Research Article

Mappings of Type Special Space of Sequences

Awad A. Bakery^{1,2}

¹Department of Mathematics, Faculty of Science and Arts, University of Jeddah (UJ), P.O. Box 355, Khulais 21921, Saudi Arabia

²Department of Mathematics, Faculty of Science, Ain Shams University, P.O. Box 1156, Abbassia, Cairo 11566, Egypt

Correspondence should be addressed to Awad A. Bakery; awad.bakery@yahoo.com

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We give sufficient conditions on a special space of sequences defined by Mohamed and Bakery (2013) such that the finite rank operators are dense in the complete space of operators whose approximation numbers belong to this sequence space. Hence, under a few conditions, every compact operator would be approximated by finite rank operators. We apply it on the sequence space defined by Tripathy and Mahanta (2003). Our results match those known for p -absolutely summable sequences of reals.

1. Introduction and Basic Definitions

By ω and $L(V, W)$, we will denote the spaces of all real sequences and all bounded linear operators between two Banach spaces V into W , respectively. In [1], Pietsch, by using the approximation numbers and p -absolutely summable sequences of real numbers, formed the operator ideals. In [2], Mohamed and Bakery have considered the space ℓ_M , when $M(t) = t^p$ ($0 < p < \infty$), which matches especially ℓ^p . A subclass U of $L = \{L(V, W)\}$ is an operator ideal if its components verify the following conditions:

- (i) The space $F(V, W)$ of all finite rank operators is a subset of $U(V, W)$.
- (ii) The space $U(V, W)$ is linear.
- (iii) For two Banach spaces V_0 and W_0 , if $T \in L(V_0, V)$, $S \in U(V, W)$, and $R \in L(W, W_0)$, then $RST \in U(V_0, W_0)$. See [3, 4].

An Orlicz function is a function $M : [0, \infty) \rightarrow [0, \infty)$ which is convex, positive, nondecreasing, and continuous, where $M(0) = 0$ and $\lim_{x \rightarrow \infty} M(x) = \infty$. An Orlicz function M is said to satisfy Δ_2 -condition for all values of $x \geq 0$ if there exists a constant $k > 0$, such that $M(2x) \leq kM(x)$.

Lindenstrauss and Tzafriri [5] used the idea of an Orlicz function to define Orlicz sequence spaces as follows:

$$\ell_M = \left\{ x \in \omega : \exists \lambda > 0 \text{ with } \rho(\lambda x) = \sum_{k=1}^{\infty} M(|\lambda x(k)|) < \infty \right\}. \quad (1)$$

$(\ell_M, \|x\|)$ is a Banach space, where $\|x\| = \inf \{ \lambda > 0 : \rho(x/\lambda) \leq 1 \}$. The space ℓ^p is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.

Remark 1. For any Orlicz function M , we have $M(\lambda x) \leq \lambda M(x)$, for all λ with $0 < \lambda < 1$.

Let P_s be the class of all subsets of $\mathbb{N} = \{0, 1, 2, \dots\}$ that do not contain more than s number of elements and let $\{\phi_n\}$ be a nondecreasing sequence of positive reals such that $n\phi_{n+1} \leq (n+1)\phi_n$, for all $n \in \mathbb{N}$. Tripathy and Mahanta [6] defined and studied the following sequence space:

$$m(\phi, M) = \left\{ x = (x_k) \in \omega : \exists \zeta > 0 \text{ with } \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M\left(\frac{|x_k|}{\zeta}\right) < \infty \right\}, \quad (2)$$

with the norm

$$\rho(x) = \inf \left\{ \zeta > 0 : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{|x_k|}{\zeta} \right) \leq 1 \right\}. \quad (3)$$

Lemma 2. (i) $\ell_M \subseteq m(\phi, M)$.

(ii) $\ell_M = m(\phi, M)$ if and only if $\sup_s \phi_s < \infty$.

As of late, different classes of sequences have been presented using Orlicz functions by Braha [7], Raj and Sharma Sunil [8], Raj et al. [9], and many others ([10–13]).

Definition 3 (see [14, 15]). A special space of sequences (sss) is a linear space E with the following:

- (1) $e_n \in E$ for all $n \in \mathbb{N}$, where $e_n = \{0, 0, \dots, 1, 0, 0, \dots\}$ with 1 appearing at n th place for all $n \in \mathbb{N}$.
- (2) E is solid''.
- (3) $(x_0, x_0, x_1, x_1, \dots) \in E$, if $(x_n)_{n \in \mathbb{N}} \in E$.

A premodular (sss) E_ρ is a (sss) and there is a function $\rho : E \rightarrow [0, \infty[$ with the following:

- (i) $\rho(x) \geq 0$, for each $x \in E$ and $\rho(x) = 0 \Leftrightarrow x = \theta$, where θ is the zero element of E .
- (ii) ρ satisfies Δ_2 -condition.
- (iii) For each $x, y \in E$, $\rho(x + y) \leq k(\rho(x) + \rho(y))$ holds for some $k \geq 1$.
- (iv) The space E is ρ -solid; that is, $\rho((x_n)) \leq \rho((y_n))$, whenever $|x_n| \leq |y_n|$, for all $n \in \mathbb{N}$.
- (v) For some numbers $k_0 \geq 1$, the inequality $\rho((x_n)_{n \in \mathbb{N}}) \leq \rho((x_0, x_0, x_1, x_1, \dots)) \leq k_0 \rho((x_n)_{n \in \mathbb{N}})$ holds.
- (vi) $\bar{F} = E_\rho$; that is, the set of all finite sequences F is ρ -dense in E .
- (vii) For each $\lambda > 0$, there is a constant $\xi > 0$ such that $\rho(\lambda, 0, 0, 0, \dots) \geq \xi \lambda \rho(1, 0, 0, 0, \dots)$.

Condition (ii) says that ρ is continuous at θ . The function ρ defines a metrizable topology in E and the linear space E enriched with this topology is denoted by E_ρ .

Definition 4 (see [16]). Consider the following:

$$U_E^{\text{app}} := \{U_E^{\text{app}}(V, W)\}, \quad (4)$$

where

$$U_E^{\text{app}}(V, W) := \{T \in L(V, W) : (\alpha_n(T))_{n \in \mathbb{N}} \in E\}. \quad (5)$$

Theorem 5 (see [2]). If E is a (sss), then U_E^{app} is an operator ideal.

We explain some results related to the operator spaces.

2. Main Results

In this part, we give sufficient conditions on E such that the finite rank operators are dense in the complete space of operators $U_E^{\text{app}}(V, W)$.

Lemma 6. If E_ρ is a premodular (sss) and $(x_n) \in E_\rho$ is a decreasing sequence of positive reals, then

$$\begin{aligned} & \rho \left(\overbrace{0, 0, 0, \dots, 0}^{2n}, x_n, x_{n+1}, x_{n+2}, \dots \right) \\ & \leq k_0 \rho \left(\overbrace{0, 0, 0, \dots, 0}^n, x_n, x_{n+1}, x_{n+2}, \dots \right). \end{aligned} \quad (6)$$

Proof. By using Definition 3, conditions (iv) and (v), and since the elements of E are decreasing, we get

$$\begin{aligned} & \rho \left(\overbrace{0, 0, 0, \dots, 0}^{2n}, x_n, x_{n+1}, x_{n+2}, \dots \right) \\ & \leq \rho \left(\overbrace{0, 0, 0, \dots, 0}^{2n}, x_n, x_n, x_{n+1}, x_{n+1}, \dots \right) \\ & \leq k_0 \rho \left(\overbrace{0, 0, 0, \dots, 0}^n, x_n, x_{n+1}, x_{n+2}, \dots \right). \end{aligned} \quad (7)$$

□

Theorem 7. Let E_ρ be a premodular (sss); then $\overline{F(V, W)^g} = U_E^{\text{app}}(V, W)$, where $g(T) = \rho(\alpha_n(T))_{n \in \mathbb{N}}$.

Proof. To prove that $F(V, W) \subseteq U_E^{\text{app}}(V, W)$, since $e_m \in E$ for each $m \in \mathbb{N}$, from the linearity of E and $T \in F(V, W)$, then finitely many elements of $(\alpha_n(T))_{n \in \mathbb{N}}$ are different from zero. Hence, $T \in U_E^{\text{app}}(V, W)$. For the other inclusion $U_E^{\text{app}}(V, W) \subseteq \overline{F(V, W)}$, let $T \in U_E^{\text{app}}(V, W)$ and, from the definition of approximation numbers, there is $N \in \mathbb{N}$, $N > 0$, A_N with $\text{rank}(A_N) \leq N$ and also

$$\|T - A_N\| \leq 2\alpha_N(T). \quad (8)$$

Since $\alpha_N(T) \rightarrow 0$ as $N \rightarrow \infty$, then $\|T - A_N\| \rightarrow 0$ as $N \rightarrow \infty$; we have to prove that $\rho((\alpha_n(T - A_N))_{n \in \mathbb{N}}) \rightarrow 0$ as

$N \rightarrow \infty$, by taking $N = 8\eta$, where η is a natural number. From Definition 3, condition (iii), we have

$$\begin{aligned}
 d(T, A_N) &= \rho((\alpha_n(T - A_N))_{n \in \mathbb{N}}) \\
 &= \rho \left[\left(\alpha_0(T - A_N), \alpha_1(T - A_N), \dots, \right. \right. \\
 &\quad \alpha_{8\eta-1}(T - A_N), 0, 0, 0, \dots) + \left(\overline{0, 0, 0, \dots, 0}, \right. \\
 &\quad \alpha_{8\eta}(T - A_N), \alpha_{8\eta+1}(T - A_N), \dots, \alpha_{12\eta-1}(T - A_N), \\
 &\quad 0, 0, 0, \dots) + \left(\overline{0, 0, 0, \dots, 0}, \alpha_{12\eta}(T - A_N), \right. \\
 &\quad \left. \left. \alpha_{12\eta+1}(T - A_N), \dots \right) \right] \leq k^2 \left[\rho(\alpha_0(T - A_N), \right. \\
 &\quad \left. \alpha_1(T - A_N), \dots, \alpha_{8\eta-1}(T - A_N), 0, 0, 0, \dots) \right. \\
 &\quad + \rho \left(\overline{0, 0, 0, \dots, 0}, \alpha_{8\eta}(T - A_N), \alpha_{8\eta+1}(T - A_N), \right. \\
 &\quad \left. \dots, \alpha_{12\eta-1}(T - A_N), 0, 0, 0, \dots) + \rho \left(\overline{0, 0, 0, \dots, 0}, \right. \right. \\
 &\quad \left. \left. \alpha_{12\eta}(T - A_N), \alpha_{12\eta+1}(T - A_N), \dots \right) \right] = k^2 [I_1(N) \\
 &\quad + I_2(\eta) + I_3(\eta)].
 \end{aligned} \tag{9}$$

Since $\alpha_n(A_N) = 0$ for $n \geq N$, then

$$\alpha_n(T - A_N) \leq \alpha_{n-N}(T). \tag{10}$$

By using Lemma 6, inequality (10), and Definition 3, condition (v), we get

$$\begin{aligned}
 I_3(\eta) &= \rho \left(\overline{0, 0, 0, \dots, 0}, \alpha_{12\eta}(T - A_N), \right. \\
 &\quad \left. \alpha_{12\eta+1}(T - A_N), \dots \right) \leq \rho \left(\overline{0, 0, 0, \dots, 0}, \alpha_{4\eta}(T), \right. \\
 &\quad \left. \alpha_{4\eta+1}(T), \dots \right) \leq k_0 \rho \left(\overline{0, 0, 0, \dots, 0}, \alpha_{4\eta}(T), \right. \\
 &\quad \left. \alpha_{4\eta+1}(T), \dots \right) \leq k_0^2 \rho \left(\overline{0, 0, 0, \dots, 0}, \alpha_{4\eta}(T), \right. \\
 &\quad \left. \alpha_{4\eta+1}(T), \dots \right) \leq k_0^2 \rho \left(\overline{0, 0, 0, \dots, 0}, \alpha_{3\eta}(T), \right. \\
 &\quad \left. \alpha_{3\eta+1}(T), \dots \right) = k_0^2 \rho((\alpha_n(T))_{n=3\eta}^\infty) \rightarrow 0 \\
 &\quad \text{as } \eta \rightarrow \infty.
 \end{aligned} \tag{11}$$

Now, using Lemma 6 and Definition 3, condition (v), we have

$$\begin{aligned}
 I_2(\eta) &= \rho \left(\overline{0, 0, 0, \dots, 0}, \alpha_{8\eta}(T - A_N), \right. \\
 &\quad \left. \alpha_{8\eta+1}(T - A_N), \dots, \alpha_{12\eta-1}(T - A_N), 0, 0, 0, \dots \right) \\
 &\leq k_0 \rho \left(\overline{0, 0, 0, \dots, 0}, \alpha_{8\eta}(T - A_N), \right. \\
 &\quad \left. \alpha_{8\eta+1}(T - A_N), \dots, \alpha_{12\eta-1}(T - A_N), 0, 0, 0, \dots \right) \\
 &\leq k_0 \rho(\alpha_0(T - A_N), \alpha_1(T - A_N), \dots, \\
 &\quad \alpha_{8\eta-1}(T - A_N), 0, 0, 0, \dots) = k_0 I_1(N).
 \end{aligned} \tag{12}$$

Finally, we have to show that $I_1(N) \rightarrow 0$ as $N \rightarrow \infty$. Since $T \in U_E^{\text{app}}(V, W)$ and ρ is continuous at θ , we have $\rho(\alpha_k(T))_{k=n}^\infty \rightarrow 0$ as $n \rightarrow \infty$. Then, for each $\varepsilon > 0$, there exists $N_0(\varepsilon)$ such that for all $n \geq N_0(\varepsilon)$ we have

$$\rho((\alpha_k(T))_{k=n}^\infty) < \varepsilon. \tag{13}$$

By taking $\varepsilon_1 = \varepsilon/3lk$ for each $n \geq N_0(\varepsilon_1)$ and using inequality (13) and Definition 3, conditions (ii) and (iii), then we have

$$\begin{aligned}
 I_1(N) &= \rho(\alpha_0(T - A_N), \alpha_1(T - A_N), \dots, \alpha_{N-1}(T - A_N), \\
 &\quad 0, 0, 0, \dots) \leq k \left[\rho(\alpha_0(T - A_N), \right. \\
 &\quad \left. \alpha_1(T - A_N), \dots, \alpha_{N_0-1}(T - A_N), 0, 0, 0, \dots) \right. \\
 &\quad + \rho \left(\overline{0, 0, 0, \dots, 0}, \alpha_{N_0}(T - A_N), \alpha_{N_0+1}(T - A_N), \right. \\
 &\quad \left. \dots, \alpha_{N-1}(T - A_N), 0, 0, 0, \dots \right) \leq k \left[\rho(\|T - A_N\|, \right. \\
 &\quad \left. \|T - A_N\|, \dots, \|T - A_N\|, 0, 0, 0, \dots) \right. \\
 &\quad + \rho \left(\overline{0, 0, 0, \dots, 0}, 2\alpha_N(T), 2\alpha_N(T), \dots, 2\alpha_N(T), \right. \\
 &\quad \left. 0, 0, 0, \dots \right) \leq k \left[\|T - A_N\| l \rho \left(\overline{1, 1, 1, \dots, 1}, 0, 0, 0, \right. \right. \\
 &\quad \left. \left. \dots \right) + 2l \rho \left(\overline{0, 0, 0, \dots, 0}, \alpha_{N_0}(T), \alpha_{N_0+1}(T), \dots, \right. \right. \\
 &\quad \left. \left. \alpha_N(T), 0, 0, 0, \dots \right) \right] \leq k [\|T - A_N\| lk_1(\varepsilon) + 2l\varepsilon_1],
 \end{aligned} \tag{14}$$

where $k_1(\varepsilon) = \rho(\overline{1, 1, 1, \dots, 1}, 0, 0, 0, \dots)$, and since $\|T - A_N\| \rightarrow 0$ as $N \rightarrow \infty$, then for each $\varepsilon > 0$ there exists N

such that $\|T - A_N\|_{k_1}(\varepsilon) \leq \varepsilon_1$, for that we have $I_1(N) \leq k[\varepsilon_1 + 2\varepsilon_1] = 3k\varepsilon_1 = \varepsilon$. This completes the proof. $\square$

We give here the sufficient conditions on the sequence spaces $m(\phi, M)$ such that the class of all bounded linear operators between any arbitrary Banach spaces with $(\alpha_n(T))_{n \in \mathbb{N}}$ in these sequence spaces form an ideal operator; the ideal of the finite rank operators in the class of Banach spaces is dense in $U_{m(\phi, M)}^{app}(V, W)$.

Theorem 8. *Let M be an Orlicz function satisfying Δ_2 -condition. Then,*

- (a) $U_{m(\phi, M)}^{app}$ is an operator ideal,
- (b) $\overline{F(V, W)} = U_{m(\phi, M)}^{app}(V, W)$.

Proof. We first prove that the space $m(\phi, M)$ is a (sss).

(1) Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and $x, y \in m(\phi, M)$; then there exist $\zeta_1 > 0$ and $\zeta_2 > 0$ such that

$$\begin{aligned} \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M\left(\frac{|x_k|}{\zeta_1}\right) &< \infty, \\ \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M\left(\frac{|x_k|}{\zeta_2}\right) &< \infty. \end{aligned} \quad (15)$$

Let $\zeta_3 = \max(2|\lambda_1|\zeta_1, 2|\lambda_2|\zeta_2)$. Since M is nondecreasing convex function with Δ_2 -condition, we have

$$\begin{aligned} \sum_{k \in \sigma} M\left(\frac{|\lambda_1 x_k + \lambda_2 y_k|}{\zeta_3}\right) \\ \leq \frac{1}{2} \left[\sum_{k \in \sigma} M\left(\frac{|x_k|}{\zeta_1}\right) + \sum_{k \in \sigma} M\left(\frac{|y_k|}{\zeta_2}\right) \right]. \end{aligned} \quad (16)$$

So, we get

$$\begin{aligned} \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M\left(\frac{|\lambda_1 x_k + \lambda_2 y_k|}{\zeta_3}\right) \\ \leq \frac{1}{2} \left[\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M\left(\frac{|x_k|}{\zeta_1}\right) \right. \\ \left. + \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M\left(\frac{|y_k|}{\zeta_2}\right) \right]. \end{aligned} \quad (17)$$

Thus, $\lambda_1 x + \lambda_2 y \in m(\phi, M)$. Hence, $m(\phi, M)$ is a linear space over the field of real numbers. Also, since $e_n \in \ell_M$ and $\ell_M \subseteq m(\phi, M)$, we have $e_n \in m(\phi, M)$ for all $n \in \mathbb{N}$.

(2) Let $x \in \omega$ and $y = (y_k)_{k=0}^\infty \in m(\phi, M)$ with $|x_k| \leq |y_k|$ for each $k \in \mathbb{N}$; since M is nondecreasing, then we get

$$\begin{aligned} \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M\left(\frac{|x_k|}{\zeta}\right) &\leq \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M\left(\frac{|y_k|}{\zeta}\right) \\ &< \infty; \end{aligned} \quad (18)$$

then $x = (x_k)_{k=0}^\infty \in m(\phi, M)$.

(3) Let $x = (x_k)_{k=0}^\infty \in m(\phi, M)$; then we have

$$\begin{aligned} \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M\left(\frac{|x_{[k/2]}|}{\zeta}\right) \\ \leq 2 \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M\left(\frac{|x_k|}{\zeta}\right) < \infty; \end{aligned} \quad (19)$$

then $x = (x_{[k/2]})_{k=0}^\infty \in m(\phi, M)$.

Finally, we have proved that the space $m(\phi, M)$ with $\rho(x)$ is a premodular (sss).

(i) Clearly, $\rho(x) \geq 0$ for all $x \in m(\phi, M)$ and $\rho(x) = 0 \Leftrightarrow x = \theta$.

(ii) Let $\lambda \in \mathbb{R}$ and $x \in m(\phi, M)$; then for $\lambda \neq 0$ we have

$$\begin{aligned} \rho(\lambda x) \\ = \inf \left\{ \zeta > 0 : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M\left(\frac{|\lambda x_k|}{\zeta}\right) \leq 1 \right\} \\ = \inf \left\{ |\lambda| \mu > 0 : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M\left(\frac{|x_k|}{\mu}\right) \leq 1 \right\}, \end{aligned} \quad (20)$$

where $\mu = \zeta/|\lambda|$. Thus, $\rho(\lambda x) = |\lambda| \inf \{ \mu > 0 : \sup_{s \geq 1, \sigma \in P_s} (1/\phi_s) \sum_{k \in \sigma} M(|x_k|/\mu) \leq 1 \} = |\lambda| \rho(x)$.

Also, for $\lambda = 0$, we have $\rho(\lambda x) = \lambda \rho(x) = 0$.

(iii) Let $x, y \in m(\phi, M)$; then there exist $\zeta_1 > 0$ and $\zeta_2 > 0$ such that

$$\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M\left(\frac{|x_k|}{\zeta_1}\right) \leq 1, \quad (21)$$

$$\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M\left(\frac{|x_k|}{\zeta_2}\right) \leq 1.$$

Let $\zeta = \zeta_1 + \zeta_2$, and since M is nondecreasing and convex, then we have

$$\begin{aligned} \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M\left(\frac{|x_k + y_k|}{\zeta}\right) &\leq \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \\ &\cdot \sum_{k \in \sigma} M\left(\frac{|x_k| + |y_k|}{\zeta_1 + \zeta_2}\right) \leq \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \\ &\cdot \sum_{k \in \sigma} \left[\left(\frac{\zeta_1}{\zeta_1 + \zeta_2} \right) M\left(\frac{|x_k|}{\zeta_1}\right) \right. \\ &\left. + \left(\frac{\zeta_2}{\zeta_1 + \zeta_2} \right) M\left(\frac{|y_k|}{\zeta_2}\right) \right] \leq \left(\frac{\zeta_1}{\zeta_1 + \zeta_2} \right) \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \\ &\cdot \sum_{k \in \sigma} M\left(\frac{|x_k|}{\zeta_1}\right) + \left(\frac{\zeta_2}{\zeta_1 + \zeta_2} \right) \sum_{k \in \sigma} M\left(\frac{|y_k|}{\zeta_2}\right) \leq 1. \end{aligned} \quad (22)$$

Since ζ 's are nonnegative, we have

$$\begin{aligned}
 & \rho(x + y) \\
 &= \inf \left\{ \zeta > 0 : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{|x_k + y_k|}{\zeta} \right) \leq 1 \right\} \\
 &\leq \inf \left\{ \zeta_1 > 0 : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{|x_k|}{\zeta_1} \right) \leq 1 \right\} \quad (23) \\
 &\quad + \inf \left\{ \zeta_2 > 0 : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{|y_k|}{\zeta_2} \right) \leq 1 \right\} \\
 &= \rho(x) + \rho(y).
 \end{aligned}$$

(iv) Let $|x_k| \leq |y_k|$ for each $k \in \mathbb{N}$, and since M is nondecreasing, then we get

$$\sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{|x_k|}{\zeta} \right) \leq \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{|y_k|}{\zeta} \right); \quad (24)$$

thus,

$$\begin{aligned}
 & \inf \left\{ \zeta > 0 : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{|x_k|}{\zeta} \right) \right\} \\
 &\leq \inf \left\{ \zeta > 0 : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{|y_k|}{\zeta} \right) \right\}. \quad (25)
 \end{aligned}$$

So, $\rho(x) \leq \rho(y)$.

(v) Since

$$\begin{aligned}
 & \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{|x_{[k/2]}|}{\zeta} \right) \\
 &\leq 2 \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{|x_k|}{\zeta} \right), \quad (26)
 \end{aligned}$$

we have

$$\begin{aligned}
 & \inf \left\{ \zeta > 0 : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{|x_{[k/2]}|}{\zeta} \right) \right\} \\
 &\leq 2 \inf \left\{ \zeta > 0 : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{|x_k|}{\zeta} \right) \right\}. \quad (27)
 \end{aligned}$$

So, $\rho((x_k)) \leq \rho((x_{[k/2]})) \leq 2\rho((x_k))$.

(vi) For each $x = (x_k)_{k=0}^\infty \in m(\phi, M)$, then

$$\begin{aligned}
 & \rho((x_k)_{k=0}^\infty) \\
 &= \inf \left\{ \zeta > 0 : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{|x_{[k/2]}|}{\zeta} \right) < \infty \right\}; \quad (28)
 \end{aligned}$$

we can find $t \in \mathbb{N}$ such that $\rho((x_k)_{k=t}^\infty) < \infty$. This means the set of all finite sequences is ρ -dense in $m(\phi, M)$.

(vii) For any $\lambda > 0$, there exists a constant $\zeta \in]0, 1]$ such that

$$\rho(\lambda, 0, 0, 0, \dots) \geq \zeta \lambda \rho(1, 0, 0, 0, \dots). \quad (29)$$

By using Theorems 5 and 7, the proof follows. $\square$

As special cases of the above theorem, we obtain the following corollaries.

Corollary 9. If $\sup_s \phi_s < \infty$, one gets that

- (a) $U_{\ell_M}^{app}$ is an operator ideal,
- (b) $\overline{F(V, W)} = U_{\ell_M}^{app}(V, W)$.

Corollary 10. If $\sup_s \phi_s < \infty$ and $M(t) = t^p$ with $0 < p < \infty$, one gets that

- (a) $U_{\ell_p}^{app}$ is an operator ideal,
- (b) $\overline{F(V, W)} = U_{\ell_p}^{app}(V, W)$. See [1].

Theorem 11. If E_ρ is a premodular (sss), then $U_{E_\rho}^{app}(V, W)$ is complete.

Proof. Let (T_m) be a Cauchy sequence in $U_{E_\rho}^{app}(V, W)$; then, by using Definition 3, condition (vii), and since $U_{E_\rho}^{app}(V, W) \subseteq L(V, W)$, we get

$$\begin{aligned}
 & \rho((\alpha_n(T_i - T_j))_{n \in \mathbb{N}}) \geq \rho(\alpha_0(T_i - T_j), 0, 0, 0, \dots) \\
 &= \rho(\|T_i - T_j\|, 0, 0, 0, \dots) \quad (30) \\
 &\geq \xi \|T_i - T_j\| \rho(1, 0, 0, 0, \dots);
 \end{aligned}$$

then (T_m) is also a Cauchy sequence in $L(V, W)$. Since the space $L(V, W)$ is a Banach space, then there exists $T \in L(V, W)$ such that $\|T_m - T\| \rightarrow 0$, as $m \rightarrow \infty$, and since $(\alpha_n(T_m))_{n \in \mathbb{N}} \in E$, for each $m \in \mathbb{N}$, then from Definition 3, conditions (iii) and (v), and since ρ is continuous at θ , we have

$$\begin{aligned}
 & \rho((\alpha_n(T))_{n \in \mathbb{N}}) = \rho(\alpha_n(T - T_m + T_m))_{n \in \mathbb{N}} \\
 &\leq k\rho(\alpha_{[n/2]}(T - T_m))_{n \in \mathbb{N}} \\
 &\quad + k\rho(\alpha_{[n/2]}(T_m))_{n \in \mathbb{N}} \quad (31) \\
 &\leq k\rho(\|T_m - T\|)_{n \in \mathbb{N}} \\
 &\quad + k\rho(\alpha_n(T_m))_{n \in \mathbb{N}} < \varepsilon;
 \end{aligned}$$

we get $(\alpha_n(T))_{n \in \mathbb{N}} \in E$, and then $T \in U_{E_\rho}^{app}(V, W)$. This finishes the proof. $\square$

By applying Theorem 11 on $m(\phi, M)$, we can easily conclude the next corollaries.

Corollary 12. Pick up an Orlicz function M which satisfies Δ_2 -condition. Then, M is continuous from the right at 0 and $U_{m(\phi, M)}^{app}(V, W)$ is complete.

Corollary 13. Pick up an Orlicz function M which satisfies Δ_2 -condition with $\sup_s \phi_s < \infty$. Then, M is continuous from the right at 0 and $U_{l_M}^{app}(X, Y)$ is complete.

Corollary 14. $U_{\ell_p}^{app}(V, W)$ is complete if $M(t) = t^p$ and $p \in (0, \infty)$.

Competing Interests

The author declares that he has no competing interests.

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