Research Article

A Sharp Lower Bound for Toader-Qi Mean with Applications

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We prove that the inequality $TQ(a, b) > L_p(a, b)$ holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq 3/2$, where $TQ(a, b) = (2/\pi) \int_0^{\pi/2} a^{\cos^2 \theta} b^{\sin^2 \theta} d\theta$, $L_p(a, b) = [(b^p - a^p)/(p(b - a))]^{1/p}$ ($p \neq 0$), and $L_0(a, b) = \sqrt{ab}$ are, respectively, the Toader-Qi and p -order logarithmic means of a and b . As applications, we find two fine inequalities chains for certain bivariate means.

1. Introduction

Let $p \in \mathbb{R}$ and $a, b > 0$ with $a \neq b$. Then the Toader-Qi mean $TQ(a, b)$ [1–3] and p -order logarithmic mean $L_p(a, b)$ are defined by

$$TQ(a, b) = \frac{2}{\pi} \int_0^{\pi/2} a^{\cos^2 \theta} b^{\sin^2 \theta} d\theta, \quad (1)$$

$$L_p(a, b) = \left[\frac{b^p - a^p}{p(\log b - \log a)} \right]^{1/p} \quad (p \neq 0), \quad (2)$$

$$L_0(a, b) = \lim_{p \rightarrow 0} L_p(a, b) = \sqrt{ab},$$

respectively. In particular, $L_1(a, b) = L(a, b)$ is the classical logarithmic mean of a and b .

It is well-known that the p -order logarithmic mean $L_p(a, b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Recently, the Toader-Qi and p -order logarithmic means have been the subject of intensive research. In particular, many remarkable inequalities for the Toader-Qi and p -order logarithmic means can be found in the literature [2–7].

In [2], Qi et al. proved that the identity

$$TQ(a, b) = \sqrt{ab} I_0 \left(\frac{1}{2} \log \frac{b}{a} \right) \quad (3)$$

and the inequalities

$$\begin{aligned} L(a, b) < TQ(a, b) &< \frac{A(a, b) + G(a, b)}{2} \\ &< \frac{2A(a, b) + G(a, b)}{3} < I(a, b) \end{aligned} \quad (4)$$

hold for all $a, b > 0$ with $a \neq b$, where

$$I_0(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{2^{2n} (n!)^2} \quad (5)$$

is the modified Bessel function of the first kind [8] and $A(a, b) = (a + b)/2$, $G(a, b) = \sqrt{ab}$ and $I(a, b) = (b^b/a^a)^{1/(b-a)}/e$ are, respectively, the classical arithmetic, geometric, and identric means of a and b .

In [3], Yang proved that the double inequalities

$$\sqrt{\frac{2A(a, b)L(a, b)}{\pi}} < TQ(a, b) < \sqrt{A(a, b)L(a, b)}, \quad (6)$$

$$\begin{aligned} A^{1/4}(a, b) L^{3/4}(a, b) &< TQ(a, b) \\ &< \frac{1}{4} A(a, b) + \frac{3}{4} L(a, b) \end{aligned} \quad (7)$$

and conjectured that the inequalities

$$TQ(a, b) < I^{1/2}(a, b) L^{1/2}(a, b), \quad (8)$$

$$TQ(a, b) > L_{3/2}(a, b) \quad (9)$$

hold for all $a, b > 0$ with $a \neq b$. Inequality (8) was proved by Yang et al. in [9].

Let $b > a > 0$ and $t = (\log b - \log a)/2 > 0$. Then from (1)–(3) we clearly see that

$$\begin{aligned} L_p(a, b) &= \sqrt{ab} \left[\frac{\sinh(pt)}{pt} \right]^{1/p} \quad (p \neq 0), \\ TQ(a, b) &= \frac{2\sqrt{ab}}{\pi} \int_0^{\pi/2} e^{t \cos(2\theta)} d\theta = \sqrt{ab} I_0(t) \\ &= \frac{2\sqrt{ab}}{\pi} \int_0^{\pi/2} \cosh(t \cos \theta) d\theta \\ &= \frac{2\sqrt{ab}}{\pi} \int_0^{\pi/2} \cosh(t \sin \theta) d\theta. \end{aligned} \quad (10)$$

The main purpose of this paper is to give a positive answer to the conjecture given by (9). As applications, we present two fine inequalities chains for certain bivariate means and a lower bound for the kernel function of the Szász–Mirakjan–Durrmeyer operator.

2. Lemmas

In order to prove our main result we need several lemmas, which we present in this section.

Lemma 1 (see [10]). *The double inequality*

$$\frac{1}{(x+a)^{1-a}} < \frac{\Gamma(x+a)}{\Gamma(x+1)} < \frac{1}{x^{1-a}} \quad (11)$$

holds for all $x > 0$ and $a \in (0, 1)$, where $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ is the classical Euler gamma function.

Lemma 2 (see [3]). *Let $I_0(t)$ be defined by (5). Then the identity*

$$I_0^2(t) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (n!)^4} t^{2n} \quad (12)$$

holds for all $t \in \mathbb{R}$.

Lemma 3 (see [3]). *The Wallis ratio*

$$W_n = \frac{(2n-1)!!}{(2n)!!} = \frac{(2n)!}{2^{2n} (n!)^2} = \frac{\Gamma(n+1/2)}{\Gamma(1/2) \Gamma(n+1)} \quad (13)$$

is strictly decreasing and log-convex with respect to all integers $n \geq 0$.

Lemma 4. *The identity*

$$\sum_{k=0}^n \frac{a^{2k}}{(2k)! (2n-2k)!} = \frac{(a+1)^{2n} + (a-1)^{2n}}{2(2n)!} \quad (14)$$

holds for all $a \in \mathbb{R}$ and $n \in \mathbb{N}$.

Proof. Let $\binom{n}{k} = n!/k!(n-k)!$ be the number of combinations of n objects taken k at a time. Then from the well-known binomial theorem we have

$$\begin{aligned} (a+1)^{2n} &= \sum_{k=0}^{2n} \binom{2n}{k} a^k \\ &= \sum_{k=0}^n \binom{2n}{2k} a^{2k} + \sum_{k=1}^n \binom{2n}{2k-1} a^{2k-1}, \\ (a-1)^{2n} &= \sum_{k=0}^{2n} \binom{2n}{k} (-1)^{2n-k} a^k \\ &= \sum_{k=0}^n \binom{2n}{2k} a^{2k} - \sum_{k=1}^n \binom{2n}{2k-1} a^{2k-1}. \end{aligned} \quad (15)$$

Equation (15) leads to

$$\begin{aligned} \frac{(a+1)^{2n} + (a-1)^{2n}}{2} &= \sum_{k=0}^n \binom{2n}{2k} a^{2k} \\ &= \sum_{k=0}^n \frac{(2n)! a^{2k}}{(2k)! (2n-2k)!}. \end{aligned} \quad (16)$$

□

Lemma 5. *Let $k, n \in \mathbb{N}$ with $k \leq n$ and*

$$u_{k,n} = \frac{(2k)!}{2^{2n} (k!)^4 [(n-k)!]^2}. \quad (17)$$

Then

$$u_{k,n} > \frac{2\sqrt{2}}{\pi\sqrt{\pi}(n+1)\sqrt{2n+1}} \frac{2^{2k}}{(2k)!(2n-2k)!} \quad (18)$$

for all $n \geq 8$.

Proof. Let W_n be defined by (13). Then it follows from Lemmas 1 and 3 together with (17) and $\Gamma(1/2) = \sqrt{\pi}$ that

$$\begin{aligned} u_{k,n} &= W_k^2 W_{n-k} \frac{2^{2k}}{(2k)!(2n-2k)!} \geq W_k W_{n/2}^2 \\ &\cdot \frac{2^{2k}}{(2k)!(2n-2k)!} \geq W_n W_{n/2}^2 \frac{2^{2k}}{(2k)!(2n-2k)!} \\ &= \frac{1}{\pi\sqrt{\pi}} \frac{\Gamma(n+1/2)}{\Gamma(n+1)} \left[\frac{\Gamma(n/2+1/2)}{\Gamma(n/2+1)} \right]^2 \\ &\cdot \frac{2^{2k}}{(2k)!(2n-2k)!} > \frac{1}{\pi\sqrt{\pi}} \\ &\cdot \frac{1}{\sqrt{n+1/2}} \left[\frac{1}{\sqrt{n/2+1/2}} \right]^2 \frac{2^{2k}}{(2k)!(2n-2k)!} \\ &= \frac{2\sqrt{2}}{\pi\sqrt{\pi}(n+1)\sqrt{2n+1}} \frac{2^{2k}}{(2k)!(2n-2k)!} \end{aligned} \quad (19)$$

for all $n \geq 8$ and $0 \leq k \leq n$.

□

3. Main Result

Theorem 6. *The inequality*

$$TQ(a, b) > L_p(a, b) \quad (20)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq 3/2$.

Proof. Since both the Toader-Qi mean $TQ(a, b)$ and p -order logarithmic mean $L_p(a, b)$ are symmetric and homogeneous and $TQ(a, b) > L_p(a, b)$ and $L_p(a, b)$ is strictly increasing with respect to $p \in \mathbb{R}$ for all $a, b > 0$ with $a \neq b$, without loss of generality, we assume that $p > 1$ and $b > a > 0$. Let $t = (\log b - \log a)/2 > 0$. Then it follows from (10) that inequality (20) is equivalent to

$$I_0(t) > \left[\frac{\sinh(pt)}{pt} \right]^{1/p} \quad (21)$$

for all $t > 0$.

If inequality (21) holds for all $t > 0$. Then (5) and (21) lead to

$$\lim_{t \rightarrow 0^+} \frac{I_0(t) - [\sinh(pt)/pt]^{1/p}}{t^2} = -\frac{1}{6} \left(p - \frac{3}{2} \right) \geq 0, \quad (22)$$

which gives $p \leq 3/2$.

Next, we only need to prove that inequality (21) holds for $p = 3/2$ and all $t > 0$; that is

$$I_0^3(t) > \left[\frac{\sinh(3t/2)}{3t/2} \right]^2. \quad (23)$$

It follows from (5) and Lemma 2 that

$$\begin{aligned} I_0^3(t) &= \left(\sum_{n=0}^{\infty} \frac{t^{2n}}{2^{2n} (n!)^2} \right)^3 = \sum_{n=0}^{\infty} \frac{(2n)! t^{2n}}{2^{2n} (n!)^4} \sum_{n=0}^{\infty} \frac{t^{2n}}{2^{2n} (n!)^2} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{(2k)!}{2^{2k} (k!)^4} \frac{1}{2^{2(n-k)} [(n-k)!]^2} \right) t^{2n} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n u_{k,n} \right) t^{2n}, \end{aligned} \quad (24)$$

where $u_{k,n}$ is defined as in (17).

Note that

$$\left[\frac{\sinh(3t/2)}{3t/2} \right]^2 = \frac{2 [\cosh(3t) - 1]}{9t^2} = 2 \sum_{n=0}^{\infty} \frac{3^{2n} t^{2n}}{(2n+2)!}. \quad (25)$$

Let

$$v_n = \sum_{k=0}^n u_{k,n} - \frac{2 \times 3^{2n}}{(2n+2)!}. \quad (26)$$

Then simple computations lead to

$$\begin{aligned} v_0 &= v_1 = 0, \\ v_2 &= \frac{3}{320}, \\ v_3 &= \frac{113}{26880}, \\ v_4 &= \frac{2057}{2867200}, \\ v_5 &= \frac{1741}{25231360}, \\ v_6 &= \frac{4335377}{991895224320}, \\ v_7 &= \frac{2186227}{11021058048000}. \end{aligned} \quad (27)$$

From Lemmas 4 and 5 together with (24)–(26), we have

$$\begin{aligned} I_0^3(t) - \left[\frac{\sinh(3t/2)}{3t/2} \right]^2 &= \sum_{n=0}^{\infty} v_n t^{2n}, \\ v_n &> \frac{2\sqrt{2}}{\pi\sqrt{\pi}(n+1)\sqrt{2n+1}} \frac{3^{2n}+1}{2(2n)!} - \frac{2 \times 3^{2n}}{(2n+2)!} \\ &= \frac{[\sqrt{2(2n+1)} - \pi\sqrt{\pi}] 3^{2n} + \sqrt{2(2n+1)}}{\pi\sqrt{\pi}(n+1)(2n+1)!} > 0 \end{aligned} \quad (28)$$

for all $n \geq 8$.

Therefore, inequality (23) follows from (27) and (28). $\square$

Remark 7. Theorem 6 gives a positive answer to the conjecture given by (9).

Remark 8. It follows from (23) that the inequality

$$I_0^3(t) > \frac{2 [\cosh(3t) - 1]}{9t^2} \quad (29)$$

holds for all $t \neq 0$.

4. Applications

For $a, b > 0$, the Toader mean $T(a, b)$ [1] and arithmetic-geometric mean $AGM(a, b)$ [11] are, respectively, defined by

$$T(a, b) = \frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta, \quad (30)$$

$$AGM(a, b) = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n,$$

where a_n and b_n are given by

$$\begin{aligned} a_0 &= a, \\ b_0 &= b, \\ a_{n+1} &= \frac{(a_n + b_n)}{2} = A(a_n, b_n), \\ b_{n+1} &= \sqrt{a_n b_n} = G(a_n, b_n). \end{aligned} \quad (31)$$

Let $T_p(a, b) = T^{1/p}(a^p, b^p)$ and $I_q(a, b) = I^{1/q}(a^q, b^q)$ be the p -order Toader and q -order identric means of a and b , respectively. Then Theorem 6 leads to two fine inequalities chains for certain bivariate means.

Theorem 9. *The inequalities*

$$\begin{aligned} L(a, b) &< AGM(a, b) < A^{1/4}(a, b) L^{3/4}(a, b) \\ &< L_{3/2}(a, b) < TQ(a, b) \\ &< \frac{1}{4}A(a, b) + \frac{3}{4}L(a, b) \\ &< \frac{1}{2}L(a, b) + \frac{1}{2}I(a, b) \\ &< \frac{1}{2}A(a, b) + \frac{1}{2}G(a, b) < T_{1/3}(a, b) \\ &< I_{3/4}(a, b), \end{aligned} \quad (32)$$

$$\begin{aligned} L(a, b) &< AGM(a, b) < A^{1/4}(a, b) L^{3/4}(a, b) \\ &< L_{3/2}(a, b) < TQ(a, b) \\ &< L^{1/2}(a, b) I^{1/2}(a, b) < \frac{1}{2}L(a, b) + \frac{1}{2}I(a, b) \\ &< \frac{1}{2}A(a, b) + \frac{1}{2}G(a, b) < T_{1/3}(a, b) \\ &< I_{3/4}(a, b) \end{aligned}$$

hold for all $a, b > 0$ with $a \neq b$.

Proof. The following inequalities can be found in the literature [3, 4, 7, 12–14]:

$$\frac{A(a, b) + G(a, b)}{2} < T_{1/3}(a, b) < I_{3/4}(a, b), \quad (33)$$

$$\begin{aligned} L(a, b) &< AGM(a, b) \\ &< L^{3/4}(a, b) A^{1/4}(a, b) \\ &< L_{3/2}(a, b), \end{aligned} \quad (34)$$

$$I(a, b) > \frac{L(a, b) + A(a, b)}{2}, \quad (35)$$

$$L(a, b) + I(a, b) < A(a, b) + G(a, b)$$

for all $a, b > 0$ with $a \neq b$.

It follows from (35) that

$$\begin{aligned} \frac{A(a, b) + G(a, b)}{2} &> \frac{I(a, b) + L(a, b)}{2} \\ &> \frac{3}{4}L(a, b) + \frac{1}{4}A(a, b) \end{aligned} \quad (36)$$

for all $a, b > 0$ with $a \neq b$.

Therefore, inequality (32) follows easily from (7), (8), (33), (34), (36), and Theorem 6. $\square$

Remark 10. Let $b > a > 0$ and $t = (\log b - \log a)/2 > 0$. Then simple computations lead to

$$\begin{aligned} \frac{L(a, b)}{\sqrt{ab}} &= \frac{\sinh(t)}{t}, \\ \frac{I(a, b)}{\sqrt{ab}} &= e^{t \cosh(t)/\sinh(t)-1}, \\ \frac{A(a, b)}{\sqrt{ab}} &= \cosh(t). \end{aligned} \quad (37)$$

Note that

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{(\sinh(t)/t) e^{t \cosh(t)/\sinh(t)-1} - (3 \sinh(t)/4t + \cosh(t)/4)^2}{t^4} &= \frac{1}{720}, \\ \lim_{t \rightarrow \infty} \left[\frac{\sinh(t)}{t} e^{t \cosh(t)/\sinh(t)-1} - \left(\frac{3 \sinh(t)}{4t} + \frac{\cosh(t)}{4} \right)^2 \right] &= -\infty. \end{aligned} \quad (38)$$

Inequalities (37) and (38) imply that there exist small enough $\delta > 0$ and large enough $M > 1$ such that

$$I^{1/2}(a, b) L^{1/2}(a, b) > \frac{1}{4}A(a, b) + \frac{3}{4}L(a, b) \quad (39)$$

for all $b > a > 0$ with $b/a \in (1, 1 + \delta)$ and

$$I^{1/2}(a, b) L^{1/2}(a, b) < \frac{1}{4}A(a, b) + \frac{3}{4}L(a, b) \quad (40)$$

for all $b > a > 0$ with $b/a \in (M, \infty)$.

Let $x \in [0, \infty)$, $n > 0$, $k \geq 0$, $p_{n,k}(x) = (nx)^k e^{-nx}/k!$, and $f \in L_p([0, \infty))$ ($1 \leq p \leq \infty$). Then the kernel function $T_n(x, y)$ of the Szász-Mirakjan-Durrmeyer operator [15]

$$\begin{aligned} M_n(f; x) &= \sum_{k=0}^{\infty} \frac{\langle f, p_{n,k} \rangle}{\langle 1, p_{n,k} \rangle} p_{n,k}(x) \\ &= n \langle f, T_n(x, \cdot) \rangle \int_0^{\infty} T_n(x, y) f(y) dy \end{aligned} \quad (41)$$

is given by

$$T_n(x, y) = \sum_{k=0}^{\infty} p_{n,k}(x) p_{n,k}(y) = e^{-n(x+y)} I_0(2n\sqrt{xy}). \quad (42)$$

Berdysheva [16] proved that $T_n(x, y)$ is completely monotonic with respect to $n > 0$ for fixed $x, y \in [0, \infty)$ and

$$T_n(x, y) \leq e^{-n(\sqrt{x} - \sqrt{y})^2} \quad (43)$$

for all $x, y \in [0, \infty)$.

From Remark 8 and (42), we get a lower bound for the kernel function $T_n(x, y)$ immediately.

Corollary 11. *The inequality*

$$T_n(x, y) > e^{-n(x+y)} \left[\frac{\cosh(6n\sqrt{xy}) - 1}{18n^2 xy} \right]^{1/3} \quad (44)$$

holds for all $x, y \in (0, \infty)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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