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Research Article

On the Asymptotic Properties of Nonlinear Third-Order Neutral Delay Differential Equations with Distributed Deviating Arguments

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This paper is concerned with the asymptotic properties of solutions to a third-order nonlinear neutral delay differential equation with distributed deviating arguments. Several new theorems are obtained which ensure that every solution to this equation either is oscillatory or tends to zero. Two illustrative examples are included.

1. Introduction

In this paper, we consider the asymptotic properties of solutions to a class of differential equations of the form

$$\left[r(t)\left(\left(x(t)+\int_a^b p(t,\xi)\,x(\tau(t,\xi))\,d\xi\right)''\right)^{\alpha}\right]' + \int_c^d q(t,\xi)\,f(x(\sigma(t,\xi)))\,d\xi = 0, \quad t \geq t_0 > 0, \tag{1}$$

where $\alpha > 0$ is a ratio of odd integers, $a < b$, and $c < d$. Throughout, the following hypotheses are tacitly supposed to hold:

(h_1) $r(t) \in C^1([t_0,\infty),(0,\infty))$, $p(t,\xi) \in C([t_0,\infty)\times[a,b],[0,\infty))$, $q(t,\xi) \in C([t_0,\infty)\times[c,d],[0,\infty))$, $f(x) \in C(\mathbb{R},\mathbb{R})$, $r'(t) \geq 0$, $\int_{t_0}^{\infty} r^{-1/\alpha}(t)dt = \infty$, $0 \leq \int_a^b p(t,\xi)d\xi \leq P < 1$, $q(t,\xi)$ is not identically zero eventually, and $f(x)/x^{\alpha} \geq 1$ for all $x \neq 0$;

(h_2) $\tau(t,\xi) \in C([t_0,\infty)\times[a,b],\mathbb{R})$ and $\sigma(t,\xi) \in C^1([t_0,\infty)\times[c,d],\mathbb{R})$ are nondecreasing functions for ξ satisfying $\tau(t,\xi) \leq t$ and $\liminf_{t\to\infty}\tau(t,\xi) = \infty$ for $\xi \in [a,b]$, $\sigma(t,\xi) \leq t$ and $\liminf_{t\to\infty}\sigma(t,\xi) = \infty$ for $\xi \in [c,d]$, and $\sigma_1'(t) > 0$, where $\sigma_1(t) = \sigma(t,c)$.

During the last few decades, many researches have been done concerning the study of oscillation and asymptotic behavior of various classes of neutral differential equations, we refer the reader to the monograph [1], the papers [2–11], and the references cited there. The investigation of asymptotic behavior of (1) is important for practical reasons and the development of asymptotic theory; see Wang [10]. Tian et al. [9] explored asymptotic properties of (1) assuming that conditions $\alpha \geq 1$, (h_1), and (h_2) hold. Very recently, applying the Riccati transformation and Lebesgue's monotone convergence theorem, Candan [5] established several oscillation criteria for a class of second-order neutral delay differential equations with distributed deviating arguments. Motivated by the method reported in the paper by Candan [5], the aim of this paper is to derive some new results on the oscillation and asymptotic behavior of solutions to (1) which can be applied in the case where $0 < \alpha < 1$ as well. These criteria provide answers to a question posed in [9, Remark 4.4].

We use the notation $z(t) = x(t) + \int_a^b p(t,\xi)x(\tau(t,\xi))d\xi$. By a solution to (1) we mean a nontrivial function $x(t) \in$

$C([T_x,\infty),\mathbb{R})$ satisfying (1) which possesses the properties $z(t) \in C^2([T_x,\infty),\mathbb{R})$ and $r(t)(z''(t))^\alpha \in C^1([T_x,\infty),\mathbb{R})$ for $T_x \geq t_0$. The focus of this paper is restricted to those solutions of (1) which have the property $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. A solution $x(t)$ of (1) is termed *oscillatory* if it does not have the largest zero on the interval $[T_x,\infty)$; $x(t)$ is said to be *nonoscillatory* if it is either eventually positive or eventually negative. In what follows, all functional inequalities are assumed to hold for all sufficiently large t.

2. Auxiliary Lemmas

In order to establish our main results, we need the following auxiliary lemmas which are extracted from the paper by Agarwal et al. [2] and the paper by Tian et al. [9].

Lemma 1 (see [9, Lemma 2.1]). *Let hypotheses (h_1) and (h_2) be satisfied and suppose that $x(t)$ is an eventually positive solution of (1). Then, for all sufficiently large $t \geq t_1 \geq t_0$, $z(t)$ satisfies either*

(I) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, *and* $z'''(t) \leq 0$

or

(II) $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$, *and* $z'''(t) \leq 0$.

Lemma 2 (see [2, Lemma 2.3]). *If $z(t)$ satisfies case (I), then $z'(t) \geq tz''(t)$ eventually.*

Lemma 3 (see [9, Lemma 2.2]). *Suppose that $x(t)$ is an eventually positive solution of (1). If $z(t)$ satisfies case (II), then $\lim_{t\to\infty} x(t) = 0$ provided that*

$$\int_{t_0}^{\infty}\int_{v}^{\infty}\left(\frac{1}{r(u)}\int_{u}^{\infty}\int_{c}^{d} q(s,\xi)\,d\xi\,ds\right)^{1/\alpha} du\,dv = \infty. \tag{2}$$

3. Main Results

For a compact presentation of our results, we use the following notation:

$$\begin{aligned} Q(t) &= (1-P)^\alpha \int_c^d q(t,\xi)\,d\xi, \\ \widetilde{Q}(t) &= \int_t^\infty Q(s)\,ds, \\ \widetilde{R}(t) &= \alpha\frac{\sigma_1(t)\,\sigma_1'(t)}{r^{1/\alpha}(\sigma_1(t))}, \end{aligned} \tag{3}$$

where $\widetilde{Q}(t)$ is well defined.

Theorem 4. *Let hypotheses (h_1), (h_2), and (2) be satisfied. Then all solutions of (1) either are oscillatory or converge to zero asymptotically if*

$$\int_{t_0}^{\infty} Q(t)\,dt = \infty. \tag{4}$$

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution to (1). Without loss of generality, it may be assumed that $x(t)$ is an eventually positive solution of (1). Then there is a $t_1 \geq t_0$ such that, for $t \geq t_1$, $x(t) > 0$, $x(\tau(t,\xi)) > 0$, and $x(\sigma(t,\xi)) > 0$, for $\xi \in [a,b]$ and $\xi \in [c,d]$, respectively. In view of (1), we have

$$\begin{aligned} \left[r(t)\left(z''(t)\right)^\alpha\right]' &= -\int_c^d q(t,\xi)\,f(x(\sigma(t,\xi)))\,d\xi \\ &\leq -\int_c^d q(t,\xi)\,x^\alpha(\sigma(t,\xi))\,d\xi \leq 0. \end{aligned} \tag{5}$$

On the basis of Lemma 1, we observe that $z(t)$ satisfies either case (I) or case (II) for $t \geq t_1$.

Let $z(t)$ satisfy case (I). It follows from the definition of $z(t)$ that

$$\begin{aligned} z(t) &= x(t) + \int_a^b p(t,\xi)\,x(\tau(t,\xi))\,d\xi \\ &\leq x(t) + \int_a^b p(t,\xi)\,z(\tau(t,\xi))\,d\xi \\ &\leq x(t) + z(t)\int_a^b p(t,\xi)\,d\xi \leq x(t) + Pz(t), \end{aligned} \tag{6}$$

which yields

$$x(t) \geq (1-P)\,z(t), \tag{7}$$

and so

$$x^\alpha(\sigma(t,\xi)) \geq \left[(1-P)\,z(\sigma(t,\xi))\right]^\alpha. \tag{8}$$

Substitution of (8) into (5) and the definition of $Q(t)$ imply that

$$\left[r(t)\left(z''(t)\right)^\alpha\right]' \leq -Q(t)\,z^\alpha(\sigma_1(t)) \leq 0. \tag{9}$$

Integrating (9) from t_1 to t, we arrive at

$$\begin{aligned} r(t)\left(z''(t)\right)^\alpha &\leq r(t_1)\left(z''(t_1)\right)^\alpha \\ &\quad - \int_{t_1}^t Q(s)\,z^\alpha(\sigma_1(s))\,ds. \end{aligned} \tag{10}$$

Taking into account that $z(t) > 0$ and $z'(t) > 0$, there exists a constant $k_1 > 0$ such that $z(t) \geq k_1$. Therefore, we deduce that

$$r(t)\left(z''(t)\right)^\alpha \leq r(t_1)\left(z''(t_1)\right)^\alpha - k_1^\alpha\int_{t_1}^t Q(s)\,ds, \tag{11}$$

and hence

$$\int_{t_1}^t Q(s)\,ds \leq \frac{r(t_1)\left(z''(t_1)\right)^\alpha}{k_1^\alpha}, \tag{12}$$

which is a contradiction with (4).

Let $z(t)$ satisfy case (II). Then $\lim_{t\to\infty} x(t) = 0$ when using Lemma 3. The proof is complete. □

Now, we establish some oscillation criteria for (1) by utilizing the Riccati transformation and Lebesgue's monotone convergence theorem. To this end, we give the following lemmas.

Lemma 5. *Let $x(t)$ be an eventually positive solution of (1) and let $z(t)$ satisfy case (I). Define the Riccati transformation by*

$$\omega(t) = r(t)\frac{\left(z''(t)\right)^{\alpha}}{z^{\alpha}\left(\sigma_1(t)\right)}. \tag{13}$$

Then

$$\omega'(t) + \widetilde{R}(t)\,\omega^{(\alpha+1)/\alpha}(t) + Q(t) \le 0. \tag{14}$$

Proof. By $[r(t)(z''(t))^{\alpha}]' \le 0$, we conclude that

$$r(t)\left(z''(t)\right)^{\alpha} \le r\left(\sigma_1(t)\right)\left(z''\left(\sigma_1(t)\right)\right)^{\alpha}, \tag{15}$$

which yields

$$z''(t) \le \frac{r^{1/\alpha}\left(\sigma_1(t)\right)}{r^{1/\alpha}(t)} z''\left(\sigma_1(t)\right). \tag{16}$$

It follows from (13) and case (I) that $\omega(t) > 0$. Differentiation of (13) and applications of (9), (13), (16), and Lemma 2 imply that

$$\begin{aligned}
\omega'(t) &= \frac{\left[r(t)\left(z''(t)\right)^{\alpha}\right]'}{z^{\alpha}\left(\sigma_1(t)\right)} \\
&\quad - \frac{\alpha r(t)\left(z''(t)\right)^{\alpha} z'\left(\sigma_1(t)\right)\sigma_1'(t)}{z^{\alpha+1}\left(\sigma_1(t)\right)} \\
&\le -Q(t) - \alpha\frac{z'\left(\sigma_1(t)\right)\sigma_1'(t)}{r^{1/\alpha}(t)\,z''(t)}\omega^{(\alpha+1)/\alpha}(t) \\
&\le -Q(t) - \alpha\frac{z'\left(\sigma_1(t)\right)\sigma_1'(t)}{r^{1/\alpha}\left(\sigma_1(t)\right)z''\left(\sigma_1(t)\right)}\omega^{(\alpha+1)/\alpha}(t) \\
&\le -Q(t) - \alpha\frac{\sigma_1(t)\,\sigma_1'(t)}{r^{1/\alpha}\left(\sigma_1(t)\right)}\omega^{(\alpha+1)/\alpha}(t) \\
&= -Q(t) - \widetilde{R}(t)\,\omega^{(\alpha+1)/\alpha}(t),
\end{aligned} \tag{17}$$

which completes the proof. □

Define a sequence of functions $\{\widetilde{y}_n(t)\}_{n=0}^{\infty}$ by $\widetilde{y}_0(t) = \widetilde{Q}(t)$ and

$$\widetilde{y}_n(t) = \widetilde{y}_0(t) + \int_t^{\infty}\widetilde{R}(s)\,\widetilde{y}_{n-1}^{(\alpha+1)/\alpha}(s)\,ds, \qquad t \ge t_0,\ n = 1, 2, \ldots, \tag{18}$$

where $\widetilde{y}_n(t)$ are well defined. By induction, $\widetilde{y}_n(t) \le \widetilde{y}_{n+1}(t)$ for $t \ge t_0$ and $n = 1, 2, \ldots$.

Lemma 6. *Let $x(t)$ be an eventually positive solution of (1) and suppose that $z(t)$ satisfies case (I). Then $\widetilde{y}_n(t) \le \omega(t)$ for $n = 0, 1, \ldots$, and there exists a function $\widetilde{y}(t) \in C([T,\infty),(0,\infty))$ such that, for $t \ge T \ge t_0$, $\lim_{n\to\infty}\widetilde{y}_n(t) = \widetilde{y}(t)$ and*

$$\widetilde{y}(t) = \widetilde{y}_0(t) + \int_t^{\infty}\widetilde{R}(s)\,\widetilde{y}^{(\alpha+1)/\alpha}(s)\,ds, \tag{19}$$

where $\omega(t)$ and $\widetilde{y}_n(t)$ are as in (13) and (18), respectively.

Proof. Integrating (14) from t to l, we deduce that

$$\omega(l) - \omega(t) + \int_t^{l}\widetilde{R}(s)\,\omega^{(\alpha+1)/\alpha}(s)\,ds + \int_t^{l}Q(s)\,ds \le 0. \tag{20}$$

For every fixed $t \ge T$, we claim that

$$\int_t^{\infty}\widetilde{R}(s)\,\omega^{(\alpha+1)/\alpha}(s)\,ds < \infty. \tag{21}$$

If (21) does not hold, then, for every fixed $t \ge T$,

$$\omega(l) \le \omega(t) - \int_t^{l}\widetilde{R}(s)\,\omega^{(\alpha+1)/\alpha}(s)\,ds \longrightarrow -\infty, \qquad l \longrightarrow \infty, \tag{22}$$

which is a contradiction to $\omega(t) > 0$. It follows now from (14) that there exists a constant $k_2 \ge 0$ such that $\lim_{t\to\infty}\omega(t) = k_2 \ge 0$. Taking into account (21) and the condition $\int_{t_0}^{\infty} r^{-1/\alpha}(t)dt = \infty$, $k_2 = 0$. An application of (20) yields

$$\widetilde{y}_0(t) = \widetilde{Q}(t) \le \widetilde{Q}(t) + \int_t^{\infty}\widetilde{R}(s)\,\omega^{(\alpha+1)/\alpha}(s)\,ds \le \omega(t). \tag{23}$$

By induction, $\widetilde{y}_n(t) \le \omega(t)$ for $t \ge t_0$ and $n = 1, 2, \ldots$. Hence, $\lim_{n\to\infty}\widetilde{y}_n(t) = \widetilde{y}(t)$ when using the fact that the sequence $\{\widetilde{y}_n(t)\}_{n=0}^{\infty}$ is nondecreasing and bounded above. Passing to the limit as $n \to \infty$ in (18) and applying Lebesgue's monotone convergence theorem, one arrives at (19). The proof is complete. □

Theorem 7. *Let hypotheses (h_1), (h_2), and (2) be satisfied. If*

$$\liminf_{t\to\infty}\frac{1}{\widetilde{Q}(t)}\int_t^{\infty}\widetilde{R}(s)\,\widetilde{Q}^{(\alpha+1)/\alpha}(s)\,ds > \frac{\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}}, \tag{24}$$

then every solution $x(t)$ of (1) either is oscillatory or satisfies $\lim_{t\to\infty}x(t) = 0$.

Proof. Without loss of generality, suppose that $x(t)$ is an eventually positive solution of (1). By virtue of Lemma 1, $z(t)$ satisfies either case (I) or case (II) eventually. Assume first that

$z(t)$ satisfies case (I). Proceeding as in the proof of Lemma 6, we obtain (23) and so

$$\begin{aligned}\frac{\omega(t)}{\widetilde{Q}(t)} &\geq 1 \\ &+ \frac{1}{\widetilde{Q}(t)}\int_t^{\infty}\widetilde{R}(s)\widetilde{Q}^{(\alpha+1)/\alpha}(s)\left(\frac{\omega(s)}{\widetilde{Q}(s)}\right)^{(\alpha+1)/\alpha}ds.\end{aligned} \tag{25}$$

Let $\lambda = \inf_{t\geq T}\omega(t)/\widetilde{Q}(t)$. Then $\lambda \geq 1$. Combining (24) and (25), we deduce that

$$\alpha\left(\frac{1}{\alpha+1}\right)^{(\alpha+1)/\alpha}\lambda^{(\alpha+1)/\alpha} - \lambda < -1. \tag{26}$$

However, an application of the inequality (see [3])

$$Au^{(\alpha+1)/\alpha} - Bu \geq -\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}, \quad A > 0 \tag{27}$$

yields

$$\alpha\left(\frac{1}{\alpha+1}\right)^{(\alpha+1)/\alpha}\lambda^{(\alpha+1)/\alpha} - \lambda \geq -1, \tag{28}$$

and so a contradiction is presented.

Assume now that $z(t)$ satisfies case (II). It follows from Lemma 3 that $\lim_{t\to\infty}x(t) = 0$. This completes the proof. □

Theorem 8. *Let hypotheses (h_1), (h_2), and (2) be satisfied and suppose that $\widetilde{y}_n(t)$ is as in (18). If for some $\widetilde{y}_n(t)$ ($n \in \{0, 1, \ldots\}$),*

$$\limsup_{t\to\infty}\widetilde{y}_n(t)\left(\int_{t_0}^{\sigma_1(t)} sr^{-1/\alpha}(s)\,ds\right)^{\alpha} > 1, \tag{29}$$

then all solutions of (1) either are oscillatory or tend to zero asymptotically.

Proof. Assume the opposite. Let $x(t)$ be an eventually positive solution of (1). By virtue of Lemma 1, $z(t)$ satisfies either case (I) or case (II) eventually. Suppose first that $z(t)$ satisfies case (I). Define $\omega(t)$ by (13). Using Lemma 2 and the monotonicity of $r^{1/\alpha}(t)z''(t)$, we conclude that, for $t \geq T \geq t_0$,

$$\begin{aligned}\frac{1}{\omega(t)} &= \frac{1}{r(t)}\left(\frac{z(\sigma_1(t))}{z''(t)}\right)^{\alpha} \\ &= \frac{1}{r(t)}\left(\frac{z(T) + \int_T^{\sigma_1(t)} r^{-1/\alpha}(s)\,r^{1/\alpha}(s)\,z'(s)\,ds}{z''(t)}\right)^{\alpha} \\ &\geq \frac{1}{r(t)}\left(\frac{z(T) + \int_T^{\sigma_1(t)} sr^{-1/\alpha}(s)\,r^{1/\alpha}(s)\,z''(s)\,ds}{z''(t)}\right)^{\alpha} \\ &\geq \frac{1}{r(t)}\left(\frac{r^{1/\alpha}(t)\,z''(t)\int_T^{\sigma_1(t)} sr^{-1/\alpha}(s)\,ds}{z''(t)}\right)^{\alpha} \\ &= \left(\int_T^{\sigma_1(t)} sr^{-1/\alpha}(s)\,ds\right)^{\alpha},\end{aligned} \tag{30}$$

which yields

$$\begin{aligned}\omega(t)\left(\int_{t_0}^{\sigma_1(t)} sr^{-1/\alpha}(s)\,ds\right)^{\alpha} &\leq \left(\frac{\int_{t_0}^{\sigma_1(t)} sr^{-1/\alpha}(s)\,ds}{\int_T^{\sigma_1(t)} sr^{-1/\alpha}(s)\,ds}\right)^{\alpha}.\end{aligned} \tag{31}$$

Hence, we deduce that

$$\limsup_{t\to\infty}\omega(t)\left(\int_{t_0}^{\sigma_1(t)} sr^{-1/\alpha}(s)\,ds\right)^{\alpha} \leq 1. \tag{32}$$

On the other hand, by Lemma 6, $\widetilde{y}_n(t) \leq \omega(t)$ for $n = 0, 1, \ldots$, and so

$$\limsup_{t\to\infty}\widetilde{y}_n(t)\left(\int_{t_0}^{\sigma_1(t)} sr^{-1/\alpha}(s)\,ds\right)^{\alpha} \leq 1, \tag{33}$$

which contradicts (29).

Assume now that $z(t)$ satisfies case (II). By virtue of Lemma 3, $\lim_{t\to\infty}x(t) = 0$. The proof is complete. □

Remark 9. Our results complement and improve those obtained by Tian et al. [9] since these results can be applied to (1) in the case where $0 < \alpha < 1$.

4. Examples

The following examples are included to show applications of the results obtained in this work.

Example 1. For $t \geq 1$ and $q_0 > 0$, consider the nonlinear differential equation

$$\begin{aligned}&\left[\left(\left(x(t) + \frac{1}{2}\int_{-1}^{0} x\left(t + \frac{\xi}{2}\right)d\xi\right)''\right)^{\alpha}\right]' \\ &\quad + \int_0^1 \frac{q_0\xi}{t}x^{\alpha}\left(\frac{t+\xi}{3}\right)d\xi = 0,\end{aligned} \tag{34}$$

where $\alpha > 0$ is the quotient of odd integers. Let $a = -1$, $b = 0$, $c = 0$, $d = 1$, $r(t) = 1$, $p(t,\xi) = P = 1/2$, $q(t,\xi) = q_0\xi/t$, $f(x) = x^{\alpha}$, $\tau(t,\xi) = t + \xi/2$, and $\sigma(t,\xi) = (t+\xi)/3$. It follows from Theorem 4 that every solution $x(t)$ of (34) either is oscillatory or converges to zero asymptotically. Observe that results obtained in [9] cannot be applied to (34) in the case when $0 < \alpha < 1$.

Example 2. For $t \geq 1$, consider the differential equation

$$\begin{aligned}&\left[t\left(x(t) + \int_{1/2}^{1}\frac{4\xi}{3t^2}x\left(\frac{t+\xi}{3}\right)d\xi\right)''\right]' \\ &\quad + \int_0^1 \frac{4\xi}{t^2}x\left(\frac{t+\xi}{2}\right)d\xi = 0.\end{aligned} \tag{35}$$

Let $\alpha = 1$, $a = 1/2$, $b = 1$, $c = 0$, $d = 1$, $r(t) = t$, $p(t,\xi) = 4\xi/(3t^2)$, $q(t,\xi) = 4\xi/t^2$, $f(x) = x$, $\tau(t,\xi) = (t+\xi)/3$, and $\sigma(t,\xi) = (t+\xi)/2$. Then

$$\int_a^b p(t,\xi)\,d\xi = \int_{1/2}^1 \frac{4\xi}{3t^2}d\xi = \frac{1}{2t^2} \le \frac{1}{2} = P. \tag{36}$$

It is not hard to verify that hypotheses (h_1), (h_2), and (2) hold; $\widetilde{R}(t) = 1/2$, and

$$\begin{aligned}\widetilde{Q}(t) &= \int_t^\infty Q(s)\,ds = (1-P)^\alpha \int_t^\infty \int_c^d q(t,\xi)\,d\xi\,ds \\ &= \frac{1}{2}\int_t^\infty \int_0^1 \frac{4\xi}{s^2}d\xi\,ds = \frac{1}{t}.\end{aligned} \tag{37}$$

Hence

$$\begin{aligned}&\liminf_{t\to\infty} \frac{1}{\widetilde{Q}(t)} \int_t^\infty \widetilde{R}(s)\,\widetilde{Q}^{(\alpha+1)/\alpha}(s)\,ds \\ &\quad = \liminf_{t\to\infty} \frac{t}{2}\int_t^\infty \frac{1}{s^2}ds = \frac{1}{2} > \frac{1}{4}.\end{aligned} \tag{38}$$

An application of Theorem 7 implies that all solutions $x(t)$ of (35) either are oscillatory or satisfy $\lim_{t\to\infty} x(t) = 0$.

Competing Interests

The authors declare that they have no competing interests.

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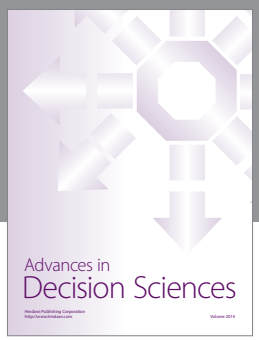

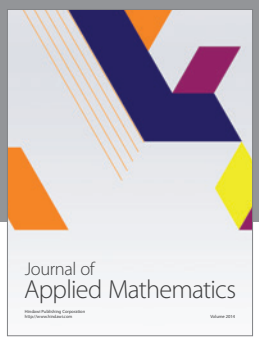

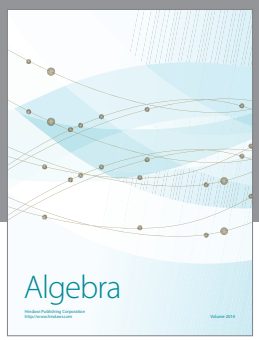

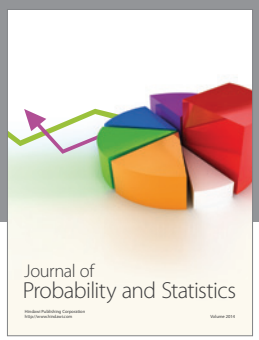

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