 0, \]
\[ \kappa_i(s) = \int_0^1 k(t, s) d\Lambda_i(t) \geq 0, \]  \( (10) \)
\[ s \in [0, 1], \quad i = 1, 2. \]

Then, \( \rho_i \geq 0, \tau_i \geq 0 (i = 1, 2) \) and \( \Delta > 0 \).

**Lemma 2** (see [21]). One has \( 0 \leq k(t, s) \leq (1/2)(1 + s)(1 - s)^2 \), \( (t, s) \in [0, 1] \times [0, 1] \).

**Lemma 3.** For any \( y \in C[0, 1] \), the BVP

\[ u'''(t) = -y(t), \quad t \in (0, 1), \]
\[ u(0) = \gamma u(\eta_1) + \lambda_1 u, \]
\[ u''(0) = 0, \]
\[ u(1) = \beta u(\eta_2) + \lambda_2 u, \]  \( (11) \)

for all \( u \in K(\phi, d) \). Suppose \( K(\phi, d) \rightarrow K(\phi, d) \) is completely continuous and there exist positive numbers \( a, b, c \) with \( a < b \), such that

\[ (C1) \{ u \in K(\phi, \Theta, \Phi, b, c, d) : \Phi(u) > b \} \neq \emptyset \text{ and } \Phi(Su) < b \text{ for } u \in K(\phi, \Theta, \Phi, b, c, d); \]
\[ (C2) \Phi(Su) > b \text{ for } u \in K(\phi, \Phi, b, d) \text{ with } \Theta(Su) > c; \]
\[ (C3) \theta \notin R(\phi, \Psi, a, d) \text{ and } \Psi(Su) < a \text{ for } u \in R(\phi, \Psi, a, d) \text{ with } \Psi(u) = a. \]
has the unique solution

\begin{equation}
\begin{split}
    u(t) &= \frac{1 - \eta_2 \beta - t (1 - \beta)}{\Delta} \lambda_1 [u] \\
    &\quad + \frac{\eta_1 \nu + t (1 - \nu)}{\Delta} \lambda_2 [u] \\
    &\quad + \frac{(1 - \eta_2 \beta) y - t y (1 - \beta)}{\Delta} \int_0^1 k(\eta_1, s) y(s) ds \\
    &\quad + \frac{\eta_1 \beta y + t \beta (1 - \nu)}{\Delta} \int_0^1 k(\eta_2, s) y(s) ds \\
    &\quad + \int_0^1 k(t, s) y(s) ds, \quad t \in [0, 1].
\end{split}
\end{equation}

Proof. By integrating the differential equation in (82) three times from 0 to \(t\) and using the boundary condition \(u''(0) = 0\), we know that

\begin{equation}
    u(t) = u(0) + u'(0) t - \frac{1}{2} \int_0^t (t - s)^2 y(s) ds, \quad t \in [0, 1].
\end{equation}

And so,

\begin{equation}
    u'(0) = u(1) - u(0) + \frac{1}{2} \int_0^1 (1 - s)^2 y(s) ds.
\end{equation}

In view of (13), (14), and the boundary conditions \(u(0) = \gamma u(\eta_1) + \lambda_1 [u]\) and \(u(1) = \beta u(\eta_2) + \lambda_2 [u]\), we have

\begin{equation}
    u(t) = (1 - t) y u(\eta_1) + t \beta u(\eta_2) + (1 - t) \lambda_1 [u] \\
    + t \lambda_2 [u] + \int_0^1 k(t, s) y(s) ds, \quad t \in [0, 1].
\end{equation}

Therefore,

\begin{equation}
    u(\eta_1) = \frac{1 - \eta_1 + \eta_1 \beta - \eta_2 \beta}{\Delta} \lambda_1 [u] + \frac{\eta_1 \nu - \eta_2 \nu}{\Delta} \lambda_2 [u] \\
    + \frac{1 - \eta_2 \beta}{\Delta} \int_0^1 k(\eta_1, s) y(s) ds \\
    + \frac{\eta_1 \beta}{\Delta} \int_0^1 k(\eta_2, s) y(s) ds,
\end{equation}

\begin{equation}
    u(\eta_2) = \frac{1 - \eta_2 \beta - \eta_1 \beta - \eta_2 \beta}{\Delta} \lambda_1 [u] + \frac{\eta_1 \nu - \eta_2 \nu}{\Delta} \lambda_2 [u] \\
    + \frac{y - \eta_2 \nu}{\Delta} \int_0^1 k(\eta_1, s) y(s) ds \\
    + \frac{1 - \nu + \eta_1 \nu}{\Delta} \int_0^1 k(\eta_2, s) y(s) ds.
\end{equation}

Substituting (16) into (15), we get

\begin{equation}
    u(t) = \frac{1 - \eta_2 \beta - t (1 - \beta)}{\Delta} \lambda_1 [u] \\
    + \frac{\eta_1 \nu + t (1 - \nu)}{\Delta} \lambda_2 [u] \\
    + \frac{(1 - \eta_2 \beta) y - t y (1 - \beta)}{\Delta} \int_0^1 k(\eta_1, s) y(s) ds \\
    + \frac{\eta_1 \beta y + t \beta (1 - \nu)}{\Delta} \int_0^1 k(\eta_2, s) y(s) ds \\
    + \int_0^1 k(t, s) y(s) ds, \quad t \in [0, 1].
\end{equation}

Let \(C[0, 1]\) be equipped with the maximum norm. Then \(C[0, 1]\) is a Banach space. If we let

\begin{equation}
    K = \left\{ u \in C([0, 1]) : u(t) \geq 0, \; t \in [0, 1], \; \min_{t \in [\eta_1, \eta_2]} u(t) \geq \Gamma \| u \|, \; \lambda_i [u] \geq 0, \; i = 1, 2 \right\},
\end{equation}

where

\begin{equation}
    \Gamma = \min \left\{ \frac{\eta_1}{1 - \nu + \eta_1 \nu}, \frac{1 - \eta_2 \nu}{1 - \eta_2 \beta} \right\},
\end{equation}

then \(K\) is a cone in \(C[0, 1]\). Now, we define operators \(T\) and \(S\) on \(K\) by

\begin{equation}
    (Tu)(t) = \frac{1 - \eta_2 \beta - t (1 - \beta)}{\Delta} \lambda_1 [u] + \frac{\eta_1 \nu + t (1 - \nu)}{\Delta} \lambda_2 [u] \\
    \cdot \lambda_1 [u] + (Fu)(t), \quad t \in [0, 1],
\end{equation}

\begin{equation}
    (Su)(t) = \left( \frac{1 - \eta_2 \beta - t (1 - \beta)}{\Delta - \tau_1} \right) \Delta - \tau_1 \\
    \cdot \frac{\eta_1 \nu + t (1 - \nu)}{\Delta - \tau_1} \cdot \lambda_2 [u] + (Fu)(t), \quad t \in [0, 1],
\end{equation}

\begin{equation}
    \Delta \rho_2 = \frac{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1}{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1},
\end{equation}

\begin{equation}
    \Delta \rho_2 = \frac{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1}{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1},
\end{equation}

\begin{equation}
    \Delta \rho_2 = \frac{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1}{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1},
\end{equation}
where

\[(F_u)(t) = (1 - \eta_2 \beta) \gamma - \eta_2 \beta \gamma (1 - \beta) \Delta \int_0^1 k(\eta_1, s) f (s, u (\alpha(s))) ds + \frac{\eta_1 \beta \gamma}{\Delta} \int_0^1 k(\eta_2, s) f (s, u (\alpha(s))) ds + \int_0^1 k(t, s) f (s, u (\alpha(s))) ds, \quad t \in [0, 1].\]

\[(21)\]

**Lemma 4.** \(T, S : K \to K\).

**Proof.** Let \(u \in K\). Then it is easy to know that

\[(Tu)' = - \int_0^1 f (s, u (\alpha(s))) ds \leq 0, \quad t \in [0, 1],\]

(22)

which shows that \(Tu\) is concave down on \([0, 1]\). In view of

\[(Fu)(0) = \frac{(1 - \eta_2 \beta) \gamma}{\Delta} \int_0^1 k(\eta_1, s) f (s, u (\alpha(s))) ds + \frac{\eta_1 \beta \gamma}{\Delta} \int_0^1 k(\eta_2, s) f (s, u (\alpha(s))) ds + \int_0^1 k(t, s) f (s, u (\alpha(s))) ds, \quad t \in [0, 1].\]

\[(23)\]

\[(Fu)(1) = \frac{(1 - \eta_2 \beta) \gamma}{\Delta} \int_0^1 k(\eta_1, s) f (s, u (\alpha(s))) ds + \frac{1 - \gamma + \eta_2 \beta \gamma}{\Delta} \int_0^1 k(\eta_2, s) f (s, u (\alpha(s))) ds + \int_0^1 k(t, s) f (s, u (\alpha(s))) ds \geq 0,\]

we have

\[(Tu)(0) = \frac{1 - \eta_2 \beta}{\Delta} \lambda_1 [u] + \frac{\eta_1 \beta}{\Delta} \lambda_2 [u] + (Fu)(0) \geq 0,\]

\[(Tu)(1) = \frac{1 - \eta_2 \beta}{\Delta} \lambda_1 [u] + \frac{1 - \gamma + \eta_2 \beta \gamma}{\Delta} \lambda_2 [u] + (Fu)(1) \geq 0.\]

\[(24)\]

Case 1. Let \((Tu)(\eta_1) \leq (Tu)(\eta_2)\). Then \(\min_{t \in [\eta_1, \eta_2]} (Tu)(t) = (Tu)(\eta_1)\) and there exists \(\overline{t} \in [\eta_1, 1]\) such that \(\|Tu\| = (Tu)(\overline{t})\). If \(\overline{t} \in [\eta_1, \eta_2]\), then

\[\frac{(Tu)(\overline{t}) - (Tu)(0)}{\overline{t} - 1} \leq \frac{(Tu)(\eta_1) - (Tu)(0)}{\eta_1} \] (25)

So,

\[\|Tu\| \leq \frac{\eta_2}{\eta_1} \frac{(Tu)(\eta_1)}{(Tu)(\eta_2)} - \frac{\eta_2 - \eta_1}{\eta_1} (Tu)(0),\]

(26)

which together with

\[(Tu)(0) = y(Tu)(\eta_1) + \lambda_1 [u],\]

(27)

implies that

\[\|Tu\| \leq \frac{\eta_2}{\eta_1} \frac{(Tu)(\eta_1)}{(Tu)(\eta_2)} - \frac{\eta_2 - \eta_1}{\eta_1} (Tu)(0).\]

(28)

If \(\overline{t} \in (\eta_2, 1]\), then

\[\frac{(Tu)(\overline{t}) - (Tu)(\eta_2)}{\overline{t} - \eta_2} \leq \frac{(Tu)(\eta_1) - (Tu)(\eta_2)}{\eta_1 - \eta_2}\]

(30)

So,

\[\|Tu\| \leq \frac{1 - \eta_2}{\eta_2 - \eta_1} (Tu)(\eta_2) - \frac{1 - \eta_2}{\eta_2 - \eta_1} (Tu)(\eta_1).\]

(31)

On the other hand, it follows from

\[\frac{(Tu)(\eta_1) - (Tu)(0)}{\eta_1} \geq \frac{(Tu)(\eta_2) - (Tu)(0)}{\eta_2}\]

(29)

and (27) that

\[\frac{(Tu)(\eta_2)}{\eta_1} \leq \frac{(Tu)(\eta_2) - (Tu)(0)}{\eta_2}\]

(32)

which together with (31) implies that

\[\|Tu\| \leq \frac{1 - \gamma + \eta_1 \gamma}{\eta_1} (Tu)(\eta_1);\]

(33)

that is,

\[\min_{t \in [\eta_1, \eta_2]} (Tu)(t) \geq \frac{\eta_1}{1 - \gamma + \eta_1 \gamma} \|Tu\|.\]

(34)

(35)
Case 2. Let \((Tu)(\eta_1) > (Tu)(\eta_2)\). Then \(\min_{t \in [\eta_1, \eta_2]} (Tu)(t) = (Tu)(\eta_2)\) and there exists \(\bar{t} \in [0, \eta_1]\) such that \(\|Tu\| = (Tu)(\bar{t})\). If \(\bar{t} \in [0, \eta_1]\), then

\[
\frac{(Tu)(\eta_2) - (Tu)(\bar{t})}{\eta_2 - \bar{t}} \geq \frac{(Tu)(\eta_2) - (Tu)(\eta_1)}{\eta_2 - \eta_1}.
\]

(36)

So,

\[
\|Tu\| \leq \frac{\eta_2}{\eta_2 - \eta_1} (Tu)(\eta_1) - \frac{\eta_1}{\eta_2 - \eta_1} (Tu)(\eta_2).
\]

(37)

At the same time, since

\[
\frac{(Tu)(\eta_2) - (Tu)(\eta_1)}{\eta_2 - \eta_1} \geq \frac{(Tu)(1) - (Tu)(\eta_1)}{1 - \eta_1},
\]

(38)

we have

\[
(Tu)(\eta_1) \leq \frac{1 - \eta_1}{1 - \eta_2} (Tu)(\eta_2) - \frac{\eta_2 - \eta_1}{1 - \eta_2} (Tu)(1),
\]

(39)

which together with

\[
(Tu)(1) = \beta(Tu)(\eta_2) + \lambda_2 [u]
\]

(40)

implies that

\[
(Tu)(\eta_1) \leq \frac{1 - \eta_1}{1 - \eta_2} (Tu)(\eta_2) - \frac{\eta_2 - \eta_1}{1 - \eta_2} (Tu)(1).
\]

(41)

In view of (37) and (41), we have

\[
\|Tu\| \leq \frac{1 - \eta_2}{1 - \eta_2} (Tu)(\eta_2);
\]

(42)

that is,

\[
\min_{t \in [\eta_1, \eta_2]} (Tu)(t) \geq \frac{1 - \eta_2}{1 - \eta_2} \|Tu\|.
\]

(43)

If \(\bar{t} \in (\eta_1, \eta_2)\), then

\[
\frac{(Tu)(1) - (Tu)(\bar{t})}{1 - \bar{t}} \geq \frac{(Tu)(1) - (Tu)(\eta_2)}{1 - \eta_2}.
\]

(44)

So,

\[
\|Tu\| \leq \frac{1 - \eta_1}{1 - \eta_2} (Tu)(\eta_2) - \frac{\eta_2 - \eta_1}{1 - \eta_2} (Tu)(1),
\]

(45)

which together with (40) implies that

\[
\|Tu\| \leq \frac{1 - \eta_1}{1 - \eta_2} (Tu)(\eta_2) - \frac{\eta_2 - \eta_1}{1 - \eta_2} (Tu)(\eta_2); \tag{46}
\]

that is,

\[
\min_{t \in [\eta_1, \eta_2]} (Tu)(t) \geq \frac{1 - \eta_2}{1 - \eta_1} \left(1 - \frac{(Tu)(\eta_1)}{\|Tu\|}\right) \|Tu\|. \tag{47}
\]

It follows from (29), (35), (43), and (47) that

\[
\min_{t \in [\eta_1, \eta_2]} (Tu)(t) \geq \Gamma \|Tu\|. \tag{48}
\]

Finally, we need to show that \(\lambda_i [Tu] \geq 0, \ i = 1, 2\). Since

\[
\lambda_i [Fu] = \int_0^1 \left(1 - \eta_2 \beta \right) \gamma - \gamma \left(1 - \beta \right) \Delta \cdot \int_0^1 k(\eta_1, s) f(s, u(\alpha(s))) \, ds \, d\Lambda_i(t)
\]

\[
+ \int_0^1 \eta_1 \beta \gamma + \beta \gamma \left(1 - \beta \right) \Delta \cdot \int_0^1 k(\eta_2, s) f(s, u(\alpha(s))) \, ds \, d\Lambda_i(t)
\]

\[
+ \int_0^1 \int_0^1 k(t, s) f(s, u(\alpha(s))) \, ds \, d\Lambda_i(t) = \frac{\gamma \beta_i}{\Delta} \Delta \cdot \int_0^1 k(\eta_1, s) f(s, u(\alpha(s))) \, ds + \frac{\beta \gamma_i}{\Delta} \Delta \cdot \int_0^1 k(\eta_2, s) f(s, u(\alpha(s))) \, ds
\]

\[
+ \int_0^1 \kappa_1(s) f(s, u(\alpha(s))) \, ds \geq 0, \quad i = 1, 2,
\]

we have

\[
\lambda_i [Tu] = \frac{\beta_i}{\Delta} \lambda_1 [u] + \frac{\gamma_i}{\Delta} \lambda_2 [u] + \lambda_1 [Fu] \geq 0,
\]

(50)

for \(i = 1, 2\).

Therefore, \(T : K \rightarrow K\). Similarly, we may prove that \(S : K \rightarrow K\).

Lemma 5. \(T\) and \(S\) have the same fixed points in \(K\).
Proof. On the one hand, if \( u \in K \) is a fixed point of \( S \), that is, \( u = Su \), then
\[
\lambda_1 [u] = \lambda_1 [Su] = \int_0^1 \left( \frac{1 - \eta_2 \beta - t (1 - \beta)}{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1} (\Delta - \tau_2) + \eta_1 y + t (1 - y) \right) \rho_2 (\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1 \\
\cdot \lambda_1 [Fu] + \frac{1 - \eta_2 \beta - t (1 - \beta)}{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1} (\Delta - \tau_2) + \eta_1 y + t (1 - y) \right) \rho_2 (\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1 \\
\cdot \lambda_2 [Fu] + (Fu) (t) \right) d\lambda_1 (t)
\]
which shows that
\[
\lambda_1 [Fu] = \frac{(\Delta - \rho_1) \lambda_1 [u] - \tau_1 \lambda_2 [u]}{\Delta},
\]
\[
\lambda_2 [Fu] = \frac{(\Delta - \tau_2) \lambda_2 [u] - \rho_2 \lambda_1 [u]}{\Delta},
\]
So,
\[
u (t) = (Su) (t) = \frac{1 - \eta_2 \beta - t (1 - \beta)}{\Delta} \lambda_1 [u] + \frac{\eta_1 y + t (1 - y)}{\Delta} \lambda_2 [u] + (Fu) (t)
\]
which indicates that \( u \) is a fixed point of \( S \).

On the other hand, if \( u \in K \) is a fixed point of \( T \), that is, \( u = Tu \), then
\[
\lambda_1 [u] = \lambda_1 [Tu] = \int_0^1 \frac{1 - \eta_2 \beta - t (1 - \beta)}{\Delta} \lambda_1 [u] + \frac{\eta_1 y + t (1 - y)}{\Delta} \lambda_2 [u] + (Fu) (t) \right) d\lambda_1 (t)
\]
which shows that
\[
\lambda_1 [u] = \frac{(\Delta - \tau_2) \lambda_1 [Fu] + \Delta \tau_2 \lambda_2 [Fu]}{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1},
\]
\[
\lambda_2 [u] = \frac{\Delta \rho_2 \lambda_1 [Fu] + \Delta (\Delta - \rho_1) \lambda_2 [Fu]}{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1}.
\]
So,
\[
u (t) = (Tu) (t) = \frac{1 - \eta_2 \beta - t (1 - \beta)}{\Delta} \lambda_1 [u] + \frac{\eta_1 y + t (1 - y)}{\Delta} \lambda_2 [u] + (Fu) (t)
\]
which indicates that \( u \) is a fixed point of \( T \).

Lemma 6. \( T, S : K \rightarrow K \) is completely continuous.

Proof. First, by Lemma 4, we know that \( T(K) \subset K \).

Next, we show that \( T \) is compact.

Let \( D \subset K \) be a bounded set. Then there exists \( M_1 > 0 \) such that \( \| u \| \leq M_1 \) for any \( u \in D \). Since \( \Lambda_1 \) and \( \Lambda_2 \) are functions of bounded variation, there exists \( M_2 > 0 \) such that
\[
\sum_{j=1}^n \left| \Lambda_1 (t_j) - \Lambda_1 (t_{j-1}) \right| \leq M_2, \quad i = 1, 2
\]
for any partition \( \Delta' : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1 \). Let
\[
M_3 = \sup \left\{ f (t, u) : (t, u) \in [0, 1] \times [0, M_1] \right\}.
\]

For any partition \( \Delta' : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1 \),
\[
\sum_{j=1}^n \left| \Lambda_1 (t_j) - \Lambda_1 (t_{j-1}) \right| \leq M_2, \quad i = 1, 2
\]
for any partition \( \Delta' : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1 \). Let
\[
M_3 = \sup \left\{ f (t, u) : (t, u) \in [0, 1] \times [0, M_1] \right\}.
\]

That means \( T(K) \subset K \) is uniformly bounded and equicontinuous.

Therefore, \( T \) is compact.

Hence, \( T, S \) is completely continuous.
Then for any $u \in D$, we have
\[
\|Tu\| = \max_{t \in [0,1]} (Tu)(t)
\leq \frac{1 - \eta_2 \beta}{\Delta} \lambda_1 [u] + \frac{1 - \gamma + \eta_1 \gamma}{\Delta} \lambda_2 [u]
+ \frac{(1 - \eta_2 \beta) \gamma}{\Delta} \int_0^1 k(\eta_1, s) f(s, u(\alpha(s))) \, ds
+ \frac{(1 - \gamma + \eta_1 \gamma) \beta M_3}{\Delta} \int_0^1 k(\eta_2, s) \, ds
+ \frac{(1 - \gamma) \beta M_2}{\Delta} \int_0^1 \int_0^1 k(\eta_2, s) \, ds + \frac{5}{24} M_3,
\]
which shows that $T(D)$ is uniformly bounded.

On the other hand, for any $\epsilon > 0$, since $k(t, s)$ is uniformly continuous on $[0,1] \times [0,1]$, there exists $\delta_1(\epsilon) > 0$ such that, for any $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta_1(\epsilon)$,
\[
|k(t_1, s) - k(t_2, s)| < \frac{\epsilon}{5 (M_3 + 1)}, \quad s \in [0,1]. \tag{60}
\]
Let
\[
\delta = \min \left\{ \delta_1(\epsilon), \frac{\epsilon \Delta}{5 ((1 - \beta) M_1 M_2 + 1)} \right\}, \tag{61}
\]
which shows that $T(D)$ is equicontinuous. It follows from Arzela-Ascoli theorem that $T(D)$ is relatively compact. Thus, we have shown that $T$ is a compact operator.

Finally, we prove that $T$ is continuous.

Assume that $u_n, u \in K$ and $\lim_{n \to \infty} u_n = u$. Then there exists $M_4 > 0$ such that $\|u\| \leq M_4$ and $\|u_n\| \leq M_4$, $n = 1, 2, \ldots$. For any $\epsilon > 0$, since $f(s, x)$ is uniformly continuous on $[0,1] \times [0, M_4]$, there exists $\delta > 0$ such that, for any $x_1, x_2 \in [0, M_4]$ with $|x_1 - x_2| < \delta$,
\[
|f(s, x_1) - f(s, x_2)| < \frac{\epsilon}{(3(2 - \eta_2 \beta - \beta) \gamma / \Delta) \int_0^1 k(\eta_1, s) \, ds + (3(1 - \gamma + \eta_1 \gamma) \beta / \Delta) \int_0^1 k(\eta_2, s) \, ds + 5/8}, \quad s \in [0,1]. \tag{63}
\]
At the same time, since \( \lim_{n \to \infty} u_n = u \), there exists positive integer \( N \) such that, for any \( n > N \),
\[
\|u_n - u\| < \min \left\{ \delta, \frac{\varepsilon \Delta}{3(2 - \eta_2 \beta - \beta) M_2} \right\},
\]
(64)

It follows from (63) and (64) that, for any \( n > N \),
\[
\|Tu_n - Tu\| = \max_{t \in [0,1]} \| (Tu_n)(t) - (Tu)(t) \|
\leq \frac{2 - \eta_2 \beta - \beta}{\Delta} \|u_n - u\| + \frac{1 + \gamma + \eta_1 \gamma}{\Delta} (1 - \frac{2 - \eta_2 \beta - \beta}{\Delta} \|u_n - u\|)
+ \int_{0}^{1} k(\eta_1, s) \left| f(s, u_n(\alpha(s))) - f(s, u(\alpha(s))) \right| ds
+ \frac{1 + \gamma + \eta_1 \gamma}{\Delta} \int_{0}^{1} k(\eta_2, s) \left| f(s, u_n(\alpha(s))) - f(s, u(\alpha(s))) \right| ds
\leq \frac{2 - \eta_2 \beta - \beta}{\Delta} \|u_n - u\| + \frac{1 + \gamma + \eta_1 \gamma}{\Delta} \|u_n - u\|
+ \frac{1 + \gamma + \eta_1 \gamma}{\Delta} \int_{0}^{1} k(\eta_1, s) f(s, u(\alpha(s))) + \frac{1 + \gamma + \eta_1 \gamma}{\Delta} \int_{0}^{1} k(\eta_2, s) f(s, u(\alpha(s)))
\leq \frac{2 - \eta_2 \beta - \beta}{\Delta} \|u_n - u\| + \frac{1 + \gamma + \eta_1 \gamma}{\Delta} \|u_n - u\|
\cdot M_2 + \int_{0}^{1} \left( \frac{2 - \eta_2 \beta - \beta}{\Delta} \gamma k(\eta_1, s)
+ \frac{1 + \gamma + \eta_1 \gamma}{\Delta} \int_{0}^{1} k(\eta_2, s) \left( 1 + \gamma + \eta_1 \gamma \right) \right)
\cdot \left| f(s, u_n(\alpha(s))) - f(s, u(\alpha(s))) \right| ds < \varepsilon,
\]
which indicates that \( T \) is continuous.

Therefore, \( T : K \to K \) is completely continuous. Similarly, we can prove that \( S : K \to K \) is also completely continuous.

For convenience, we denote
\[
D_1 = \frac{\eta_2}{\Delta} \int_{0}^{1} k(\eta_1, s) ds + \frac{\beta \tau_1}{\Delta} \int_{0}^{1} k(\eta_2, s) ds
+ \int_{0}^{1} \kappa_1(s) ds,
D_2 = \frac{\eta_2}{\Delta} \int_{0}^{1} k(\eta_1, s) ds + \frac{\beta \tau_2}{\Delta} \int_{0}^{1} k(\eta_2, s) ds
+ \int_{0}^{1} \kappa_2(s) ds,
D_3 = \frac{(1 - \eta_2 \beta) \gamma}{\Delta} \int_{0}^{1} k(\eta_1, s) ds
+ \frac{(1 + \gamma + \eta_1 \gamma) \beta}{\Delta} \int_{0}^{1} k(\eta_2, s) ds + \frac{5}{24},
D_4 = \frac{\eta_2}{\Delta} \int_{0}^{1} k(\eta_1, s) ds + \frac{\beta \tau_1}{\Delta} \int_{0}^{1} k(\eta_2, s) ds
+ \int_{0}^{1} \kappa_1(s) ds,
D_5 = \frac{\eta_2}{\Delta} \int_{0}^{1} k(\eta_1, s) ds + \frac{\beta \tau_2}{\Delta} \int_{0}^{1} k(\eta_2, s) ds
+ \int_{0}^{1} \kappa_2(s) ds,
D_6 = \frac{\gamma (1 - \eta_2 \beta)}{\Delta} \int_{0}^{1} k(\eta_1, s) ds
+ \frac{1 + \gamma + \eta_1 \gamma}{\Delta} \int_{0}^{1} k(\eta_2, s) ds.
\]
(66)

Let
\[
\mu > \frac{(1 - \eta_2 \beta)(\Delta - \tau_2) + (1 + \gamma + \eta_1 \gamma) \rho_2 \rho_D}{(\Delta - \rho_1)(\Delta - \tau_2) - \rho_2 \tau_1},
\frac{(1 - \eta_2 \beta)(\Delta - \tau_2) + (1 + \gamma + \eta_1 \gamma) \rho_2 \rho_D}{(\Delta - \rho_1)(\Delta - \tau_2) - \rho_2 \tau_1},
0 < L
\]
(67)

Theorem 7. Suppose that there exist positive constants \( a, b, \) and \( d \) with \( a < b < b/\Gamma \leq d \) such that the following conditions are fulfilled:
\[
(A_1) \ f(t, u) \leq d/\mu, (t, u) \in [0,1] \times [0, d],
(A_2) \ f(t, u) \geq b/L, (t, u) \in [\eta_1, \eta_2] \times [b, b/\Gamma],
(A_3) \ f(t, u) \leq a/\mu, (t, u) \in [0,1] \times [0, a].
\]

Then the BVP (4) has at least three positive solutions \( u_1, u_2, u_3 \) satisfying \( \|u_i\| \leq d \) \( (i = 1, 2, 3) \) and
\[
\min_{t \in [\eta_1, \eta_2]} u_1(t) > b,
\max_{t \in [\eta_1, \eta_2]} u_2 > b \quad \text{with} \quad \min_{t \in [\eta_1, \eta_2]} u_4(t) < b,
\max_{t \in [\eta_1, \eta_2]} u_3 < a.
\]
(68)

Proof. For \( u \in K \), we define
\[
\Phi(u) = \min_{t \in [\eta_1, \eta_2]} u(t),
\varphi(u) = \Theta(u) = \Psi(u) = \|u\|.
\]
(69)
Then it is easy to know that $\Phi$ is a nonnegative continuous concave functional on $K$ and $\varphi$, $\Theta$ and $\Psi$ are nonnegative continuous convex functionals on $K$. In order to apply Theorem 1 to prove our main results, we use the operator $S$ and take $c = b/\Gamma$.

First, we assert that $S : K(\varphi, d) \rightarrow K(\varphi, d)$.

In fact, if $u \in K(\varphi, d)$, then $0 \leq u(t) \leq d$, $t \in [0, 1]$, which together with (A.1) implies that

\[
\lambda_1 [Fu] = \frac{\gamma \eta_1}{\Delta} \int_0^1 k(\eta_1, s) f(s, u(\alpha(s))) ds + \frac{\beta r_1}{\Delta} \\
\cdot \left[ \int_0^1 \kappa_1(s) f(s, u(\alpha(s))) ds \right] \\
+ \int_0^1 \kappa_1(s) \left( \frac{1}{\mu} \right) d = \frac{D_3 d}{\mu}.
\]

It follows from (70) that

\[
\varphi(Su) = \|Su\| \\
\leq \left( \frac{1 - \eta_2 \beta}{\Delta} \right) (\Delta - \tau_2) + \left( 1 - \gamma + \eta_1 \gamma \right) \rho_2 \lambda_1 [Fu] \\
+ \left( \frac{1 - \eta_2 \beta}{\Delta} \right) \tau_1 + \left( 1 - \gamma + \eta_1 \gamma \right) \left( \frac{\rho_2}{\Delta} \right) \lambda_2 [Fu] \\
+ \|Fu\| \\
\leq \left( \frac{1 - \eta_2 \beta}{\Delta} \right) (\Delta - \tau_2) + \left( 1 - \gamma + \eta_1 \gamma \right) \rho_2 D_1 \\
+ \left( \frac{1 - \eta_2 \beta}{\Delta} \right) \tau_1 + \left( 1 - \gamma + \eta_1 \gamma \right) \left( \frac{\rho_2}{\Delta} \right) D_2 \\
+ D_3 \left( \frac{d}{\mu} \right) \leq d.
\]

This shows that $S : K(\varphi, d) \rightarrow K(\varphi, d)$.

Next, we claim that $\{u \in K(\varphi, \Theta, \Phi, b, c, d) : \Phi(u) > b\}$ is a nonnegative continuous concave functional on $K$. Moreover, if $u \in K(\varphi, \Theta, \Phi, b, c, d)$, then $b \leq u(t) \leq c$, $t \in [\eta_1, \eta_2]$, which together with $\eta_1 \leq t \leq \alpha(t) \leq \eta_2$ for $t \in [\eta_1, \eta_2]$, implies that $b \leq u(\alpha(t)) \leq c$, $t \in [\eta_1, \eta_2]$. In view of (A.2), we have

\[
\lambda_1 [Fu] = \frac{\gamma \eta_1}{\Delta} \int_0^1 k(\eta_1, s) f(s, u(\alpha(s))) ds + \frac{\beta r_1}{\Delta} \\
\cdot \left[ \int_0^1 \kappa_1(s) f(s, u(\alpha(s))) ds \right] \\
+ \int_0^1 \kappa_1(s) \left( \frac{1}{\mu} \right) d = \frac{D_3 d}{\mu}.
\]

(70)

It follows from (70) that

\[
\varphi(Su) = \|Su\| \\
\leq \left( \frac{1 - \eta_2 \beta}{\Delta} \right) (\Delta - \tau_2) + \left( 1 - \gamma + \eta_1 \gamma \right) \rho_2 \lambda_1 [Fu] \\
+ \left( \frac{1 - \eta_2 \beta}{\Delta} \right) \tau_1 + \left( 1 - \gamma + \eta_1 \gamma \right) \left( \frac{\rho_2}{\Delta} \right) \lambda_2 [Fu] \\
+ \|Fu\| \\
\leq \left( \frac{1 - \eta_2 \beta}{\Delta} \right) (\Delta - \tau_2) + \left( 1 - \gamma + \eta_1 \gamma \right) \rho_2 D_1 \\
+ \left( \frac{1 - \eta_2 \beta}{\Delta} \right) \tau_1 + \left( 1 - \gamma + \eta_1 \gamma \right) \left( \frac{\rho_2}{\Delta} \right) D_2 \\
+ D_3 \left( \frac{d}{\mu} \right) \leq d.
\]
\[
+ \frac{1 - y + \eta_1 y}{\Delta} \int_{\eta_1}^{\eta_2} k(\eta_2, s) f(s, u(\alpha(s))) \, ds \\
\geq \left( \frac{1 - \eta_2}{\Delta} \right) \int_{\eta_1}^{\eta_2} k(\eta_1, s) \, ds \\
+ \frac{1 - y + \eta_1 y}{\Delta} \int_{\eta_1}^{\eta_2} k(\eta_2, s) \, ds \right) \frac{b}{L} = D_2 b.
\]

Since \( Su \) is concave down on \([0,1]\], we have
\[
\frac{(Su)(\eta_2) - (Su)(0)}{\eta_2} \leq \frac{(Su)(\eta_1) - (Su)(0)}{\eta_1}.
\]
So,
\[
(Su)(\eta_2) \leq \frac{\eta_2}{\eta_1} (Su)(\eta_1) - \frac{\eta_2 - \eta_1}{\eta_1} (Su)(0),
\]
which together with
\[
(Su)(0) = y (Su)(\eta_1) + \frac{(\Delta - \tau_2) \lambda_1 [Fu] + \Delta \tau_1 \lambda_2 [Fu]}{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1} \cdot \lambda_1 [Fu] + \frac{(1 - \eta_2) \tau_1 + (\eta_2 y + \eta_2 - \eta_3 y)(\Delta - \rho_1)}{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1} \lambda_1 [Fu]
\]
implies that
\[
(Su)(\eta_1) \geq \frac{\eta_1}{\eta_2 - (\eta_2 - \eta_1) y} (Su)(\eta_2).
\]
Therefore, it follows from (72) and (76) that
\[
\Phi(Su) = \min_{t \in [\eta_1, \eta_2]} (Su)(t) = \min \{ (Su)(\eta_1), (Su)(\eta_2) \}
\]
\[
\geq \min \left\{ \frac{\eta_1}{\eta_2 - (\eta_2 - \eta_1) y} (Su)(\eta_2), (Su)(\eta_1) \right\}
\]
\[
= \frac{\eta_1}{\eta_2 - (\eta_2 - \eta_1) y} (Su)(\eta_2)
\]
\[
\geq \frac{\eta_1}{\eta_2 - (\eta_2 - \eta_1) y} \left( \frac{(1 - \eta_2) (\Delta - \tau_2) + (\eta_2 y + \eta_2 - \eta_3 y)(\Delta - \rho_1)}{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1} \lambda_1 [Fu] + \frac{(1 - \eta_2) \tau_1 + (\eta_2 y + \eta_2 - \eta_3 y)(\Delta - \rho_1)}{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1} \lambda_1 [Fu] \right)
\]
\[
\geq \left( \frac{\eta_1}{\eta_2 - (\eta_2 - \eta_1) y} \right) \left( \frac{(1 - \eta_2) (\Delta - \tau_2) + (\eta_2 y + \eta_2 - \eta_3 y)(\Delta - \rho_1)}{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1} \lambda_1 [Fu] + \frac{(1 - \eta_2) \tau_1 + (\eta_2 y + \eta_2 - \eta_3 y)(\Delta - \rho_1)}{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1} \lambda_1 [Fu] \right)
\]
\[
+ (Fu)(\eta_2) \right)
\]
\[
\geq \left( \frac{\eta_1}{\eta_2 - (\eta_2 - \eta_1) y} \right) \left( \frac{(1 - \eta_2) (\Delta - \tau_2) + (\eta_2 y + \eta_2 - \eta_3 y)(\Delta - \rho_1)}{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1} \lambda_1 [Fu] + \frac{(1 - \eta_2) \tau_1 + (\eta_2 y + \eta_2 - \eta_3 y)(\Delta - \rho_1)}{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1} \lambda_1 [Fu] \right)
\]
\[
+ \frac{(1 - \eta_2) \tau_1 + (\eta_2 y + \eta_2 - \eta_3 y)(\Delta - \rho_1)}{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1} \lambda_1 [Fu] + \left( \frac{D_4 + \frac{(1 - \eta_2) \tau_1 + (\eta_2 y + \eta_2 - \eta_3 y)(\Delta - \rho_1)}{(\Delta - \rho_1) (\Delta - \tau_2) - \rho_2 \tau_1} D_5 + D_6}{L} b \right)
\]
\[
> b.
\]
Finally, we prove that \( \theta \notin R(\phi, \Psi, a, d) \) and \( \Psi(Su) < a \) for \( u \in R(\phi, \Psi, a, d) \) with \( \Psi(u) = a \).
Indeed, it follows from \( \Psi(\theta) = 0 < a \) that \( \theta \notin R(\phi, \Psi, a, d) \). Moreover, if \( u \in R(\phi, \Psi, a, d) \) and \( \Psi(u) = a \), then \( 0 \leq u(t) \leq a \), \( t \in [0,1] \), which together with \( (A_3) \) implies that
\[
\lambda_i [Fu] = \frac{\gamma_i a}{\Delta} \int_0^1 k(\eta_i, s) f(s, u(\alpha(s))) \, ds + \frac{\beta \tau_1}{\Delta}
\]
\[
\cdot \int_0^1 k(\eta_2, s) f(s, u(\alpha(s))) \, ds + \frac{1}{2} \left( \frac{1 - \eta_2 \beta}{\Delta} \int_0^1 k(\eta_1, s) \, ds \right)
\]
\[
+ \frac{1 - y + \eta_1 y}{\Delta} \int_0^1 k(\eta_2, s) \, ds + \frac{5}{24} \frac{a}{\mu} = D_3 a.
\]
In view of (79), we have
\[
\Psi(Su) = \|Su\| \leq \left( \frac{1 - \eta_2 \beta}{\Delta} \right) \left( \frac{(1 - \eta_2 \beta)(\Delta - \tau_2) + (1 - y + \eta_1 y) \rho_2}{(\Delta - \rho_1)(\Delta - \tau_2) - \rho_2 \tau_1} \lambda_1 [Fu] \right)
\]
\[
+ \frac{(1 - \eta_2 \beta)(\Delta - \tau_2) + (1 - y + \eta_1 y) \rho_2}{(\Delta - \rho_1)(\Delta - \tau_2) - \rho_2 \tau_1} \lambda_2 [Fu]
\]
\[
+ \frac{1 - \eta_2 \beta}{\Delta} \int_0^1 k(\eta_1, s) \, ds + \frac{5}{24} \frac{a}{\mu} = D_3 a.
\]

Thirdly, we assert that \( \Phi(Su) > b \) for \( u \in K(\phi, \Phi, b, d) \) with \( \Theta(Su) > c \).
To see this, we suppose \( u \in K(\phi, \Phi, b, d) \) and \( \Theta(Su) = \|Su\| > c \). Then
\[
\Phi(Su) = \min_{t \in [\eta_1, \eta_2]} (Su)(t) \geq \Gamma \|Su\| > \Gamma c = b.
\]
To sum up, all the hypotheses of Theorem 1 are satisfied. Hence, the BVP (4) has at least three positive solutions \( u_1, u_2, u_3 \) satisfying \( \| u_i \| \leq d \) (i = 1, 2, 3) and

\[
\begin{align*}
\min_{t \in [\eta_1, \eta_2]} u_i(t) &> b, \\
\| u_2 \| &> a \quad \text{with} \quad \min_{t \in [\eta_1, \eta_2]} u_2(t) < b, \quad (81) \\
\| u_3 \| &< a.
\end{align*}
\]

3. An Example

Example 1. Consider the following BVP:

\[
\begin{align*}
\dddot{u}(t) + f(t, u(\alpha(t))) &= 0, \quad t \in (0,1), \\
\dot{u}(0) &= 1 \\
u(0) &= \frac{2}{5}u(\frac{1}{4}) + \int_0^1 u(t) \cdot \left(\frac{3}{2}t^2 - t\right) dt, \\
\end{align*}
\]

(82)

\[
\begin{align*}
\ddot{u}(0) &= 0, \\
u(1) &= \frac{1}{4}u(\frac{1}{2}) + \int_0^1 u(t) \cdot \left(\frac{1}{4}t^2\right) dt, \\
\end{align*}
\]

(83)

where

\[
\begin{align*}
f(t, u) &= \begin{cases} 
20u^2 + \left(\frac{1}{20} - u\right) t(1 - t), & (t, u) \in [0,1] \times [0, \frac{1}{20}], \\
\frac{1}{20} + 1980\left(u - \frac{1}{20}\right)^2 + \left(u - \frac{1}{20}\right) \left(\frac{1}{10} - u\right) t(1 - t), & (t, u) \in [0,1] \times (\frac{1}{20}, \frac{1}{10}), \\
5 + \frac{1}{10} \left(u - \frac{1}{10}\right)^2 t(1 - t), & (t, u) \in [0,1] \times [\frac{1}{10}, +\infty),
\end{cases}
\end{align*}
\]

\[
\alpha(t) = \begin{cases} 
\sqrt{2t}, & t \in [0, \frac{1}{2}], \\
\sqrt{2t - 1} + 1, & t \in (\frac{1}{2}, 1].
\end{cases}
\]

Since \( \Lambda_1(t) = (3/2)t^2 - t \) and \( \Lambda_2(t) = (1/4)t^2, \ t \in [0,1] \), a simple calculation shows that

\[
\begin{align*}
\int_0^1 d\Lambda_1(t) &= \int_0^1 t \cdot d\Lambda_1(t) = \frac{1}{2}, \\
\int_0^1 d\Lambda_2(t) &= \frac{1}{4}, \\
\int_0^1 t \cdot d\Lambda_2(t) &= \frac{1}{6}, \\
\kappa_1(s) &= \frac{1}{8} s^4 - \frac{1}{6} s^3 + \frac{1}{24} s, \\
\kappa_2(s) &= \frac{1}{48} \left(1 - s^2\right)^2,
\end{align*}
\]

(84)

\[ s \in [0,1]. \]

At the same time, in view of \( \eta_1 = \beta = 1/4 \) and \( \eta_2 = \gamma = 1/2 \), we get

\[
\begin{align*}
\Delta &= \frac{17}{32}, \\
\rho_1 &= \frac{1}{16}, \\
\tau_1 &= \frac{5}{16}, \\
\rho_2 &= \frac{3}{32},
\end{align*}
\]

(85)

If we choose \( \mu = 1, \ L = 1/50, \ a = 1/20, \ b = 1/10, \) and \( d = 6 \), then all the conditions of Theorem 7 are fulfilled. Therefore, it follows from Theorem 7 that the BVP (82) has at least three positive solutions \( u_1, u_2, u_3 \) satisfying \( \| u_i \| \leq 6 \) (i = 1, 2, 3) and

\[
\begin{align*}
\min_{t \in [1/4,1/2]} u_i(t) &> \frac{1}{10}, \\
\| u_2 \| &> \frac{1}{20} \quad \text{with} \quad \min_{t \in [1/4,1/2]} u_2(t) < \frac{1}{10}, \\
\| u_3 \| &< \frac{1}{20}.
\end{align*}
\]

(86)
Competing Interests
The authors declare that they have no competing interests.

References


