Research Article

Adjoining a Constant Function to $n$-Dimensional Chebyshev Space

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This paper is concerned with extending a Chebyshev system of $n$ continuous nonconstant functions into a set of $n+1$ functions including a constant function. Necessary and sufficient conditions for the new set to be a Chebyshev system are discussed and some results are obtained.

1. Introduction

Most of the material in this section and Section 2 can be found in any standard book in approximation theory and related topics; see, for example, [1–5]. The finite set of functions \( \{g_1, \ldots, g_n\} \subset C([a, b]) \) is called a Chebyshev system on \([a, b]\) if it is linearly independent and \( D^*(\frac{g_1}{g_1}, \ldots, \frac{g_n}{g_n}) = \text{Det} \{g_i(x_i)\} \neq 0 \), \( i, j = 1, \ldots, n \), for all \( \{x_i\}_{i=1}^n \) such that \( a \leq x_1 < x_2 < \cdots < x_n \leq b \), and the \( n \)-dimensional subspace \( G = \langle g_1, \ldots, g_n \rangle \) of \( C([a, b]) \) will be called a Chebyshev subspace or Haar subspace. Using the continuity of the determinant, it can be shown that the sign of the determinant is constant (see [6]), so we will assume that the determinant is always positive throughout this paper (replace \( g_1 \) by \(-g_1\) if necessary). If each \( g_i \) is continuously differentiable function on \([a, b], i = 1, \ldots, n \) and \( a \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq b \), then the determinant \( D^*(\frac{g_1}{g_1}, \ldots, \frac{g_n}{g_n}) \) is defined as follows:

\[
D^*(\frac{g_1}{x_1}, \ldots, \frac{g_n}{x_n}) = \begin{bmatrix}
g_1(t_1) & \cdots & g_n(t_1) \\
g_1^{'(r-1)}(t_1) & \cdots & g_n^{'(r-1)}(t_1) \\
\vdots & \ddots & \vdots \\
g_1(t_p) & \cdots & g_n(t_p) \\
g_1^{'(r-1)}(t_p) & \cdots & g_n^{'(r-1)}(t_p)
\end{bmatrix},
\]

where \( x_i \) is repeated \( r_i \) times, \( i = 1, \ldots, p \), \( a \leq t_1 < t_2 < \cdots < t_p \leq b \), and \( \{t_1, t_2, \ldots, t_p\} = \{x_1, x_2, \ldots, x_n\} \). The set of functions \( \{g_1, \ldots, g_n\} \) is called an extended Chebyshev system on \([a, b]\) if \( D^*(\frac{g_1}{x_1}, \ldots, \frac{g_n}{x_n}) > 0 \), and the \( n \)-dimensional subspace \( G = \langle g_1, \ldots, g_n \rangle \) of \( C([a, b]) \) will be called an extended Chebyshev subspace.

In this paper we will consider the following problem. If \( G = \langle g_1, \ldots, g_n \rangle \) is a Chebyshev subspace of \( C([a, b]) \) such that \( 1 \notin G \) then what property must \( G \) have so that the subspace \( U = \langle u_0, u_1, \ldots, u_n \rangle \) is \((n + 1)\)-dimensional Chebyshev subspace of \( C([a, b]) \), where \( u_0 = 1, u_i = g_i, i = 1, \ldots, n \)? We will present some results in Section 3 which give a partial answer to this question.

2. Preliminary

Let \( G = \langle g_1, \ldots, g_n \rangle \) be a Chebyshev subspace of \( C([a, b]) \) and let \( \{x_i\}_{i=1}^{n+1} \) be a set of points such that \( a \leq x_1 < x_2 < \cdots < x_{n+1} \leq b \), and then for any \( g \in G \) we have

\[
0 = D\left(\frac{g_1}{x_1}, \ldots, \frac{g_n}{x_n+1}\right) = \sum_{i=1}^{n+1} (-1)^{i+1} \Delta_i g(x_i),
\]

where

\[
\Delta_1 = D\left(\frac{g_1}{x_2}, \ldots, \frac{g_n}{x_{n+1}}\right),
\]

\[
\Delta_i = D\left(\frac{g_1}{x_{i+1}}, \ldots, \frac{g_n}{x_{i+1}+1}\right) = \sum_{k=1}^{i} (-1)^{k+1} \Delta_k g(x_k)\text{,}
\]

and so forth.
\[ \Delta_{n+1} = D\left( g_1, \ldots, g_n, x_1, \ldots, x_n \right), \]
\[ \Delta_i = D\left( g_1, \ldots, g_n, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1} \right), \]
\[ i = 2, \ldots, n. \]  

Taking \( \theta_i = \Delta_i / \sum_{j=1}^{n+1} \Delta_j \) then \( \theta_i > 0 \) for all \( i = 1, \ldots, n+1 \) with \( \sum_{j=1}^{n+1} \theta_i = 1 \) and \( \sum_{j=1}^{n+1} (-1)^j \theta_j g(x_j) = 0 \) for every \( g \in G \).

This discussion proves the existence part of the following lemma.

**Lemma 1.** Let \( G = \langle g_1, \ldots, g_n \rangle \) be a Chebyshev subspace of \( C[\alpha, \beta] \) and let \( \{x_j\}_{j=1}^{n+1} \) be a set of points such that \( \alpha \leq x_1 < x_2 < \cdots < x_{n+1} \leq \beta \), and then there exists a unique set of positive numbers \( \{\theta_j\}_{j=1}^{n+1} \) with \( \sum_{j=1}^{n+1} \theta_j = 1 \) such that \( \sum_{j=1}^{n+1} (-1)^j \theta_j g(x_j) = 0 \) for every \( g \in G \).

**Proof.** We only need to prove the uniqueness part of this lemma. Suppose that there are two sets of positive real numbers \( \{\theta_j\}_{j=1}^{n+1} \) and \( \{\lambda_j\}_{j=1}^{n+1} \) with \( \sum_{j=1}^{n+1} \lambda_j = 1 \) such that
\[ \sum_{i=1}^{n+1} (-1)^i \theta_i g(x_i) = \sum_{i=1}^{n+1} (-1)^i \lambda_i g(x_i), \]
for every \( g \in G \).

Since \( G \) is a Chebyshev subspace, then for each \( k \in \{2, \ldots, n+1\} \) there exists a unique function \( h^{(k)} \in G \) such that \( h^{(k)}(x_k) = 1 \) and \( h^{(k)}(x_j) = 0, \ j \neq k \) (see [6]), and from (4) we have
\[ -\theta_1 + (-1)^k \theta_k h^{(k)}(x_k) = 0, \]
\[ -\lambda_1 + (-1)^k \lambda_k h^{(k)}(x_k) = 0, \]
where \( k = 2, \ldots, n+1 \).

Clearly \( h^{(k)}(x_k) \neq 0 \), and therefore (5) yield
\[ \frac{\theta_k}{\theta_1} = \frac{\lambda_k}{\lambda_1}, \ k = 2, \ldots, n+1. \]  

Hence \( (1/\theta_1) \sum_{k=2}^{n+1} \theta_k = (1/\lambda_1) \sum_{k=2}^{n+1} \lambda_k \Rightarrow \sum_{i=1}^{n+1} (-1)^i \theta_i / \theta_1 \Rightarrow \sum_{i=1}^{n+1} (-1)^i \lambda_i / \lambda_1 \Rightarrow \) \( \theta_i = \lambda_i, \ i = 2, \ldots, n+1, \) and the proof is complete.

3. **The Main Result**

We start this section by the following theorem.

**Theorem 2.** Let \( G = \langle g_1, \ldots, g_n \rangle \), where \( \{g_i\}_{i=1}^{n} \subset C[\alpha, \beta] \) is a Chebyshev system on \([\alpha, \beta]\). Then \( \langle g_0 = 1, g_1, \ldots, g_n \rangle \) is a Chebyshev system on \([\alpha, \beta]\) if and only if, for each set of points \( \{t_i\}_{i=1}^{n} \) such that \( \alpha \leq t_0 < t_1 < \cdots < t_n \leq \beta \) and the corresponding set of positive real numbers \( \{\theta_j\}_{j=0}^{n} \) with
\[ \sum_{i=0}^{n} \theta_i = 1 \] satisfying \( \sum_{i=0}^{n} (-1)^i \theta_i g(t_i) = 0 \) for every \( g \in G \), we have
\[ \sum_{i \in I} \theta_i \neq \sum_{j \in J} \theta_j, \] where \( I = \{i \in A : i \text{ is odd}\}, A = \{0, 1, \ldots, n\}, \) and \( J = A \setminus I \).

**Proof.** Let \( \{t_i\}_{i=0}^{n} \) be a set of points such that \( \alpha \leq t_0 < t_1 < \cdots < t_n \leq \beta \) and the corresponding set of positive numbers \( \{\theta_j\}_{j=0}^{n} \) with \( \sum_{j=0}^{n} \theta_j = 1 \) satisfying \( \sum_{j=0}^{n} (-1)^j \theta_j g(t_j) = 0 \) for every \( g \in G \). From Lemma 1 \( \theta_1 = \Delta_1/d \), where \( d = \sum_{j=0}^{n} \Delta_j \) and \( \Delta_j = D\left( \sum_{i=0}^{j-1} \frac{g_i - g_j}{g_j} \right) \), \( i = 0, \ldots, n \).

\[ D\left( g_0, g_1, \ldots, g_n \right), t_0, \ldots, t_n = \text{Det} \left[ \begin{array}{cccc} g_1(t_0) & \cdots & g_n(t_0) \\ \vdots & \ddots & \vdots \\ g_1(t_n) & \cdots & g_n(t_n) \end{array} \right], \]
\[ = \sum_{i=0}^{n} (-1)^i \Delta_i = \sum_{i=0}^{n} (-1)^i d \theta_i \]
and
\[ = d \left( \sum_{i \in I} \theta_i - \sum_{j \in J} \theta_j \right) \neq 0, \]
if and only if \( \sum_{i \in I} \theta_i \neq \sum_{j \in J} \theta_j \), where \( I \) and \( J \) are as defined above, and the theorem is proved.

**Assumption 3.** Let \( g_i \in C^1[\alpha, \beta] \) for all \( i = 1, \ldots, n \) and let \( G = \langle g_1, \ldots, g_n \rangle \) be a Chebyshev subspace of \( C[\alpha, \beta] \). We say that \( G \) satisfies Assumption 3 if, for each nontrivial element \( g \) of \( G \), \( g' \) can have at most \( n-1 \) distinct zeros on \([\alpha, \beta]\). That is, if \( g'(x) = 0, \alpha < x_1 < x_2 < \cdots < x_k < \beta \), and \( k \geq n \), then \( g \) is identically zero.

**Theorem 4.** Suppose \( g_i \in C^1[\alpha, \beta] \) for all \( i = 1, \ldots, n \) and \( G = \langle g_1, \ldots, g_n \rangle \) is a Chebyshev subspace of \( C[\alpha, \beta] \). If \( G \) satisfies Assumption 3, then \( \langle g_0 = 1, g_1, \ldots, g_n \rangle \) is a Chebyshev system on \([\alpha, \beta]\).

**Proof.** If there is a function \( \tilde{h} = \tilde{a}_0 + \tilde{a}_1 g_1 + \cdots + \tilde{a}_n g_n \) such that \( \tilde{h}(t_i) = 0 \) at some set of points \( \alpha \leq t_0 < t_1 < \cdots < t_k \leq \beta \); then by Rolle’s theorem there exists a set of points \( \{x_j\}_{j=1}^{k+1}, x_j \in (t_j-1, t_j) \), such that \( \tilde{h}'(x_j) = 0 = (g_j)'(x_j), j = 1, \ldots, k \), where \( \bar{g} = \tilde{a}_1 g_1 + \cdots + \tilde{a}_n g_n \in G \). So if \( k \geq n \) then \( \bar{g} = 0 \), and hence \( \bar{h} = \tilde{a}_0 = \tilde{h} = 0 \), and this shows that \( \langle g_0 = 1, g_1, \ldots, g_n \rangle \) is a Chebyshev system on \([\alpha, \beta]\).

When \( n = 1, u \in C[\alpha, \beta] \), and \( G = \langle u \rangle \) is a Chebyshev subspace of \( C[\alpha, \beta] \) if \( u(x) \neq 0 \) for all \( x \in [\alpha, \beta] \). For this special case we have the following results which can be found in [3].

**Proposition 5.** Let \( u \in C[\alpha, \beta] \), and then \( H = \langle 1, u \rangle \) is a Chebyshev subspace of \( C[\alpha, \beta] \) if and only if \( u \) is strictly monotonic function on \([\alpha, \beta]\).
Remark 6. If \( u \) is an even function on \([-a, a], a > 0\), then \( \{1, u\} \) is not a Chebyshev system on \([-a, a]\), that is, since \( D \left( \frac{1}{\sqrt{1-u^2}} \right) = 0, \forall t \in (0, a]. \)

Remark 7. Assumption 3 is not a necessary condition for Theorem 4 as the following example illustrates.

Example 8. Take \( u = e^{x^3} \) and then \( \{1, u\} \) is a Chebyshev system on \([-a, a], a > 0\) although \( u'(0) = 0 \).

Finally, we will give an example of a set of continuously differentiable functions \( \{g_i\}_{i=1}^{n} \) which is an extended Chebyshev system on \([a, b]\) and \( \{g_0 = 1, g_1, \ldots, g_n\} \) is a Chebyshev system on \([a, b]\) but not an extended Chebyshev system.

Example 9. Let \( g_1 = x \) and \( g_2 = -\cos x \), then \( D^* \left( \frac{g_1 g_2}{1 \times 2} \right) = t \sin t + \cos t \) > 0, \( \forall t \in [0, \pi/2] \), and hence \( G = \langle g_1, g_2 \rangle \) is an extended Chebyshev system on \([0, \pi/2]\). And if \( g \in G \) then \( g' \) can have at most one zero on \([0, \pi/2]\); this means that \( G \) satisfies Assumption 3 and by Theorem 4 \( \{1, g_1, g_2\} \) is a Chebyshev system on \([0, \pi/2]\). Taking \( v(x) = -\pi/2 + x + \cos x \), then \( v(\pi/2) = v'(\pi/2) = v''(\pi/2) = 0 \); that is, \( \{1, g_1, g_2\} \) is not an extended Chebyshev system on \([0, \pi/2]\).

Competing Interests
The author declares no competing interests.

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References