Research Article

On the Theory of Multilinear Singular Operators with Rough Kernels on the Weighted Morrey Spaces

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We study some multilinear operators with rough kernels. For the multilinear fractional integral operators $T_{\Omega,\alpha}^A$ and the multilinear fractional maximal integral operators $M_{\Omega,\alpha}^A$, we obtain their boundedness on weighted Morrey spaces with two weights $L^{p,\kappa}(u, v)$ when $D^\gamma A \in \dot{\Lambda}_\beta$ ($|\gamma| = m - 1$) or $D^\gamma A \in \text{BMO}$ ($|\gamma| = m - 1$). For the multilinear singular integral operators T_Ω^A and the multilinear maximal singular integral operators M_Ω^A , we show they are bounded on weighted Morrey spaces with two weights $L^{p,\kappa}(u, v)$ if $D^\gamma A \in \dot{\Lambda}_\beta$ ($|\gamma| = m - 1$) and bounded on weighted Morrey spaces with one weight $L^{p,\kappa}(w)$ if $D^\gamma A \in \text{BMO}$ ($|\gamma| = m - 1$) for $m = 1, 2$.

1. Introduction and Main Results

Let us consider the following multilinear fractional integral operator,

$$T_{\Omega,\alpha}^A f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha+m-1}} R_m(A; x, y) f(y) dy, \quad (1)$$

$$0 < \alpha < n,$$

and the multilinear fractional maximal operator:

$$M_{\Omega,\alpha}^A f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha+m-1}} \cdot \int_{|x-y|<r} |\Omega(x-y) R_m(A; x, y) f(y)| dy, \quad (2)$$

$$0 < \alpha < n,$$

where $\Omega \in L^s(S^{n-1})$ ($s > 1$) is homogeneous of degree zero in $\mathbb{R}^n$, A is a function defined on $\mathbb{R}^n$, and $R_m(A; x, y)$ denotes the m th order Taylor series remainder of A at x expanded about y ; that is,

$$R_m(A; x, y) = A(x) - \sum_{|\gamma|<m} \frac{1}{\gamma!} D^\gamma A(y) (x-y)^\gamma, \quad (3)$$

$\gamma = (\gamma_1, \dots, \gamma_n)$, each γ_i , $i = 1, \dots, n$, is a nonnegative integer, $|\gamma| = \sum_{i=1}^n \gamma_i$, $\gamma! = \gamma_1! \cdots \gamma_n!$, $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$, and $D^\gamma = \partial^{|\gamma|} / \partial^{|\gamma|} x_1 \cdots \partial^{|\gamma|} x_n$.

We notice that if $\alpha = 0$, the above two operators $T_{\Omega,\alpha}^A$, $M_{\Omega,\alpha}^A$ are the multilinear singular integral operator T_Ω^A and the multilinear maximal singular integral operator M_Ω^A whose definitions are given as follows, respectively:

$$T_\Omega^A f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A; x, y) f(y) dy, \quad (4)$$

$$M_\Omega^A f(x) = \sup_{r>0} \frac{1}{r^{n+m-1}} \cdot \int_{|x-y|<r} |\Omega(x-y) R_m(A; x, y) f(y)| dy. \quad (5)$$

For $m = 1$, $T_{\Omega,\alpha}^A$ is obviously the commutator $[A, T_{\Omega,\alpha}]$ of $T_{\Omega,\alpha}$ and A : $[A, T_{\Omega,\alpha}] f(x) = A(x) T_{\Omega,\alpha} f(x) - T_{\Omega,\alpha}(Af)(x)$, where $T_{\Omega,\alpha}$ is the fractional integral operator given by

$$T_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y) dy, \quad 0 < \alpha < n. \quad (6)$$

There are numerous works on the study of multilinear operators with rough kernels. If $D^\gamma A \in \text{BMO}$ ($|\gamma| = m - 1$),

the L^p boundedness of T_Ω^A was obtained by means of a good- λ inequality by Cohen and Gosselin [1]. In 1994, Hofmann [2] proved that T_Ω^A is a bounded operator on $L^p(w)$ when $\nabla A \in BMO$ and $w \in A_{p,\beta}$. Recently, Lu et al. [3] proved T_Ω^A and M_Ω^A are bounded from L^p to L^q ($1/p - 1/q = \beta/n$) when $D^\gamma A \in \dot{\Lambda}_\beta$ ($|\gamma| = m - 1$), while for multilinear fractional integral operators, Ding and Lu [4] showed the $(L^p(w^p), L^q(w^q))$ boundedness of $T_{\Omega,\alpha}^{A_1,\dots,A_k}$ and $M_{\Omega,\alpha}^{A_1,\dots,A_k}$ (their definitions will be given later) if $D^\gamma A_j \in BMO$ ($|\gamma| = m - 1$), $j = 1, \dots, k$. After that, Lu and Zhang [5] proved $T_{\Omega,\alpha}^A$ is a bounded operator from L^p to L^q ($1/p - 1/q = (\alpha + \beta)/n$) when $D^\gamma A \in \dot{\Lambda}_\beta$ ($|\gamma| = m - 1$).

On the other hand, the classical Morrey spaces were first introduced by Morrey [6] to study the local behavior of solutions to second-order elliptic partial differential equations. From then on, a lot of works concerning Morrey spaces and some related spaces have been done; see [7–9] and the references therein for details. In 2009, Komori and Shirai [10] first studied the weighted Morrey spaces and investigated some classical singular integrals in harmonic analysis on them, such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator, the fractional integral operator, and the fractional maximal operator. Recently, Wang [11] discussed the boundedness of the classical singular operators with rough kernels on the weighted Morrey spaces.

We note that many works concerning $T_{\Omega,\alpha}^A$, $M_{\Omega,\alpha}^A$, T_Ω^A , and M_Ω^A have been done on L^p spaces or weighted L^p spaces when $D^\gamma A$ belongs to some function spaces for $|\gamma| = m - 1$. However, there is not any study about these operators on weighted Morrey spaces. Therefore, it is natural to ask whether they are bounded on weighted Morrey spaces. The aim of this paper is to investigate the boundedness of $T_{\Omega,\alpha}^A$, $M_{\Omega,\alpha}^A$, T_Ω^A , and M_Ω^A on weighted Morrey spaces if $D^\gamma A \in \dot{\Lambda}_\beta$ ($|\gamma| = m - 1$) or $D^\gamma A \in BMO$ ($|\gamma| = m - 1$). When $D^\gamma A \in \dot{\Lambda}_\beta$ ($|\gamma| = m - 1$), we show $T_{\Omega,\alpha}^A$ and T_Ω^A are controlled pointwisely by the fractional singular integral operators $\bar{T}_{\Omega,\alpha+\beta}$ and $\bar{T}_{\Omega,\beta}$ (their definition will be given later), respectively. Thus, the problem of studying the boundedness of $T_{\Omega,\alpha}^A$ and T_Ω^A on weighted Morrey spaces with two weights could be reduced to that of $\bar{T}_{\Omega,\alpha+\beta}$ and $\bar{T}_{\Omega,\beta}$. When $D^\gamma A \in BMO$ ($|\gamma| = m - 1$), the boundedness of $T_{\Omega,\alpha}^A$ on weighted Morrey spaces with two weights is proved by standard method. However, we could only obtain the boundedness of T_Ω^A on weighted Morrey spaces with one weight for $m = 1$ and $m = 2$, since we need the $L^p(w)$ boundedness of T_Ω^A in our proof, but to the best of our knowledge, there is not such bounds hold for T_Ω^A when $m \geq 3$. For $M_{\Omega,\alpha}^A$ and M_Ω^A , we show they are controlled pointwisely by $T_{\Omega,\alpha}^A$ and T_Ω^A , respectively. Thus, it is easy to obtain the same results for $M_{\Omega,\alpha}^A$ and M_Ω^A as those of $T_{\Omega,\alpha}^A$ and T_Ω^A .

Before stating our main results, we introduce some definitions and notations at first.

A weight is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, \infty)$ almost everywhere. For a weight w

and a measurable set E , we define $w(E) = \int_E w(x)dx$, the Lebesgue measure of E by $|E|$ and the characteristic function of E by χ_E . The weighted Lebesgue spaces with respect to the measure $w(x)dx$ are denoted by $L^p(w)$ with $0 < p < \infty$. We say a weight w satisfies the doubling condition if there exists a constant $D > 0$ such that for any ball B , we have $w(2B) \leq Dw(B)$. When w satisfies this condition, we denote $w \in \Delta_2$ for short.

Throughout this paper, $B(x_0, r)$ denotes a ball centered at x_0 with radius r . Let Q be a cube with sides parallel to the axes. For $K > 0$, KQ denotes the cube with the same center as Q and side-length being K times longer. When $\alpha = 0$, we will denote $T_{\Omega,\alpha}$, $T_{\Omega,\alpha}^A$, $M_{\Omega,\alpha}^A$ by T_Ω , T_Ω^A , M_Ω^A , respectively. And for any number a , a' stands for the conjugate of a . The letter C denotes a positive constant that may vary at each occurrence but is independent of the essential variable.

Next, we give the definition of weighted Morrey space introduced in [10].

Definition 1. Let $1 \leq p < \infty$, let $0 < \kappa < 1$, and let w be a weight. Then the weighted Morrey space is defined by

$$L^{p,\kappa}(w) := \{f \in L_{\text{loc}}^p(w) : \|f\|_{L^{p,\kappa}(w)} < \infty\}, \quad (7)$$

where

$$\|f\|_{L^{p,\kappa}(w)} = \sup_B \left(\frac{1}{w(B)^\kappa} \int_B |f(x)|^p w(x) dx \right)^{1/p}, \quad (8)$$

and the supremum is taken over all balls B in $\mathbb{R}^n$.

When we investigate the boundedness of the multilinear fractional integral operator, we need to consider the weighted Morrey space with two weights. It is defined as follows.

Definition 2. Let $1 \leq p < \infty$, let $0 < \kappa < 1$, and let u, v be two weights. The two weights weighted Morrey space is defined by

$$L^{p,\kappa}(u, v) := \{f : \|f\|_{L^{p,\kappa}(u,v)} < \infty\}, \quad (9)$$

where

$$\|f\|_{L^{p,\kappa}(u,v)} = \sup_B \left(\frac{1}{v(B)^\kappa} \int_B |f(x)|^p u(x) dx \right)^{1/p}, \quad (10)$$

and the supremum is taken over all balls B in $\mathbb{R}^n$. If $u = v$, then we denote $L^{p,\kappa}(u)$ for short.

As is pointed out in [10], we could also define the weighted Morrey spaces with cubes instead of balls. So we shall use these two definitions of weighted Morrey spaces appropriate to calculation.

Then, we give the definitions of Lipschitz space and BMO space.

Definition 3. The Lipschitz space of order β , $0 < \beta < 1$, is defined by

$$\dot{\Lambda}_\beta(\mathbb{R}^n) = \{f : |f(x) - f(y)| \leq C|x - y|^\beta\}, \quad (11)$$

and the smallest constant $C > 0$ is the Lipschitz norm $\|\cdot\|_{\dot{\Lambda}_\beta}$.

Definition 4. A locally integrable function b is said to be in $\text{BMO}(\mathbb{R}^n)$ if

$$\|b\|_* = \|b\|_{\text{BMO}} = \sup_B \frac{1}{|B|} \int_B |b(x) - b_B| dx < \infty, \quad (12)$$

where

$$b_B = \frac{1}{|B|} \int_B b(y) dy, \quad (13)$$

and the supremum is taken over all balls B in $\mathbb{R}^n$.

At last, we give the definition of two weight classes.

Definition 5. A weight function w is in the Muckenhoupt class A_p with $1 < p < \infty$ if there exists $C > 1$ such that for any ball B ,

$$\left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C. \quad (14)$$

We define $A_\infty = \bigcup_{1 < p < \infty} A_p$.

When $p = 1$, we define $w \in A_1$ if there exists $C > 1$ such that for almost every x ,

$$Mw(x) \leq Cw(x). \quad (15)$$

Definition 6. A weight function w belongs to $A(p, q)$ for $1 < p < q < \infty$ if there exists $C > 1$ such that for any ball B ,

$$\left(\frac{1}{|B|} \int_B w(x)^q dx \right)^{1/q} \cdot \left(\frac{1}{|B|} \int_B w(x)^{-p/(p-1)} dx \right)^{(p-1)/p} \leq C. \quad (16)$$

When $p = 1$, then we define $w \in A(1, q)$ with $1 < q < \infty$ if there exists $C > 1$ such that

$$\left(\frac{1}{|B|} \int_B w(x)^q dx \right)^{1/q} \left(\text{ess sup}_{x \in B} \frac{1}{w(x)} \right) \leq C. \quad (17)$$

Remark 7 (see [10]). If $w \in A(p, q)$ with $1 < p < q$, then

- (a) $w^q, w^{-p'}, w^{-q'} \in \Delta_2$.
- (b) $w^{-p'} \in A_{p'}$ with $t = 1 + q/p'$.

Now we state the main results of this paper.

Theorem 8. If $0 < \alpha + \beta < n$, $\Omega \in L^s(S^{n-1})$ ($s > 1$) is homogeneous of degree zero, $1 < s' < p < n/(\alpha + \beta)$, $1/q = 1/p - (\alpha + \beta)/n$, $0 < \kappa < p/q$, $w^{s'} \in A(p/s', q/s')$, $D^\gamma A \in \dot{\Lambda}_\beta$ ($|\gamma| = m - 1$), then

$$\|T_{\Omega, \alpha}^A f\|_{L^{q, \kappa q/p}(w^q)} \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \|f\|_{L^{p, \kappa}(w^p, w^q)}, \quad (18)$$

$$\|M_{\Omega, \alpha}^A f\|_{L^{q, \kappa q/p}(w^q)} \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \|f\|_{L^{p, \kappa}(w^p, w^q)}. \quad (19)$$

Theorem 9. If $0 < \beta < 1$, $\Omega \in L^s(S^{n-1})$ ($s > 1$) is homogeneous of degree zero, $1 < s' < p < n/\beta$, $1/q = 1/p - \beta/n$, $0 < \kappa < p/q$, $w^{s'} \in A(p/s', q/s')$, $D^\gamma A \in \dot{\Lambda}_\beta$ ($|\gamma| = m - 1$), then

$$\|T_{\Omega}^A f\|_{L^{q, \kappa q/p}(w^q)} \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \|f\|_{L^{p, \kappa}(w^p, w^q)}, \quad (20)$$

$$\|M_{\Omega}^A f\|_{L^{q, \kappa q/p}(w^q)} \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \|f\|_{L^{p, \kappa}(w^p, w^q)}. \quad (21)$$

Theorem 10. If $0 < \alpha < n$, $\Omega \in L^s(S^{n-1})$ ($s > 1$) is homogeneous of degree zero, $1 < s' < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \kappa < p/q$, $w^{s'} \in A(p/s', q/s')$, $D^\gamma A \in \text{BMO}$ ($|\gamma| = m - 1$), then

$$\|T_{\Omega, \alpha}^A f\|_{L^{q, \kappa q/p}(w^q)} \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p, \kappa}(w^p, w^q)}, \quad (22)$$

$$\|M_{\Omega, \alpha}^A f\|_{L^{q, \kappa q/p}(w^q)} \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p, \kappa}(w^p, w^q)}. \quad (23)$$

When $m = 1$ and $m = 2$, we denote T_{Ω}^A , M_{Ω}^A by $[A, T_{\Omega}]$, $[A, M_{\Omega}]$, and $\tilde{T}_{\Omega}^A$, $\tilde{M}_{\Omega}^A$, respectively, in order to distinguish from T_{Ω}^A and M_{Ω}^A that are defined for any $m \in \mathbb{N}^*$. To be more precise,

$$\begin{aligned} [A, T_{\Omega}] f(x) &= \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} (A(x) - A(y)) \\ &\quad \cdot f(y) dy, \\ [A, M_{\Omega}] f(x) &= \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} \Omega(x-y) \\ &\quad \cdot (A(x) - A(y)) f(y) dy, \\ \tilde{T}_{\Omega}^A f(x) &= \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}} (A(x) - A(y) - \nabla A(y) \\ &\quad \cdot (x-y)) f(y) dy, \\ \tilde{M}_{\Omega}^A f(x) &= \sup_{r>0} \frac{1}{r^{n+1}} \int_{|x-y|<r} \Omega(x-y) \\ &\quad \cdot (A(x) - A(y) - \nabla A(y) \cdot (x-y)) f(y) dy. \end{aligned} \quad (24)$$

Then for the above operators, we have the following results on weighted Morrey spaces with one weight.

Theorem 11. If $\Omega \in L^s(S^{n-1})$ ($s > 1$) is homogeneous of degree zero and satisfies the vanishing condition $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$, $1 < s' < p < \infty$, $0 < \kappa < 1$, $w \in A_{p/s'}$, $A \in \text{BMO}$, then

$$\|[A, T_{\Omega}] f\|_{L^{p, \kappa}(w)} \leq C \|A\|_* \|f\|_{L^{p, \kappa}(w)}, \quad (25)$$

$$\|[A, M_{\Omega}] f\|_{L^{p, \kappa}(w)} \leq C \|A\|_* \|f\|_{L^{p, \kappa}(w)}. \quad (26)$$

Theorem 12. If $\Omega \in L^\infty(S^{n-1})$ is homogeneous of degree zero and satisfies the moment condition $\int_{S^{n-1}} \theta \Omega(\theta) d\theta = 0$, $1 < p < \infty$, $0 < \kappa < 1$, $w \in A_p$, $\nabla A \in \text{BMO}$, then

$$\|\tilde{T}_\Omega^A f\|_{L^{p,\kappa}(w)} \leq C \|\nabla A\|_* \|f\|_{L^{p,\kappa}(w)}, \quad (27)$$

$$\|\tilde{M}_\Omega^A f\|_{L^{p,\kappa}(w)} \leq C \|\nabla A\|_* \|f\|_{L^{p,\kappa}(w)}. \quad (28)$$

Remark 13. Here we point out that for T_Ω^A and M_Ω^A , when $D^y A \in \text{BMO}$ ($|\gamma| = m - 1$), the analogues of Theorems 11 and 12 are open for $m \geq 3$.

Remark 14. Define

$$\begin{aligned} T_{\Omega,\alpha}^{A_1,\dots,A_k} f(x) &= \int_{\mathbb{R}^n} \prod_{i=1}^k R_{m_i}(A_i; x, y) \\ &\quad \cdot \frac{\Omega(x-y)}{|x-y|^{n-\alpha+N}} f(y) dy, \\ M_{\Omega,\alpha}^{A_1,\dots,A_k} f(x) &= \sup_{r>0} \frac{1}{r^{n-\alpha+N}} \int_{|x-y|<r} |\Omega(x-y)| \\ &\quad \cdot \prod_{i=1}^k |R_{m_i}(A_i; x, y)| |f(y)| dy, \end{aligned} \quad (29)$$

where $R_{m_i}(A_i; x, y) = A_i(x) - \sum_{|\gamma|<m_i} (1/\gamma!) D^\gamma A_i(y) (x - y)^\gamma$, $i = 1, \dots, k$, $N = \sum_{i=1}^k (m_i - 1)$. When $0 < \alpha < n$, they are a class of multilinear fractional integral operators and multilinear fractional maximal operators. When $\alpha = 0$, they are a class of multilinear singular integral operators and multilinear maximal singular integral operators. Repeating the proofs of the theorems above, we will find that for $T_{\Omega,\alpha}^{A_1,\dots,A_k}$ and $M_{\Omega,\alpha}^{A_1,\dots,A_k}$, the conclusions of Theorems 8 and 9 above with the bounds $C \prod_{i=1}^k (\sum_{|\gamma|=m_i-1} \|D^\gamma A_i\|_{\dot{A}_\beta})$ and Theorem 10 with the bounds $C \prod_{i=1}^k (\sum_{|\gamma|=m_i-1} \|D^\gamma A_i\|_*)$ also hold, respectively.

The organization of this paper is as follows. In Section 2, we give some requisite lemmas and well-known results that are important in proving the theorems. The proof of the theorems will be shown in Section 3.

2. Lemmas and Well-Known Results

Lemma 15 (see [1]). Let $A(x)$ be a function on $\mathbb{R}^n$ with m th order derivatives in $L_{\text{loc}}^l(\mathbb{R}^n)$ for some $l > n$. Then

$$\begin{aligned} |R_m(A; x, y)| \\ \leq C |x - y|^m \sum_{|\gamma|=m} \left(\frac{1}{|I_x^\gamma|} \int_{I_x^\gamma} |D^\gamma A(z)|^l dz \right)^{1/l}, \end{aligned} \quad (30)$$

where I_x^γ is the cube centered at x with sides parallel to the axes, whose diameter is $5\sqrt{n}|x - y|$.

Lemma 16 (see [12]). For $0 < \beta < 1$, $1 \leq q < \infty$, we have

$$\begin{aligned} \|f\|_{\dot{A}_\beta} &\approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - b_Q(f)| dx \\ &\approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |f(x) - b_Q(f)|^q dx \right)^{1/q}. \end{aligned} \quad (31)$$

For $q = \infty$, the formula should be interpreted appropriately.

Lemma 17 (see [13]). Let $Q_1 \subset Q_2$, $g \in \dot{A}_\beta$ ($0 < \beta < 1$). Then

$$|g_{Q_1} - g_{Q_2}| \leq C |Q_2|^{\beta/n} \|g\|_{\dot{A}_\beta}. \quad (32)$$

Theorem 18 (see [14]). Suppose that $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $\Omega \in L^s(S^{n-1})$ ($s > 1$) is homogeneous of degree zero. Then $T_{\Omega,\alpha}$ is a bounded operator from $L^p(w^p)$ to $L^q(w^q)$, if the index set $\{\alpha, p, q, s\}$ satisfies one of the following conditions:

- (a) $s' < p$ and $w(x)^{s'} \in A(p/s', q/s')$;
- (b) $s > q$ and $w(x)^{-s'} \in A(q'/s', p'/s')$;
- (c) $\alpha/n + 1/s < 1/p < 1/s'$ and there is r , $1 < r < s/(n/\alpha)'$ such that $w(x)^{r'} \in A(p, q)$.

Lemma 19 (see [10]). If $w \in \Delta_2$, then there exists a constant $D_1 > 1$, such that

$$w(2B) \geq D_1 w(B). \quad (33)$$

We call D_1 the reverse doubling constant.

Theorem 20 (see [4]). Suppose that $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $\Omega \in L^s(S^{n-1})$ ($s > 1$) is homogeneous of degree zero. Moreover, for $1 \leq i \leq k$, $|\gamma| = m_i - 1$, $m_i \geq 2$, and $D^\gamma A_i \in \text{BMO}(\mathbb{R}^n)$, if the index set $\{\alpha, p, q, s\}$ satisfies one of the following conditions:

- (a) $s' < p$ and $w(x)^{s'} \in A(p/s', q/s')$;
- (b) $s > q$ and $w(x)^{-s'} \in A(q'/s', p'/s')$;
- (c) $\alpha/n + 1/s < 1/p < 1/s'$ and there is r , $1 < r < s/(n/\alpha)'$, such that $w(x)^{r'} \in A(p, q)$.

Then there is a $C > 0$, independent of f and A_i , such that

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} |T_{\Omega,\alpha}^{A_1,\dots,A_k} f(x) w(x)|^q dx \right)^{1/q} \\ &\leq C \prod_{i=1}^k \left(\sum_{|\gamma|=m_i-1} \|D^\gamma A_i\|_* \right) \\ &\quad \cdot \left(\int_{\mathbb{R}^n} |f(x) w(x)|^p dx \right)^{1/p}. \end{aligned} \quad (34)$$

Lemma 21 (see [15]). (a) (John-Nirenberg Lemma) Let $1 \leq p < \infty$. Then $b \in \text{BMO}$ if and only if

$$\frac{1}{|Q|} \int_Q |b - b_Q|^p dx \leq C \|b\|_*^p. \quad (35)$$

(b) Assume $b \in \text{BMO}$; then for cubes $Q_1 \subset Q_2$,

$$|b_{Q_1} - b_{Q_2}| \leq C \log \left(\frac{|Q_2|}{|Q_1|} \right) \|b\|_*. \quad (36)$$

(c) If $b \in \text{BMO}$, then

$$|b_{2^{j+1}B} - b_B| \leq 2^n (j+1) \|b\|_*. \quad (37)$$

Theorem 22 (see [16]). Suppose that $\Omega \in L^s(S^{n-1})$ ($s > 1$) is homogeneous of degree zero and satisfies the vanishing condition $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$. If $b \in \text{BMO}(\mathbb{R}^n)$, then $[b, T_\Omega]$ is bounded on $L^p(w)$ if the index set $\{p, q, s\}$ satisfies one of the following conditions:

(a) $s' \leq p < \infty$, $p \neq 1$ and $w \in A_{p/s'}$;

(b) $1 \leq p \leq s$, $p \neq \infty$ and $w^{1-p'} \in A_{p'/s'}$;

(c) $1 \leq p < \infty$ and $w^{s'} \in A_p$.

Theorem 23 (see [2]). If $\Omega \in L^\infty(S^{n-1})$ is homogeneous of degree zero and satisfies the moment condition $\int_{S^{n-1}} \theta \Omega(\theta) d\theta = 0$, $w \in A_p$, $1 < p < \infty$, $\nabla A \in \text{BMO}$, then we have

$$\|\bar{T}_\Omega^A f\|_{L^p(w)} \leq C \|\Omega\|_\infty \|\nabla A\|_* \|f\|_{L^p(w)}. \quad (38)$$

Lemma 24 (see [15]). The following are true:

(1) If $w \in A_p$ for some $1 \leq p < \infty$, then $w \in \Delta_2$. More precisely, for all $\lambda > 1$ we have

$$w(\lambda Q) \leq C \lambda^{np} w(Q). \quad (39)$$

(2) If $w \in A_p$ for some $1 \leq p < \infty$, then there exist $C > 0$ and $\delta > 0$ such that for any cube Q and a measurable set $S \subset Q$,

$$\frac{w(S)}{w(Q)} \leq C \left(\frac{|S|}{|Q|} \right)^\delta. \quad (40)$$

Lemma 25 (see [17]). Let $w \in A_\infty$. Then the norm of $\text{BMO}(w)$ is equivalent to the norm of $\text{BMO}(\mathbb{R}^n)$, where

$$\begin{aligned} \text{BMO}(w) &= \left\{ b : \|b\|_{*,w} \right\} \\ &= \sup_Q \frac{1}{w(Q)} \int_Q |b(x) - m_{Q,w} b| w(x) dx, \\ m_{Q,w} b &= \frac{1}{w(Q)} \int_Q b(x) w(x) dx. \end{aligned} \quad (41)$$

3. Proofs of the Main Results

Before proving Theorem 8, we give a pointwise estimate of $T_{\Omega,\alpha}^A f(x)$ at first. Set

$$\begin{aligned} \bar{T}_{\Omega,\alpha+\beta}^A f(x) &= \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-\beta}} |f(y)| dy, \\ 0 < \alpha + \beta < n, \end{aligned} \quad (42)$$

where $\Omega \in L^s(S^{n-1})$ ($s > 1$) is homogeneous of degree zero in $\mathbb{R}^n$. Then we have the following estimate.

Theorem 26. If $\alpha \geq 0$, $0 < \alpha + \beta < n$, $D^\gamma A \in \dot{\Lambda}_\beta$ ($|\gamma| = m-1$), then there exists a constant C independent of f such that

$$|T_{\Omega,\alpha}^A f(x)| \leq C \left(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) \bar{T}_{\Omega,\alpha+\beta} f(x). \quad (43)$$

Proof. For fixed $x \in \mathbb{R}^n$, $r > 0$, let Q be a cube with center at x and diameter r . Denote $Q_k = 2^k Q$ and set

$$A_{Q_k}(y) = A(y) - \sum_{|\gamma|=m-1} \frac{1}{\gamma!} m_{Q_k}(D^\gamma A) y^\gamma, \quad (44)$$

where $m_{Q_k} f$ is the average of f on Q_k . Then we have, when $|\gamma| = m-1$,

$$D^\gamma A_{Q_k}(y) = D^\gamma A(y) - m_{Q_k}(D^\gamma A), \quad (45)$$

and it is proved in [1] that

$$R_m(A; x, y) = R_m(A_{Q_k}; x, y). \quad (46)$$

Hence,

$$\begin{aligned} |T_{\Omega,\alpha}^A f(x)| &\leq \sum_{k=-\infty}^{\infty} \int_{2^{k-1}r \leq |x-y| < 2^k r} \frac{|R_m(A_{Q_k}; x, y)|}{|x-y|^{m-1}} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)| dy \\ &:= \sum_{k=-\infty}^{\infty} T_k. \end{aligned} \quad (47)$$

By Lemma 15 we get

$$\begin{aligned} |R_m(A_{Q_k}; x, y)| &\leq |R_{m-1}(A_{Q_k}; x, y)| \\ &\quad + C \sum_{|\gamma|=m-1} |D^\gamma A_{Q_k}(y)| |x-y|^{m-1} \\ &\leq C |x-y|^{m-1} \sum_{|\gamma|=m-1} \left(\frac{1}{|I_x^\gamma|} \int_{I_x^\gamma} |D^\gamma A_{Q_k}(z)|^l dz \right)^{1/l} \\ &\quad + C |x-y|^{m-1} \sum_{|\gamma|=m-1} |D^\gamma A_{Q_k}(y)|. \end{aligned} \quad (48)$$

Note that, if $|x - y| < 2^k r$, then $I_x^\gamma \subset 5nQ_k$. By Lemmas 16 and 17 we have, when $|\gamma| = m - 1$,

$$\begin{aligned}
 & \left(\frac{1}{|I_x^\gamma|} \int_{I_x^\gamma} |D^\gamma A_{Q_k}(z)|^l dz \right)^{1/l} \\
 &= \left(\frac{1}{|I_x^\gamma|} \int_{I_x^\gamma} |D^\gamma A(z) - m_{Q_k}(D^\gamma A)|^l dz \right)^{1/l} \\
 &\leq \left(\frac{1}{|I_x^\gamma|} \int_{I_x^\gamma} |D^\gamma A(z) - m_{I_x^\gamma}(D^\gamma A)|^l dz \right)^{1/l} \\
 &\quad + |m_{I_x^\gamma}(D^\gamma A) - m_{5nQ_k}(D^\gamma A)| \\
 &\quad + |m_{5nQ_k}(D^\gamma A) - m_{Q_k}(D^\gamma A)| \\
 &\leq C |Q_k|^{\beta/n} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \leq C (2^k r)^\beta \|D^\gamma A\|_{\dot{\Lambda}_\beta}.
 \end{aligned} \tag{49}$$

It is obvious that when $|\gamma| = m - 1$,

$$\begin{aligned}
 |D^\gamma A_{Q_k}(y)| &= |D^\gamma A(y) - m_{Q_k}(D^\gamma A)| \\
 &\leq C |Q_k|^{\beta/n} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \\
 &\leq C (2^k r)^\beta \|D^\gamma A\|_{\dot{\Lambda}_\beta}.
 \end{aligned} \tag{50}$$

Thus,

$$\begin{aligned}
 & |R_m(A_{Q_k}; x, y)| \\
 &\leq C |x - y|^{m-1} (2^k r)^\beta \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}.
 \end{aligned} \tag{51}$$

Therefore,

$$\begin{aligned}
 T_k &\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \\
 &\cdot \int_{2^{k-1}r \leq |x-y| < 2^k r} \frac{(2^k r)^\beta}{|x-y|^{n-\alpha}} |\Omega(x-y)| |f(y)| dy \\
 &\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \\
 &\cdot \int_{2^{k-1}r \leq |x-y| < 2^k r} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-\beta}} |f(y)| dy.
 \end{aligned} \tag{52}$$

It follows that

$$\begin{aligned}
 |T_{\Omega, \alpha}^A f(x)| &\leq \sum_{k=-\infty}^{\infty} \left(C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right. \\
 &\cdot \int_{2^{k-1}r \leq |x-y| < 2^k r} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-\beta}} |f(y)| dy \Big) \\
 &\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \\
 &\cdot \sum_{k=-\infty}^{\infty} \int_{2^{k-1}r \leq |x-y| < 2^k r} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-\beta}} |f(y)| dy \\
 &= C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-\beta}} |f(y)| dy \\
 &= C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \bar{T}_{\Omega, \alpha+\beta} f(x).
 \end{aligned} \tag{53}$$

Thus, we finish the proof of Theorem 26. $\square$

The following theorem is a key theorem in proving (18) of Theorem 8.

Theorem 27. *Under the same assumptions of Theorem 8, $\bar{T}_{\Omega, \alpha+\beta}$ is bounded from $L^{p, \kappa}(w^p, w^q)$ to $L^{q, \kappa q/p}(w^q)$.*

Proof. Fix a ball $B(x_0, r_B)$, we decompose $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$. Then we have

$$\begin{aligned}
 & \left(\frac{1}{w^q(B)^{\kappa q/p}} \int_B |\bar{T}_{\Omega, \alpha+\beta} f(x)|^q w^q(x) dx \right)^{1/q} \\
 &\leq \frac{1}{w^q(B)^{\kappa/p}} \left(\int_B |\bar{T}_{\Omega, \alpha+\beta} f_1(x)|^q w^q(x) dx \right)^{1/q} \\
 &\quad + \frac{1}{w^q(B)^{\kappa/p}} \left(\int_B |\bar{T}_{\Omega, \alpha+\beta} f_2(x)|^q w^q(x) dx \right)^{1/q} \\
 &:= J_1 + J_2.
 \end{aligned} \tag{54}$$

We estimate J_1 at first. By Remark 7(a) we know that $w^q \in \Delta_2$. Then by Theorem 18(a) and the fact that $w^q \in \Delta_2$ we get,

$$\begin{aligned}
 J_1 &\leq \frac{1}{w^q(B)^{\kappa/p}} \|\bar{T}_{\Omega, \alpha+\beta} f_1\|_{L^q(w^q)} \\
 &\leq \frac{C}{w^q(B)^{\kappa/p}} \|f_1\|_{L^p(w^p)} \\
 &= \frac{C}{w^q(B)^{\kappa/p}} \left(\int_{2B} |f(x)|^p w(x)^p dx \right)^{1/p} \\
 &\leq C \|f\|_{L^{p, \kappa}(w^p, w^q)} \frac{w^q(2B)^{\kappa/p}}{w^q(B)^{\kappa/p}} \leq C \|f\|_{L^{p, \kappa}(w^p, w^q)}.
 \end{aligned} \tag{55}$$

Now we consider the term J_2 . By Hölder's inequality, we have

$$\begin{aligned} |\bar{T}_{\Omega, \alpha + \beta} f_2(x)| &= \sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^jB} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-\beta}} |f(y)| dy \\ &\leq C \sum_{j=1}^{\infty} \left(\int_{2^{j+1}B} |\Omega(x-y)|^s dy \right)^{1/s} \\ &\quad \cdot \left(\int_{2^{j+1}B \setminus 2^jB} \frac{|f(y)|^{s'}}{|x-y|^{(n-\alpha-\beta)s'}} dy \right)^{1/s'} \\ &:= C \sum_{j=1}^{\infty} (I_{1j} I_{2j}). \end{aligned} \quad (56)$$

We will estimate I_{1j} , I_{2j} , respectively. Let $z = x - y$; then for $x \in B$, $y \in 2^{j+1}B$, we have $z \in 2^{j+2}B$. Noticing that Ω is homogeneous of degree zero and $\Omega \in L^s(S^{n-1})$, then we have

$$\begin{aligned} I_{1j} &= \left(\int_{2^{j+2}B} |\Omega(z)|^s dz \right)^{1/s} \\ &= \left(\int_0^{2^{j+2}r_B} \int_{S^{n-1}} |\Omega(z')|^s dz' r^{n-1} dr \right)^{1/s} \\ &= C \|\Omega\|_{L^s(S^{n-1})} |2^{j+2}B|^{1/s}, \end{aligned} \quad (57)$$

where $z' = z/|z|$. For $x \in B$, $y \in (2B)^c$, we have $|x - y| \sim |x_0 - y|$. Thus,

$$I_{2j} \leq \frac{C}{|2^{j+1}B|^{1-(\alpha+\beta)/n}} \left(\int_{2^{j+1}B} |f(y)|^{s'} dy \right)^{1/s'}. \quad (58)$$

By Hölder's inequality and $w^{s'} \in A(p/s', q/s')$, we get

$$\begin{aligned} &\left(\int_{2^{j+1}B} |f(y)|^{s'} dy \right)^{1/s'} \\ &\leq C \left(\int_{2^{j+1}B} |f(y)|^p w(y)^p dy \right)^{1/p} \\ &\quad \cdot \left(\int_{2^{j+1}B} w(y)^{-ps'/(p-s')} dy \right)^{(p-s')/ps'} \\ &\leq C \|f\|_{L^{p,\kappa}(w^p, w^q)} w^q (2^{j+1}B)^{\kappa/p} \\ &\quad \cdot \left(\int_{2^{j+1}B} w(y)^{-ps'/(p-s')} dy \right)^{(p-s')/ps'} \\ &\leq C \|f\|_{L^{p,\kappa}(w^p, w^q)} w^q (2^{j+1}B)^{\kappa/p} \\ &\quad \cdot \frac{|2^{j+1}B|^{(pq-s'q+s'p)/pq s'}}{w^q (2^{j+1}B)^{1/q}}. \end{aligned} \quad (59)$$

Thus,

$$\begin{aligned} |\bar{T}_{\Omega, \alpha + \beta} f_2(x)| &\leq C \sum_{j=1}^{\infty} (I_{1j} I_{2j}) \\ &\leq C \sum_{j=1}^{\infty} \|f\|_{L^{p,\kappa}(w^p, w^q)} \frac{1}{w^q (2^{j+1}B)^{1/q-\kappa/p}}. \end{aligned} \quad (60)$$

So we get

$$J_2 \leq C \|f\|_{L^{p,\kappa}(w^p, w^q)} \sum_{j=1}^{\infty} \frac{w^q(B)^{1/q-\kappa/p}}{w^q (2^{j+1}B)^{1/q-\kappa/p}}. \quad (61)$$

We know from Remark 7(a) and Lemma 19 that w^q satisfies inequality (33), so the above series converges since the reverse doubling constant is larger than one. Hence,

$$J_2 \leq C \|f\|_{L^{p,\kappa}(w^p, w^q)}. \quad (62)$$

Therefore, the proof of Theorem 27 is completed. $\square$

Remark 28. It is worth noting that Theorem 27 is essentially verifying the multilinear fractional operator $T_{\Omega, \alpha}$ is bounded on weighted Morrey spaces.

Now we are in the position of proving Theorem 8.

We will obtain (18) immediately in combination of Theorems 26 and 27.

Then let us turn to prove (19).

Set

$$\begin{aligned} \bar{T}_{\Omega, \alpha}^A f(x) &= \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha+m-1}} |R_m(A; x, y)| |f(y)| dy, \end{aligned} \quad (63)$$

$$0 \leq \alpha < n,$$

where $\Omega \in L^s(S^{n-1})$ ($s > 1$) is homogeneous of degree zero in $\mathbb{R}^n$. It is easy to see inequality (18) also holds for $\bar{T}_{\Omega, \alpha}^A$. On the other hand, for any $r > 0$, we have

$$\begin{aligned} \bar{T}_{\Omega, \alpha}^A f(x) &\geq \int_{|x-y|<r} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha+m-1}} |R_m(A; x, y)| |f(y)| dy \\ &\geq \frac{1}{r^{n-\alpha+m-1}} \\ &\quad \cdot \int_{|x-y|<r} |\Omega(x-y)| |R_m(A; x, y)| |f(y)| dy. \end{aligned} \quad (64)$$

Taking the supremum for $r > 0$ on the inequality above, we get

$$\bar{T}_{\Omega, \alpha}^A f(x) \geq M_{\Omega, \alpha}^A f(x). \quad (65)$$

Thus, we can immediately obtain (19) from (65) and (18).

Similarly as before, we give the following theorem at first before proving Theorem 9, since it plays an important role in the proof of Theorem 9. Set

$$\bar{T}_{\Omega,\beta} f(x) = \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\beta}} |f(y)| dy, \quad (66)$$

where $\Omega \in L^s(S^{n-1})$ ($s > 1$) is homogeneous of degree zero in $\mathbb{R}^n$.

Theorem 29. *Under the assumptions of Theorem 9, $\bar{T}_{\Omega,\beta}$ is bounded from $L^{p,\kappa}(w^p, w^q)$ to $L^{q,\kappa q/p}(w^q)$.*

The proof of Theorem 29 can be treated as that of Theorem 27 with only slight modifications; we omit its proof here.

Now, let us prove Theorem 9. It is not difficult to see that (20) can be easily obtained from Theorems 26 and 29. Then we can immediately arrive at (21) from (65) and (20).

From now on, we are in the place of showing Theorem 10. We prove (22) at first. Fixing any cube Q with center at x and diameter r , denote $\bar{Q} = 2Q$ and set

$$A_{\bar{Q}}(y) = A(y) - \sum_{|\gamma|=m-1} \frac{1}{\gamma!} m_{\bar{Q}}(D^\gamma A) y^\gamma. \quad (67)$$

Noticing that equality (67) is the special case of equality (44) when $k = 1$. Thus, equalities (45) and (46) also hold for $A_{\bar{Q}}(y)$. We decompose f as $f = f\chi_{\bar{Q}} + f\chi_{\bar{Q}^c} := f_1 + f_2$. Then we have

$$\begin{aligned} & \frac{1}{w^q(Q)^{\kappa/p}} \left(\int_Q |T_{\Omega,\alpha}^A f(y)|^q w(y)^q dy \right)^{1/q} \\ & \leq \frac{1}{w^q(Q)^{\kappa/p}} \left(\int_Q |T_{\Omega,\alpha}^A f_1(y)|^q w(y)^q dy \right)^{1/q} \\ & \quad + \frac{1}{w^q(Q)^{\kappa/p}} \left(\int_Q |T_{\Omega,\alpha}^A f_2(y)|^q w(y)^q dy \right)^{1/q} \\ & := I + II. \end{aligned} \quad (68)$$

By Theorem 20(a) and Remark 7(a) that $w^q \in \Delta_2$, we have

$$\begin{aligned} I & \leq \frac{C}{w^q(Q)^{\kappa/p}} \\ & \cdot \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \left(\int_{\bar{Q}} |f(y)|^p w(y)^p dy \right)^{1/p} \\ & = C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p,\kappa}(w^p, w^q)} \left(\frac{w^q(\bar{Q})}{w^q(Q)} \right)^{\kappa/p} \\ & \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p,\kappa}(w^p, w^q)}. \end{aligned} \quad (69)$$

Next, we consider the term $T_{\Omega,\alpha}^A f_2(y)$ contained in II . By Lemma 15 and equality (45), (46), we have

$$\begin{aligned} & |T_{\Omega,\alpha}^A f_2(y)| \\ & \leq \int_{(\bar{Q})^c} \frac{|R_m(A_{\bar{Q}}; y, z)|}{|y-z|^{m-1}} \frac{|\Omega(y-z)|}{|y-z|^{n-\alpha}} |f(z)| dz \\ & \leq C \int_{(\bar{Q})^c} \sum_{|\gamma|=m-1} \left(\frac{1}{|I_y^z|} \int_{I_y^z} |D^\gamma A_{\bar{Q}}(t)|^l dt \right)^{1/l} \\ & \quad \cdot |\Omega(y-z)| \frac{|f(z)|}{|y-z|^{n-\alpha}} dz \\ & + C \int_{(\bar{Q})^c} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_{\bar{Q}}(D^\gamma A)| |\Omega(y-z)| \\ & \quad \cdot \frac{|f(z)|}{|y-z|^{n-\alpha}} dz := II_1 + II_2. \end{aligned} \quad (70)$$

We estimate II_1 and II_2 , respectively. By Lemma 21(a) and (b), Hölder's inequality, and $w^{s'} \in A(p/s', q/s')$, we get

$$\begin{aligned} II_1 & \leq C \sum_{|\gamma|=m-1} \left(\frac{1}{|I_y^z|} \int_{I_y^z} |D^\gamma A(t) - m_{\bar{Q}}(D^\gamma A)|^l dt \right)^{1/l} \\ & \quad \cdot \sum_{j=1}^{\infty} \int_{2^{j+1}Q \setminus 2^jQ} |\Omega(y-z)| \frac{|f(z)|}{|y-z|^{n-\alpha}} dz \\ & \leq C \sum_{|\gamma|=m-1} \left[\left(\frac{1}{|I_y^z|} \int_{I_y^z} |D^\gamma A(t) - m_{\bar{Q}}(D^\gamma A)|^l dt \right)^{1/l} + |m_{\bar{Q}}(D^\gamma A)| \right. \\ & \quad \left. - m_{5n\bar{Q}}(D^\gamma A) + |m_{5n\bar{Q}}(D^\gamma A) - m_{\bar{Q}}(D^\gamma A)| \right] \\ & \quad \cdot \sum_{j=1}^{\infty} \left(\int_{2^{j+1}Q} |\Omega(y-z)|^s dz \right)^{1/s} \\ & \quad \cdot \left(\int_{2^{j+1}Q \setminus 2^jQ} \frac{|f(z)|^{s'}}{|y-z|^{(n-\alpha)s'}} dz \right)^{1/s'} \\ & \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|\Omega\|_{L^s(S^{n-1})} \\ & \quad \cdot \sum_{j=1}^{\infty} \frac{|2^{j+2}Q|^{1/s}}{|2^jQ|^{1-\alpha/n}} \left(\int_{2^{j+1}Q} |f(z)|^{s'} dz \right)^{1/s'} \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \sum_{j=1}^{\infty} \frac{|2^{j+2}Q|^{1/s}}{|2^jQ|^{1-\alpha/n}} \left(\int_{2^{j+1}Q} |f(z)|^p \right. \\
&\quad \cdot w(z)^p dz \Big)^{1/p} \\
&\quad \cdot \left(\int_{2^{j+1}Q} w(z)^{-ps'/(p-s')} dz \right)^{(p-s')/ps'} \\
&\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p,\kappa}(w^p, w^q)} \\
&\quad \cdot \sum_{j=1}^{\infty} \frac{1}{w^q (2^{j+1}Q)^{1/q-\kappa/p}}.
\end{aligned} \tag{71}$$

For $y \in Q, z \in (\bar{Q})^c$, we have $|y-z| \sim |x-z|$, so we obtain

$$\begin{aligned}
II_2 &\leq C \sum_{j=1}^{\infty} \int_{2^{j+1}Q \setminus 2^jQ} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_{2^{j+1}Q}(D^\gamma A)| \\
&\quad \cdot |\Omega(y-z)| \frac{|f(z)|}{|y-z|^{n-\alpha}} dz \\
&\quad + C \sum_{j=1}^{\infty} \int_{2^{j+1}Q \setminus 2^jQ} \sum_{|\gamma|=m-1} |m_{2^{j+1}Q}(D^\gamma A) - m_{\bar{Q}}(D^\gamma A)| \\
&\quad \cdot |\Omega(y-z)| \frac{|f(z)|}{|y-z|^{n-\alpha}} dz := II_{21} + II_{22}.
\end{aligned} \tag{72}$$

By Hölder's inequality, we get

$$\begin{aligned}
II_{21} &\leq C \sum_{j=1}^{\infty} \frac{1}{|2^jQ|^{1-\frac{\alpha}{n}}} \int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_{2^{j+1}Q}(D^\gamma A)| |\Omega(y-z)| |f(z)| dz \\
&\leq C \sum_{j=1}^{\infty} \frac{1}{|2^jQ|^{1-\frac{\alpha}{n}}} \left(\int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_{2^{j+1}Q}(D^\gamma A)|^{s'} |f(z)|^{s'} dz \right)^{1/s'} \left(\int_{2^{j+1}Q} |\Omega(y-z)|^s dz \right)^{1/s} \\
&\leq C \|\Omega\|_{L^s(S^{n-1})} \sum_{j=1}^{\infty} \frac{|2^{j+2}Q|^{1/s}}{|2^jQ|^{1-\frac{\alpha}{n}}} \left(\int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_{2^{j+1}Q}(D^\gamma A)|^{s'} |f(z)|^{s'} dz \right)^{1/s'} \\
&\leq C \|\Omega\|_{L^s(S^{n-1})} \sum_{j=1}^{\infty} \frac{|2^{j+2}Q|^{1/s}}{|2^jQ|^{1-\frac{\alpha}{n}}} \left(\int_{2^{j+1}Q} |f(z)|^p w(z)^p dz \right)^{1/p} \\
&\quad \cdot \left(\int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_{2^{j+1}Q}(D^\gamma A)|^{ps'/(p-s')} w(z)^{-ps'/(p-s')} dz \right)^{(p-s')/ps'}.
\end{aligned} \tag{73}$$

We estimate the part containing the function $D^\gamma A$ as follows:

$$\begin{aligned}
&\left(\int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_{2^{j+1}Q}(D^\gamma A)|^{ps'/(p-s')} w(z)^{-ps'/(p-s')} dz \right)^{(p-s')/ps'} \\
&\leq C \left(\int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_{2^{j+1}Q, w^{-ps'/(p-s')}}(D^\gamma A)|^{ps'/(p-s')} w(z)^{-ps'/(p-s')} dz \right)^{(p-s')/ps'} \\
&\quad + \sum_{|\gamma|=m-1} |m_{2^{j+1}Q, w^{-ps'/(p-s')}}(D^\gamma A) - m_{2^{j+1}Q}(D^\gamma A)| w^{-ps'/(p-s')} (2^{j+1}Q)^{(p-s')/ps'} := III + IV.
\end{aligned} \tag{74}$$

For the term III , since $w^{s'} \in A(p/s', q/s')$, we then have $w^{-ps'/(p-s')} \in A_{p'} \subset A_\infty$ by Remark 7(b). Thus, by Lemma 25

that the norm of $BMO(w^{-ps'/(p-s')})$ is equivalent to the norm of $BMO(\mathbb{R}^n)$ and $w^{s'} \in A(p/s', q/s')$, we have

$$\begin{aligned}
III &\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* w^{-ps'/(p-s')} (2^{j+1}Q)^{(p-s')/ps'} \\
&= C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \frac{|2^{j+1}Q|^{(p-s')/ps'+1/q}}{w^q (2^{j+1}Q)^{1/q}}.
\end{aligned} \quad (75)$$

For the term IV , by Lemma 21(a), there exist $C_1, C_2 > 0$ such that for any cube Q and $s > 0$,

$$\begin{aligned}
&\left| \left\{ t \in 2^{j+1}Q : \sum_{|\gamma|=m-1} |D^\gamma A(t) - m_{2^{j+1}Q}(D^\gamma A)| > s \right\} \right| \\
&\leq C_1 |2^{j+1}Q| e^{-C_2 s / (\sum_{|\gamma|=m-1} \|D^\gamma A\|_*)},
\end{aligned} \quad (76)$$

since $\sum_{|\gamma|=m-1} (D^\gamma A) \in \text{BMO}$. Then by Lemma 24(2), we have

$$\begin{aligned}
&w \left(\left\{ t \in 2^{j+1}Q : \sum_{|\gamma|=m-1} |D^\gamma A(t) - m_{2^{j+1}Q}(D^\gamma A)| \right. \right. \\
&\quad \left. \left. > s \right\} \right) \leq C w (2^{j+1}Q) e^{-C_2 s \delta / (\sum_{|\gamma|=m-1} \|D^\gamma A\|_*)},
\end{aligned} \quad (77)$$

for some $\delta > 0$. Hence it implies

$$\begin{aligned}
&\sum_{|\gamma|=m-1} |m_{2^{j+1}Q, w^{-ps'/(p-s')}}(D^\gamma A) - m_{2^{j+1}Q}(D^\gamma A)| \\
&\leq \frac{1}{w^{-ps'/(p-s')} (2^{j+1}Q)} \int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^\gamma A(t) \\
&\quad - m_{2^{j+1}Q}(D^\gamma A)| w^{-ps'/(p-s')} (t) dt \\
&= \frac{C}{w^{-ps'/(p-s')} (2^{j+1}Q)} \int_0^\infty w^{-ps'/(p-s')} \left(\left\{ t \right. \right. \\
&\quad \left. \left. \in 2^{j+1}Q : \sum_{|\gamma|=m-1} |D^\gamma A(t) - m_{2^{j+1}Q}(D^\gamma A)| \right. \right. \\
&\quad \left. \left. > s \right\} \right) ds \leq \frac{C}{w^{-ps'/(p-s')} (2^{j+1}Q)} \\
&\quad \cdot \int_0^\infty w^{-ps'/(p-s')} (2^{j+1}Q) e^{-C_2 s \delta / (\sum_{|\gamma|=m-1} \|D^\gamma A\|_*)} ds \\
&= C \sum_{|\gamma|=m-1} \|D^\gamma A\|_*.
\end{aligned} \quad (78)$$

As a result,

$$\begin{aligned}
IV &\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* w^{-ps'/(p-s')} (2^{j+1}Q)^{(p-s')/ps'} \\
&= C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \frac{|2^{j+1}Q|^{(p-s')/ps'+1/q}}{w^q (2^{j+1}Q)^{1/q}}.
\end{aligned} \quad (79)$$

Thus,

$$\begin{aligned}
II_{21} &\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p,\kappa}(w^p, w^q)} \\
&\quad \cdot \sum_{j=1}^\infty \frac{1}{w^q (2^{j+1}Q)^{1/q-\kappa/p}}.
\end{aligned} \quad (80)$$

For the term II_{22} , by Lemma 21(c), Hölder's inequality, and $w^{s'} \in A(p/s', q/s')$, we get

$$\begin{aligned}
II_{22} &\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \\
&\quad \cdot \sum_{j=1}^\infty j \int_{2^{j+1}Q} | \Omega(y-z) | \frac{|f(z)|}{|y-z|^{n-\alpha}} dz \\
&\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|\Omega\|_{L^s(S^{n-1})} \sum_{j=1}^\infty j \\
&\quad \cdot \frac{|2^{j+2}Q|^{1/s}}{|2^jQ|^{1-\alpha/n}} \left(\int_{2^{j+1}Q} |f(z)|^{s'} dz \right)^{1/s'} \\
&\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \sum_{j=1}^\infty j \\
&\quad \cdot \frac{|2^{j+2}Q|^{1/s}}{|2^jQ|^{1-\alpha/n}} \left(\int_{2^{j+1}Q} |f(z)|^p w(z)^p dz \right)^{1/p} \\
&\quad \cdot \left(\int_{2^{j+1}Q} w(z)^{-ps'/(p-s')} dz \right)^{(p-s')/ps'} \\
&\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p,\kappa}(w^p, w^q)} \\
&\quad \cdot \sum_{j=1}^\infty \frac{j}{w^q (2^{j+1}Q)^{1/q-\kappa/p}}.
\end{aligned} \quad (81)$$

Hence,

$$\begin{aligned}
|T_{\Omega, \alpha}^A f_2(y)| &\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p,\kappa}(w^p, w^q)} \\
&\quad \cdot \sum_{j=1}^\infty \frac{j}{w^q (2^{j+1}Q)^{1/q-\kappa/p}}.
\end{aligned} \quad (82)$$

Therefore,

$$\begin{aligned}
II &\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p,\kappa}(w^p, w^q)} \\
&\quad \cdot \sum_{j=1}^\infty j \frac{w^q(Q)^{1/q-\kappa/p}}{w^q (2^{j+1}Q)^{1/q-\kappa/p}}
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p,\kappa}(w^p, w^q)} \sum_{j=1}^{\infty} \frac{j}{(D_1^{j+1})^{1/q-\kappa/p}} \\
&\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p,\kappa}(w^p, w^q)},
\end{aligned} \tag{83}$$

where $D_1 > 1$ is the reverse doubling constant. Consequently,

$$\begin{aligned}
&\frac{1}{w^q(Q)^{\kappa/p}} \left(\int_Q |T_{\Omega, \alpha}^A f(y)|^q w(y)^q dy \right)^{1/q} \\
&\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p,\kappa}(w^p, w^q)}.
\end{aligned} \tag{84}$$

Taking supremum over all cubes in $\mathbb{R}^n$ on both sides of the above inequality, we complete the proof of (22) of Theorem 10.

It is not difficult to see that inequality (23) is easy to get from (22) and (65).

Proof of Theorem 11. We consider (25) firstly. Let Q be the same as in the proof of (22) and denote $\bar{Q} = 2Q$; we decompose f as $f = f\chi_{\bar{Q}} + f\chi_{(\bar{Q})^c} := f_1 + f_2$. Then we have

$$\begin{aligned}
&\frac{1}{w(Q)^{\kappa/p}} \left(\int_Q |[A, T_\Omega] f(y)|^p w(y) dy \right)^{1/p} \\
&\leq \frac{1}{w(Q)^{\kappa/p}} \left(\int_Q |[A, T_\Omega] f_1(y)|^p w(y) dy \right)^{1/p} \\
&\quad + \frac{1}{w(Q)^{\kappa/p}} \left(\int_Q |[A, T_\Omega] f_2(y)|^p w(y) dy \right)^{1/p} \\
&:= I + II.
\end{aligned} \tag{85}$$

By Theorem 22(a) and Lemma 24(1) that $w \in \Delta_2$, we get

$$\begin{aligned}
I &\leq \frac{1}{w(Q)^{\kappa/p}} \|[A, T_\Omega] f_1\|_{L^p(w)} \\
&\leq \frac{C}{w(Q)^{\kappa/p}} \|A\|_* \|f_1\|_{L^p(w)}
\end{aligned} \tag{86}$$

$$= C \|A\|_* \|f\|_{L^{p,\kappa}(w)} \frac{w(\bar{Q})^{\kappa/p}}{w(Q)^{\kappa/p}} \leq C \|A\|_* \|f\|_{L^{p,\kappa}(w)}.$$

For $|[A, T_\Omega] f_2(y)|$, by Hölder's inequality, we obtain

$$\begin{aligned}
|[A, T_\Omega] f_2(y)| &\leq \sum_{j=1}^{\infty} \int_{2^{j+1}Q \setminus 2^jQ} \frac{|\Omega(y-z)|}{|y-z|^n} |A(y) - A(z)| \\
&\quad \cdot |f(z)| dz \\
&\leq C \sum_{j=1}^{\infty} \frac{1}{|2^jQ|} \left(\int_{2^{j+1}Q} |\Omega(y-z)|^s dz \right)^{1/s} \\
&\quad \cdot \left(\int_{2^{j+1}Q} |A(y) - A(z)|^{s'} |f(z)|^{s'} dz \right)^{1/s'} \leq C \|\Omega\|_{L^s} \\
&\quad \cdot \sum_{j=1}^{\infty} \frac{|2^{j+2}Q|^{1/s}}{|2^jQ|} |A(y) - m_{2^{j+1}Q, w^{-s'/(p-s')}}(A)| \\
&\quad \cdot \left(\int_{2^{j+1}Q} |f(z)|^{s'} dz \right)^{1/s'} + C \|\Omega\|_{L^s} \\
&\quad \cdot \sum_{j=1}^{\infty} \frac{|2^{j+2}Q|^{1/s}}{|2^jQ|} \left(\int_{2^{j+1}Q} |m_{2^{j+1}Q, w^{-s'/(p-s')}}(A) - A(z)|^{s'} \right. \\
&\quad \cdot |f(z)|^{s'} dz \left. \right)^{1/s'} := II_1(y) + II_2.
\end{aligned} \tag{87}$$

Next we estimate $II_1(y)$ and II_2 , respectively. By Hölder's inequality and $w \in A_{p/s'}$, we have

$$\begin{aligned}
&\frac{1}{w(Q)^{\kappa/p}} \left(\int_Q II_1(y)^p w(y) dy \right)^{1/p} \\
&= C \frac{\|\Omega\|_{L^s}}{w(Q)^{\kappa/p}} \sum_{j=1}^{\infty} \frac{|2^{j+2}Q|^{1/s}}{|2^jQ|} \left(\int_Q |A(y) - m_{2^{j+1}Q, w^{-s'/(p-s')}}(A)|^p \left(\int_{2^{j+1}Q} |f(z)|^{s'} dz \right)^{p/s'} w(y) dy \right)^{1/p} \\
&\leq C \frac{\|f\|_{L^{p,\kappa}(w)}}{w(Q)^{\frac{\kappa}{p}}} \sum_{j=1}^{\infty} \frac{|2^{j+2}Q|^{1/s}}{|2^jQ|} w(2^{j+1}Q)^{\kappa/p} w^{-s'/(p-s')}(2^{j+1}Q)^{(p-s')/ps'} \cdot \left(\int_Q |A(y) - m_{2^{j+1}Q, w^{-s'/(p-s')}}(A)|^p w(y) dy \right)^{1/p} \\
&\leq C \frac{\|f\|_{L^{p,\kappa}(w)}}{w(Q)^{\kappa/p}} \sum_{j=1}^{\infty} \frac{1}{w(2^{j+1}Q)^{(1-\kappa)/p}} \left(\int_Q |A(y) - m_{2^{j+1}Q, w^{-s'/(p-s')}}(A)|^p w(y) dy \right)^{1/p}.
\end{aligned} \tag{88}$$

We estimate the part containing $m_{2^{j+1}Q, w^{-s'/(p-s')}}(A)$ as follows:

$$\begin{aligned} & \left(\int_Q |A(y) - m_{2^{j+1}Q, w^{-s'/(p-s')}}(A)|^p w(y) dy \right)^{1/p} \\ & \leq \left(\int_Q |A(y) - m_{Q, w}(A)|^p w(y) dy \right)^{1/p} \\ & \quad + |m_{Q, w}(A) - m_{2^{j+1}Q, w^{-s'/(p-s')}}(A)| w(Q)^{1/p} \\ & := III + IV. \end{aligned} \quad (89)$$

For the term *III*, notice that $w \in A_{p/s'} \subset A_\infty$; we thus get by Lemma 25 that

$$III \leq \|A\|_* w(Q)^{1/p}. \quad (90)$$

Next we estimate *IV*. By Lemmas 21(c) and 25, we have

$$\begin{aligned} & |m_{Q, w}(A) - m_{2^{j+1}Q, w^{-s'/(p-s')}}(A)| \\ & \leq |m_{Q, w}(A) - m_Q(A)| + |m_Q(A) - m_{2^{j+1}Q}(A)| \\ & \quad + |m_{2^{j+1}Q}(A) - m_{2^{j+1}Q, w^{-s'/(p-s')}}(A)| \leq \frac{1}{w(Q)} \\ & \quad \cdot \int_Q |A(t) - m_Q(A)| w(t) dt + 2^n(j+1) \|A\|_* \\ & \quad + \frac{1}{w^{-s'/(p-s')}(2^{j+1}Q)} \\ & \quad \cdot \int_{2^{j+1}Q} |A(t) - m_{2^{j+1}Q}(A)| w^{-s'/(p-s')}(t) dt \\ & \leq C(j+1) \|A\|_*. \end{aligned} \quad (91)$$

Hence,

$$IV \leq C(j+1) \|A\|_* w(Q)^{1/p}. \quad (92)$$

As a result,

$$\begin{aligned} & \frac{1}{w(Q)^{\kappa/p}} \left(\int_Q II_1(y)^p w(y) dy \right)^{1/p} \\ & \leq C \|A\|_* \|f\|_{L^{p, \kappa}(w)} \sum_{j=1}^{\infty} (j+1) \frac{w(Q)^{(1-\kappa)/p}}{w(2^{j+1}Q)^{(1-\kappa)/p}} \\ & \leq C \|A\|_* \|f\|_{L^{p, \kappa}(w)}. \end{aligned} \quad (93)$$

For II_2 , by Hölder's inequality and $w \in A_{p/s'}$, we get

$$\begin{aligned} II_2 & \leq C \|\Omega\|_{L^s} \\ & \quad \cdot \sum_{j=1}^{\infty} \frac{|2^{j+2}Q|^{1/s}}{|2^jQ|} \left(\int_{2^{j+1}Q} |f(z)|^p w(z) dz \right)^{1/p} \\ & \quad \cdot \left(\int_{2^{j+1}Q} |m_{2^{j+1}Q, w^{-s'/(p-s')}}(A) - A(z)|^{ps'/(p-s')} \right. \\ & \quad \cdot w(z)^{-s'/(p-s')} dz \left. \right)^{(p-s')/ps'} \leq C \|\Omega\|_{L^s} \|A\|_* \\ & \quad \cdot \|f\|_{L^{p, \kappa}(w)} \sum_{j=1}^{\infty} \frac{|2^{j+2}Q|^{1/s}}{|2^jQ|} w(2^{j+1}Q)^{\kappa/p} \\ & \quad \cdot w^{-s'/(p-s')}(2^{j+1}Q)^{(p-s')/ps'} \leq C \|A\|_* \|f\|_{L^{p, \kappa}(w)} \\ & \quad \cdot \sum_{j=1}^{\infty} \frac{1}{w(2^{j+1}Q)^{(1-\kappa)/p}}. \end{aligned} \quad (94)$$

Therefore,

$$\begin{aligned} & \frac{1}{w(Q)^{\kappa/p}} \left(\int_Q II_2^p w(y) dy \right)^{1/p} \\ & \leq C \|A\|_* \|f\|_{L^{p, \kappa}(w)} \sum_{j=1}^{\infty} \frac{w(Q)^{(1-\kappa)/p}}{w(2^{j+1}Q)^{(1-\kappa)/p}} \\ & \leq C \|A\|_* \|f\|_{L^{p, \kappa}(w)}. \end{aligned} \quad (95)$$

So far, we have completed the proof of (25). $\square$

Inequality (26) can be immediately obtained from (65) and (25).

Proof of Theorem 12. As before, we prove (27) at first. Assume Q to be the same as in the proof of (22), denote $\bar{Q} = 2Q$, and set

$$A_{\bar{Q}}(y) = A(y) - m_{\bar{Q}}(\nabla A) y. \quad (96)$$

We also decompose f according to $\bar{Q}$: $f = f\chi_{\bar{Q}} + f\chi_{(\bar{Q})^c} := f_1 + f_2$. Then we get

$$\begin{aligned} & \frac{1}{w(Q)^{\kappa/p}} \left(\int_Q |\tilde{T}_{\Omega}^A f(y)|^p w(y) dy \right)^{1/p} \\ & \leq \frac{1}{w(Q)^{\kappa/p}} \left(\int_Q |\tilde{T}_{\Omega}^A f_1(y)|^p w(y) dy \right)^{1/p} \\ & \quad + \frac{1}{w(Q)^{\kappa/p}} \left(\int_Q |\tilde{T}_{\Omega}^A f_2(y)|^p w(y) dy \right)^{1/p} \\ & := I + II. \end{aligned} \quad (97)$$

For I, Theorem 23 and Lemma 24(1) imply

$$\begin{aligned}
 I &\leq \frac{1}{w(Q)^{\kappa/p}} \left\| \tilde{T}_{\Omega}^A f_1 \right\|_{L^p(w)} \\
 &\leq \frac{C}{w(Q)^{\kappa/p}} \|\Omega\|_{\infty} \|\nabla A\|_* \|f_1\|_{L^p(w)} \\
 &\leq C \|\nabla A\|_* \|f\|_{L^{p,\kappa}(w)} \frac{w(\overline{Q})^{\kappa/p}}{w(Q)^{\kappa/p}} \\
 &\leq C \|\nabla A\|_* \|f\|_{L^{p,\kappa}(w)}.
 \end{aligned} \tag{98}$$

We will omit the proof for II since it is similar to and even easier than the part of II in the proof of (22), except that we use the conditions $w \in A_p$, $\Omega \in L^{\infty}(S^{n-1})$, $m = 2$, and $f \in L^{p,\kappa}(w)$. For inequality (28), it can be easily proved by (27) and (65). Thus, we complete the proof of Theorem 12. $\square$

Competing Interests

The authors declare that they have no competing interests.

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References

- [1] J. Cohen and J. Gosselin, "A BMO estimate for multilinear singular integrals," *Illinois Journal of Mathematics*, vol. 30, no. 3, pp. 445–464, 1986.
- [2] S. Hofmann, "On certain nonstandard Calderón-Zygmund operators," *Studia Mathematica*, vol. 109, no. 2, pp. 105–131, 1994.
- [3] S. Z. Lu, H. X. Wu, and P. Zhang, "Multilinear singular integrals with rough kernel," *Acta Mathematica Sinica (English Series)*, vol. 19, no. 1, pp. 51–62, 2003.
- [4] Y. Ding and S. Z. Lu, "Weighted boundedness for a class of rough multilinear operators," *Acta Mathematica Sinica—English Series*, vol. 17, no. 3, pp. 517–526, 2001.
- [5] S. Z. Lu and P. Zhang, "Lipschitz estimates for generalized commutators of fractional integrals with rough kernel," *Mathematische Nachrichten*, vol. 252, no. 1, pp. 70–85, 2003.
- [6] C. B. Morrey, "On the solutions of quasi-linear elliptic partial differential equations," *Transactions of the American Mathematical Society*, vol. 43, no. 1, pp. 126–166, 1938.
- [7] F. Chiarenza and M. Frasca, "Morrey spaces and Hardy-Littlewood maximal function," *Rendiconti di Matematica e delle sue Applicazioni*, vol. 7, no. 3-4, pp. 273–279, 1987.
- [8] D. Fan, S. Lu, and D. Yang, "Regularity in Morrey spaces of strong solutions to nondivergence elliptic equations with VMO coefficients," *Georgian Mathematical Journal*, vol. 5, no. 5, pp. 425–440, 1998.
- [9] J. Peetre, "On the theory of $L_{p,\lambda}$ spaces," *Journal of Functional Analysis*, vol. 4, no. 1, pp. 71–87, 1969.
- [10] Y. Komori and S. Shirai, "Weighted Morrey spaces and a singular integral operator," *Mathematische Nachrichten*, vol. 282, no. 2, pp. 219–231, 2009.
- [11] H. Wang, "The boundedness of some operators with rough kernel on the weighted Morrey spaces," *Acta Mathematica Sinica*, vol. 55, no. 4, pp. 589–600, 2012 (Chinese).
- [12] M. Paluszynski, "Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss," *Indiana University Mathematics Journal*, vol. 44, no. 1, pp. 1–17, 1995.
- [13] R. A. DeVore and R. C. Sharpley, "Maximal functions measuring smoothness," *Memoirs of the American Mathematical Society*, vol. 47, no. 293, 1984.
- [14] Y. Ding and S. Z. Lu, "Weighted norm inequalities for fractional integral operators with rough kernel," *Canadian Journal of Mathematics*, vol. 50, no. 1, pp. 29–39, 1998.
- [15] L. Grafakos, *Modern Fourier Analysis*, vol. 250 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2nd edition, 2009.
- [16] S. Z. Lu, Y. Ding, and D. Y. Yan, *Singular Integrals and Related Topics*, World Scientific, 2007.
- [17] B. Muckenhoupt and R. L. Wheeden, "Weighted bounded mean oscillation and the Hilbert transform," *Studia Mathematica*, vol. 54, no. 3, pp. 221–237, 1976.