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Research Article

Multilinear Square Functions with Kernels of Dini's Type

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Let T be a multilinear square function with a kernel satisfying Dini(1) condition and let T^* be the corresponding multilinear maximal square function. In this paper, first, we showed that T is bounded from $L^1 \times \cdots \times L^1$ to $L^{1/m,\infty}$. Secondly, we obtained that if each $p_i > 1$, then T and T^* are bounded from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ to $L^p(\nu_{\vec{\omega}})$ and if there is $p_i = 1$, then T and T^* are bounded from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ to $L^{p,\infty}(\nu_{\vec{\omega}})$, where $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$. Furthermore, we established the weighted strong and weak type boundedness for T and T^* on weighted Morrey type spaces, respectively.

1. Introduction and Main Results

Let $m(t) \in L^\infty$ and $a \in \mathrm{BMO}$. In 1978, Coifman and Meyer [1] introduced a class of multilinear operators as a multilinearization of Littlewood-Paley g-function as follows:

$$\mathscr{B}(a, f) = \int_0^\infty (f * \phi_t)(a * \Phi_t) \frac{m(t)}{t} dt, \tag{1}$$

where $\widehat{\phi}$ and $\widehat{\Phi}$ have compact support with $0 \notin \operatorname{supp} \widehat{\Phi}$. They studied the L^2 estimate of $\mathscr{B}$ by using the notion of Carleson measures. In 1982, letting $m \geq 2$ and $p \geq 1$, Yabuta [2] obtained the L^p boundedness and BMO type estimates of $\mathscr{B}$ by weakening the assumptions in [1]. In 2002, Sato and Yabuta [3] studied the $(L^{p_1} \times \cdots \times L^{p_m}, L^p)$ boundedness of the following multilinear Littlewood-Paley g-function:

$$T_g(\vec{f})(x) = \int_0^\infty \prod_{i=1}^m ((\phi_i)_t * f)(x) \frac{dt}{t}. \tag{2}$$

The kernels in (1) and (2) are restricted to separable variable kernels. Thus, efforts have been made to study the above operators with kernels of nonseparated type. In 2015, Xue et al. [4] introduced and studied the weighted estimates for the following multilinear Littlewood-Paley g-function with convolution type kernel:

$$g(\vec{f})(x) = \left(\int_0^\infty \left| \psi_t * \vec{f}(x) \right|^2 \frac{dt}{t} \right)^{1/2}, \tag{3}$$

where

$$\psi_t * \vec{f}(x) = \int_{(\mathbb{R}^n)^m} \psi_t(y_1, \ldots, y_m) \prod_{j=1}^m f_j(x - y_j) dy_j, \tag{4}$$

with $\psi_t(y_1, \ldots, y_m) = (1/t^{mn}) \psi(y_1/t, \ldots, y_m/t)$.

The importance of multilinear Littlewood-Paley g-function and related multilinear Littlewood-Paley type estimates were shown in PDE and other fields (see [5–10]). For other recent works about multilinear Littlewood-Paley type operators, see [11, 12] and the references therein.

In this paper, our aim is to study the boundedness of multilinear square functions with more rough kernels. To begin with, we give some notations and introduce some definitions. The following multiple weights classes $A_{\vec{p}}$ were introduced and studied by Lerner et al. [13].

Definition 1 ($A_{\vec{p}}$ weights class [13]). Let $1 \le p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$. Given $\vec{\omega} = (\omega_1, \dots, \omega_m)$, set $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$. One says that $\vec{\omega}$ satisfies the $A_{\vec{p}}$ condition if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \prod_{i=1}^m \omega_i^{p/p_i} \right)^{1/p} \prod_{i=1}^m \left(\frac{1}{|Q|} \int_Q \omega_i^{1-p_i'} \right)^{1/p_i'} < \infty, \tag{5}$$

when $p_i = 1$ and $((1/|Q|)\int_Q \omega_i^{1-p_i'})^{1/p_i'}$ is understood as $(\inf_Q \omega_i)^{-1}$.

Definition 2 (Dini(a) condition). Suppose that $\omega(t) : [0,\infty) \mapsto [0,\infty)$ is a nondecreasing function with $0 < \omega(1) < \infty$. For $a > 0$, one says that $\omega \in$ Dini(a), if

$$|\omega|_{\mathrm{Dini}(a)} = \int_0^1 \frac{\omega^a(t)}{t} dt < \infty. \tag{6}$$

Definition 3 (kernels of type $\omega(t)$). For any $t \in (0,\infty)$, a locally integrable function $K_t(x, y_1, \dots, y_m)$ defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$ is called a kernel of type $\omega(t)$, if there is a positive constant $A > 0$, such that

$$\left(\int_0^\infty \left| K_t(x, y_1, \dots, y_m) \right|^2 \frac{dt}{t} \right)^{1/2} \le \frac{A}{\left(\sum_{j=1}^m |x - y_j| \right)^{mn}}; \tag{7}$$

$$\left(\int_0^\infty \left| K_t(z, y_1, \dots, y_m) - K_t(x, y_1, \dots, y_m) \right|^2 \frac{dt}{t} \right)^{1/2} \le \frac{A}{\left(\sum_{j=1}^m |x - y_j| \right)^{mn}} \omega\left(\frac{|z - x|}{\sum_{j=1}^m |x - y_j|} \right), \tag{8}$$

whenever $|z - x| \le (1/2)\max_{j=1}^m |x - y_j|$; and

$$\left(\int_0^\infty \left| K_t(x, y_1, \dots, y_j, \dots, y_m) - K_t(x, y_1, \dots, y_j', \dots, y_m) \right|^2 \frac{dt}{t} \right)^{1/2} \le \frac{A}{\left(\sum_{j=1}^m |x - y_j| \right)^{mn}} \omega\left(\frac{|y_j - y_j'|}{\sum_{j=1}^m |x - y_j|} \right), \tag{9}$$

whenever $|y_j - y_j'| \le (1/2)\max_{j=1}^m |x - y_j|$.

We say that T is a multilinear square function with a kernel of type $\omega(t)$, if

$$T(\vec{f})(x) = \left(\int_0^\infty \left| \int_{(\mathbb{R}^n)^m} K_t(x, y_1, \dots, y_m) \cdot \prod_{j=1}^m f_j(y_j)\, dy_1, \dots, dy_m \right|^2 \frac{dt}{t} \right)^{1/2}, \tag{10}$$

whenever $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$ and each $f_j \in C_c^\infty(\mathbb{R}^n)$.

The corresponding multilinear maximal square function T^* is defined by

$$T^*(\vec{f})(x) = \sup_{\delta > 0} T_\delta(\vec{f})(x), \tag{11}$$

where

$$T_\delta(\vec{f})(x) = \left(\int_0^\infty \left| \int_{\sum_{i=1}^m |x - y_i|^2 > \delta^2} K_t(x, y_1, \dots, y_m) \cdot \prod_{j=1}^m f_j(y_j)\, dy_1, \dots, dy_m \right|^2 \frac{dt}{t} \right)^{1/2}. \tag{12}$$

We always assume that T and T^* can be extended to be a bounded operator from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q for some $1 < q, q_1, \dots, q_m < \infty$ with $1/q_1 + \cdots + 1/q_m = 1/q$.

The aim of this paper is to study the bounded properties of multilinear square function T and multilinear maximal square function T^* with nonconvolution type kernels. It should be pointed out that the methods used in [4, 11, 12] do not work for Littlewood-Paley operators with more general nonconvolution type kernels, for the reason that the estimates there rely heavily on the convolution type kernels and the well-known Marcinkiewicz function studied in [14].

We formulate the main results of this paper as follows.

Theorem 4. *Let T be a multilinear square function of type $\omega(t)$ and $\omega \in$ Dini(1). Then, T can be extended to be a bounded operator from $L^1 \times \cdots \times L^1$ to $L^{1/m,\infty}$.*

Theorem 5. *Let T be a multilinear square function of type $\omega(t)$ and $\omega \in$ Dini(1). Let $\vec{\omega} \in A_{\vec{p}}$ and $1/p = 1/p_1 + \cdots + 1/p_m$. Then, one has the following weighted estimates:*

(i) *If $1 < p_1, \dots, p_m < \infty$, then*

$$\left\| T\vec{f} \right\|_{L^p(\nu_{\vec{\omega}})} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)}. \tag{13}$$

(ii) *If $1 \le p_1, \dots, p_m < \infty$, then*

$$\left\| T\vec{f} \right\|_{L^{p,\infty}(\nu_{\vec{\omega}})} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)}. \tag{14}$$

Theorem 6. *Let T^* be a multilinear maximal square function of type $\omega(t)$ and $\omega \in$ Dini(1). Let $1/p = 1/p_1 + \cdots + 1/p_m$ and $\vec{\omega} \in A_{\vec{p}}$. Then, one has the following:*

(i) *If $1 < p_1, \dots, p_m < \infty$, then*

$$\left\| T^*\vec{f} \right\|_{L^p(\nu_{\vec{\omega}})} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)}. \tag{15}$$

(ii) *If $1 \le p_1, \dots, p_m < \infty$, then*

$$\left\| T^*\vec{f} \right\|_{L^{p,\infty}(\nu_{\vec{\omega}})} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)}. \tag{16}$$

Remark 7. When $\omega(x) = x^\gamma$ for some $\gamma > 0$, Theorems 4 and 5 were proved in [15].

Remark 8. The theory on multilinear Calderón-Zygmund operators has attracted much attention recently. In 2009, Maldonado and Naibo [16] established the weighted norm inequalities, with the Muckenhoupt weights, for bilinear Calderón-Zygmund operators of type $\omega(t)$. In 2014, Lu and Zhang [17] obtained some multiple-weighted norm inequalities for multilinear Calderón-Zygmund operators of type $\omega(t)$ and related operators.

The paper is organized as follows. Section 2 contains some preliminary information and proofs of Theorems 4–6. In Section 3, we establish the weighted strong and weak type boundedness of T and T^* on Morrey type spaces.

2. Proofs of the Main Theorems

First, we give the proof of Theorem 4.

Proof. The basic idea of the following arguments is essentially taken from [15, 17].

Set $B = \|T\|_{L^{q_1}\times\cdots\times L^{q_m}\to L^q}$. Without loss of generality, we can assume that $\|f_j\|_{L^1} = 1$, where $j = 1,\ldots,m$. For any fixed number $\lambda > 0$, we need to show that there is constant $C > 0$ such that

$$\begin{aligned}&\left|\left\{x\in\mathbb{R}^n : \left|T\left(f_1,\ldots,f_m\right)(x)\right| > \lambda\right\}\right| \\ &\quad\le C\left(A+B\right)^{1/m}\lambda^{-1/m}.\end{aligned} \tag{17}$$

We perform Calderón-Zygmund decomposition to each function f_j at level $(\gamma\lambda)^{1/m}$, where γ is a positive number to be determined later. Then, we obtain a sequence of pairwise disjoint cubes $\{Q_{j,k_j}\}_{k_j=1}^{\infty}$ and decomposition $f_j = g_j + b_j = g_j + \sum_{k_j} b_{j,k_j}$. Moreover, we have

(P1) $\operatorname{supp}(b_{j,k_j}) \subset Q_{j,k_j}$,

(P2) $\int_{\mathbb{R}^n} b_{j,k_j}(x)dx = 0$,

(P3) $\int_{\mathbb{R}^n} |b_{j,k_j}(x)|dx \le C(\gamma\lambda)^{1/m}|Q_{j,k_j}|$,

(P4) $\sum_{k_j} |Q_{j,k_j}| \le C(\gamma\lambda)^{1/m}$,

(P5) $\|b_j\|_{L^1(\mathbb{R}^n)} \le C$,

(P6) $\|g_j\|_{L^s(\mathbb{R}^n)} \le C(\gamma\lambda)^{1/(ms')}$ for $1 \le s \le \infty$.

Let c_{j,k_j} be the center of cube Q_{j,k_j} and let $l(Q_{j,k_j})$ be its side length. For $j = 1,\ldots,m$, set $\Omega^* = \bigcup_{j=1}^m \Omega_j^*$, where $\Omega_j^* = \bigcup_{k_j} Q_{j,k_j}^*$ and $Q_{j,k_j}^* = 8\sqrt{n}Q_{j,k_j}$. And let

$$\begin{aligned}E_1 &= \left\{x\in\mathbb{R}^n : \left|T\left(g_1,g_2,\ldots,g_m\right)(x)\right| > \frac{\lambda}{2^m}\right\};\\ E_2 &= \left\{x\in\mathbb{R}^n\setminus\Omega^* : \left|T\left(b_1,g_2,\ldots,g_m\right)(x)\right| > \frac{\lambda}{2^m}\right\};\\ E_3 &= \left\{x\in\mathbb{R}^n\setminus\Omega^* : \left|T\left(g_1,b_2,\ldots,g_m\right)(x)\right| > \frac{\lambda}{2^m}\right\};\\ &\vdots\\ E_{2^m} &= \left\{x\in\mathbb{R}^n\setminus\Omega^* : \left|T\left(b_1,b_2,\ldots,b_m\right)(x)\right| > \frac{\lambda}{2^m}\right\}.\end{aligned} \tag{18}$$

It follows from property (P4) that

$$\left|\Omega^*\right| \le \sum_{j=1}^m \left|\Omega_j^*\right| \le C\sum_{j=1}^m\sum_{k_j}\left|Q_{j,k_j}\right| \le C\left(\gamma\lambda\right)^{-1/m}. \tag{19}$$

By the $L^{q_1}\times\cdots\times L^{q_m}\to L^{q,\infty}$ boundedness of T and property (P6), we get

$$\begin{aligned}\left|E_1\right| &\le \left(2^m B\right)^q\lambda^{-q}\left\|g_1\right\|_{L^{q_1}(\mathbb{R}^n)}^q\cdots\left\|g_m\right\|_{L^{q_m}(\mathbb{R}^n)}^q\\ &\le CB^q\gamma^{q-1}\lambda^{-1/m}.\end{aligned} \tag{20}$$

Thus, we have

$$\begin{aligned}\left|\left\{x\in\mathbb{R}^n : \left|T\left(\vec{f}\right)(x)\right| > \lambda\right\}\right| &\le C\sum_{s=1}^{2^m}\left|E_s\right| + C\left|\Omega^*\right|\\ &\le C\sum_{s=2}^{2^m}\left|E_s\right| + CB^q\gamma^{q-1/m}\lambda^{-1/m} + C\left(\gamma\lambda\right)^{-1/m}.\end{aligned} \tag{21}$$

To complete the proof, we need to estimate $|E_s|$ for $2 \le s \le 2^m$. Suppose that for some $1 \le l \le m$ there are l bad functions and $m-l$ good functions appearing in $T(h_1,\ldots,h_m)$, where $h_j \in \{g_j, b_j\}$. For simplicity, we assume that the bad functions appear at the entries $1,\ldots,l$, and denote the corresponding term by $|E_s^{(l)}|$ to distinguish it from the other terms. That is, we will consider

$$\begin{aligned}\left|E_s^{(l)}\right| = \Bigg|\Bigg\{&x\\ &\in\mathbb{R}^n\setminus\Omega^* : \left|T\left(b_1,b_2,\ldots,b_l,g_{l+1},\ldots,g_m\right)(x)\right|\\ &> \frac{\lambda}{2^m}\Bigg\}\Bigg|,\end{aligned} \tag{22}$$

and the other terms can be estimated similarly.

We will show that $|E_s^{(l)}| \le CA(\gamma\lambda)^{-1/m}$. Obviously, $T(b_1,b_2,\ldots,b_l,g_{l+1},\ldots,g_m)(x)$ can be controlled by

$$\sum_{k_1,\ldots,k_l}\left(\int_0^\infty\left|\int_{\mathbb{R}^{nm}} K_t\left(x,y_1,\ldots,y_m\right)\prod_{i=1}^l b_{i,k_i}\left(y_i\right)\prod_{i=l+1}^m g_i\left(y_i\right)d\vec{y}\right|^2\frac{dt}{t}\right)^{1/2}. \tag{23}$$

By properties (P4) and (P6) and Minkowski's inequality, we can further control the above term by

$$\begin{aligned}
&\sum_{k_1,\dots,k_l}\left(\int_0^\infty\left|\int_{\mathbb{R}^{nm}}\left(K_t\left(x,y_1,\dots,y_m\right)-K_t\left(x,c_{1,k_1},\dots,y_m\right)\right)\prod_{i=1}^{l}b_{i,k_i}\left(y_i\right)\prod_{i=l+1}^{m}g_i\left(y_i\right)d\vec{y}\right|^2\frac{dt}{t}\right)^{1/2}\\
&\quad\le\sum_{k_1,\dots,k_l}\int_{\mathbb{R}^{nm}}\left(\int_0^\infty\left|K_t\left(x,y_1,\dots,y_m\right)-K_t\left(x,c_{1,k_1},\dots,y_m\right)\right|^2\frac{dt}{t}\right)^{1/2}\prod_{i=1}^{l}b_{i,k_i}\left(y_i\right)\prod_{i=l+1}^{m}g_i\left(y_i\right)d\vec{y}\\
&\quad\le C\left(\gamma\lambda\right)^{(m-l)/m}\sum_{k_1,\dots,k_l}\int_{\mathbb{R}^{nm}}\left(\int_0^\infty\left|K_t\left(x,y_1,\dots,y_m\right)-K_t\left(x,c_{1,k_1},\dots,y_m\right)\right|^2\frac{dt}{t}\right)^{1/2}\prod_{i=1}^{l}b_{i,k_i}\left(y_i\right)d\vec{y}.
\end{aligned}\tag{24}$$

Let $\mathcal{Q}_{r,k_r}^{i_r}=(2^{i_r+2}\sqrt{n}Q_{r,k_r})\setminus(2^{i_r+1}\sqrt{n}Q_{r,k_r})$ for $r=1,\dots,l$ and $i_r=1,2,\dots$. It was proved that $\mathbb{R}^n\setminus Q^*\subset\bigcup_{i_1,\dots,i_l=1}^{\infty}\bigcap_{r=1}^{l}\mathcal{Q}_{r,k_r}^{i_r}$; see [17, p. 105]. Assume that Q_{1,k_1} has the smallest length in $\{Q_{i,k_i}\}_{i=1}^{l}$; then, by (9) one has

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus Q^*}\left(\int_0^\infty\left|K_t\left(x,y_1,\dots,y_m\right)-K_t\left(x,c_{1,k_1},\dots,y_m\right)\right|^2\frac{dt}{t}\right)^{1/2}dx\le C\int_{\mathbb{R}^n\setminus Q^*}\frac{1}{\left(\sum_{j=1}^{m}\left|x-y_j\right|\right)^{mn}}\omega\left(\frac{\left|y_1-c_{1,k_1}\right|}{\sum_{j=1}^{m}\left|x-y_j\right|}\right)dx\\
&\quad\le C\sum_{i_1,\dots,i_l=1}^{\infty}\int_{\bigcap_{r=1}^{l}\mathcal{Q}_{r,k_r}^{i_r}}\frac{1}{\left(\sum_{j=1}^{m}\left|x-y_j\right|\right)^{mn}}\omega\left(\frac{\left|y_1-c_{1,k_1}\right|}{\sum_{j=1}^{m}\left|x-y_j\right|}\right)dx\le C\sum_{i_1,\dots,i_l=1}^{\infty}\omega\left(2^{-i_l}\right)\int_{\bigcap_{r=1}^{l}\mathcal{Q}_{r,k_r}^{i_r}}\frac{1}{\left(\sum_{j=1}^{m}\left|x-y_j\right|\right)^{mn}}dx.
\end{aligned}\tag{25}$$

This together with Chebyshev's inequality gives

$$\begin{aligned}
\left|E_s^{(l)}\right|&\le\frac{2^m}{\lambda}\int_{\mathbb{R}^n\setminus Q^*}\left|T\left(b_1,b_2,\dots,b_l,g_{l+1},\dots,g_m\right)(x)\right|dx\\
&\le\frac{2^m}{\lambda}\left(\gamma\lambda\right)^{(m-l)/m}\int_{\mathbb{R}^n\setminus Q^*}\sum_{k_1,\dots,k_l}\int_{\mathbb{R}^{nm}}\left(\int_0^\infty\left|K_t\left(x,y_1,\dots,y_m\right)-K_t\left(x,c_{1,k_1},\dots,y_m\right)\right|^2\frac{dt}{t}\right)^{1/2}\prod_{i=1}^{l}b_{i,k_i}\left(y_i\right)d\vec{y}\,dx\\
&\le\frac{2^m}{\lambda}\left(\gamma\lambda\right)^{(m-l)/m}\sum_{k_1,\dots,k_l}\sum_{i_1,\dots,i_l=1}^{\infty}\omega\left(2^{-i_l}\right)\int_{(\mathbb{R}^n)^{m-l}}\int_{\prod_{r=1}^{l}Q_{r,k_r}}\int_{\bigcap_{r=1}^{l}\mathcal{Q}_{r,k_r}^{i_r}}\frac{dx}{\left(\sum_{j=1}^{m}\left|x-y_j\right|\right)^{mn}}\prod_{i=1}^{l}b_{i,k_i}\left(y_i\right)d\vec{y},
\end{aligned}\tag{26}$$

where $\prod_{r=1}^{l}Q_{r,k_r}=Q_{1,k_1}\times\cdots\times Q_{l,k_l}$.

Now, by repeating the same arguments as in [17, p. 106–108], we can easily obtain $|E_s^{(l)}|\le CA\gamma(\gamma\lambda)^{-1/m}$. Thus, setting $\gamma=(A+B)^{-1}$, we get

$$\begin{aligned}
&\left|\left\{x\in\mathbb{R}^n:\left|T\left(\vec{f}\right)(x)\right|>\lambda\right\}\right|\\
&\quad\le CA\gamma\left(\gamma\lambda\right)^{-1/m}+CB^q\gamma^{q-1/m}\lambda^{-1/m}+C\left(\gamma\lambda\right)^{-1/m}\\
&\quad=C\left(A+B\right)^{1/m}\lambda^{-1/m}.
\end{aligned}\tag{27}$$

This completes the proof of Theorem 4. □

In order to prove Theorem 5, the following lemmas are needed.

Lemma 9 (see [15]). *Suppose that* $\operatorname{supp} f_i\subset B(0,R)$*; then, there is a constant* $C<\infty$ *such that, for all* $|x|>2R$*, the following inequality holds:*

$$T\left(\vec{f}\right)(x)\le C\mathcal{M}\left(\vec{f}\right)(x).\tag{28}$$

Lemma 10. *Let* T *be a multilinear square function of type* $\omega(t)$ *and* $\omega\in Dini(1)$*. For any* $0<\delta<1/m$*, there is a constant* $C<\infty$ *such that for any bounded and compact supported* f_j*,* $j=1,\dots,m$*, the following inequality holds:*

$$M_\delta^\sharp T\left(\vec{f}\right)(x)\le C\mathcal{M}\left(\vec{f}\right)(x).\tag{29}$$

Proof. The proof of Lemma 10 involves a routine application of the method used in Lemma 4.1 in [15]. For the sake of similarity, we sketch the proof. Given $0 < \delta < 1/m$, for a fixed point $x \in \mathbb{R}^n$ and a cube $Q \ni x$, it is sufficient to show that there exists a constant c_Q such that

$$\left(\frac{1}{|Q|}\int_Q \left|T\left(\vec{f}\right)(z) - c_Q\right|^\delta dz\right)^{1/\delta} \le C\mathscr{M}\vec{f}(x). \tag{30}$$

For each $i = 1, \ldots, m$, let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m)$ satisfying $\alpha_i = 0$ or ∞. We next introduce two notations:

$$\begin{aligned} c_{Q,t} &= \sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}} \int_{\mathbb{R}^{nm}} K_t(x, y_1, \ldots, y_m) \prod_{j=1}^m f_j(y_j)\, d\vec{y}, \\ c_Q &= \left(\int_0^\infty |c_{Q,t}|^2 \frac{dt}{t}\right)^{1/2}. \end{aligned} \tag{31}$$

Therefore,

$$\left(\frac{1}{|Q|}\int_Q \left|T\left(\vec{f}\right)(z) - c_Q\right|^\delta dz\right)^{1/\delta} \le CI_{\vec{0}} + C\sum_{\vec{\alpha}\neq\vec{0}} I_{\vec{\alpha}}, \tag{32}$$

where

$$\begin{aligned} I_{\vec{0}} &= \left(\frac{1}{|Q|}\int_Q \left(\int_0^\infty \left|\int_{\mathbb{R}^{nm}} K_t(z,\vec{y}) \prod_{j=1}^m f_j^0(y_j)\, d\vec{y}\right|^2 \frac{dt}{t}\right)^{\delta/2} dz\right)^{1/\delta}, \\ I_{\vec{\alpha}} &= \sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}} \left(\frac{1}{|Q|}\int_Q \left(\int_0^\infty \left|\int_{\mathbb{R}^{nm}} \left(K_t(z,\vec{y}) - K_t(x,\vec{y})\right) \prod_{j=1}^m f_j^{\alpha_j}(y_j)\, d\vec{y}\right|^2 \frac{dt}{t}\right)^{\delta/2} dz\right)^{1/\delta}. \end{aligned} \tag{33}$$

By using the boundedness of T, we get immediately that $I_{\vec{0}} \le C\mathscr{M}(\vec{f})(x)$.

Using the smooth condition (8), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^{nm}} \left(\int_0^\infty \left|K_t(z,\vec{y}) - K_t(x,\vec{y})\right|^2 \frac{dt}{t}\right)^{1/2} \\ &\quad\cdot \prod_{j=1}^m \left|f_j^{\alpha_j}(y_j)\right| d\vec{y} \le C\int_{Q(x,\delta)^l} \prod_{i\in\mathcal{F}} \left|f_i^0(y_i)\, dy_i\right| \\ &\quad\cdot \sum_{k=1}^\infty \int_{(\mathcal{Q}_k)^{m-l}} \frac{A}{\left(\sum_{i\in\mathcal{F}^c}|z-y_i|\right)^{mn}} \omega\left(\frac{|z-x|}{\sum_{i\in\mathcal{F}^c}|z-y_i|}\right) \\ &\quad\cdot \prod_{i\in\mathcal{F}^c} \left|f_i^\infty(y_i)\right| dy_i \le C\int_{Q(x,\delta)^l} \prod_{i\in\mathcal{F}} \left|f_i^0(y_i)\, dy_i\right| \\ &\quad\cdot \sum_{k=1}^\infty \int_{(\mathcal{Q}_k)^{m-l}} \frac{\omega\left(2^{-k}\right)}{\left|2^kQ\right|^m} \prod_{i\in\mathcal{F}^c} \left|f_i^\infty(y_i)\right| dy_i \\ &\quad\le C\sum_{k=1}^\infty \omega\left(2^{-k}\right) \frac{1}{\left|2^kQ\right|^m} \prod_{i\in\mathcal{F}} \int_{Q(x,\delta)} \left|f_i^0(y_i)\right| dy_i \\ &\quad\cdot \prod_{i\in\mathcal{F}^c} \int_{2^kQ} \left|f_i^\infty(y_i)\right| dy_i \le C|\omega|_{\mathrm{Dini}(1)} \mathscr{M}\left(\vec{f}\right)(x). \end{aligned} \tag{34}$$

Then, by Minkowski's inequality, it yields that

$$I_{\vec{\alpha}} \le C\left(\frac{1}{|Q|}\int_Q \left(\int_{\mathbb{R}^{nm}} \left(\int_0^\infty \left|K_t(z,\vec{y}) - K_t(x,\vec{y})\right|^2 \frac{dt}{t}\right)^{1/2} \prod_{j=1}^m \left|f_j^{\alpha_j}(y_j)\right| d\vec{y}\right)^\delta dz\right)^{1/\delta} \le C|\omega|_{\mathrm{Dini}(1)} \mathscr{M}\left(\vec{f}\right)(x). \tag{35}$$

Thus, we have finished the proof of Lemma 10. □

Proof of Theorem 5. Theorem 5 follows from using Lemmas 9 and 10 and repeating the same steps as in [15], here, we omit the proof. □

To prove Theorem 6, we need some preliminary lemmas.

Lemma 11 (see [18]). *If $\omega \in A_p$ and $p \ge 1$, then M maps from $L^{p,\infty}(\omega)$ to $L^{p,\infty}(\omega)$.*

Lemma 12 (see [13]). *Let $\vec{p} = (p_1, \ldots, p_m)$ with $1/p = 1/p_1 + \cdots + 1/p_m$ and $1 \le p_1, \ldots, p_m$.*

(1) *If $1 < p_j < \infty$ for all $j = 1, \ldots, m$, then $\mathscr{M}$ is bounded from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ to $L^p(\nu_{\vec{\omega}})$ if and only if $\vec{\omega} = (\omega_1, \ldots, \omega_m) \in A_{\vec{p}}$.*

(2) *If $1 \le p_j < \infty$ for all $j = 1, \ldots, m$, then $\mathscr{M}$ is bounded from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ to $L^{p,\infty}(\nu_{\vec{\omega}})$ if $\vec{\omega} = (\omega_1, \ldots, \omega_m) \in A_{\vec{p}}$.*

Lemma 13 (see [19]). *If $\omega \in A_p$ and $p > 1$, then M maps from $L^p(\omega)$ to $L^p(\omega)$.*

Lemma 14. *Let T be a multilinear square function of type $\omega(t)$ and $\omega \in Dini(1)$. For any $\eta > 0$, there is a constant $C < \infty$ depending on η such that for all $\vec{f}$ in any product of $L^{q_j}(\mathbb{R}^n)$ spaces, with $1 \le q_j < \infty$, the following inequality holds for all $x \in \mathbb{R}^n$:*

$$T^*\left(\vec{f}\right)(x) \le C\left(M_\eta\left(T\left(\vec{f}\right)\right)(x) + \mathscr{M}\left(\vec{f}\right)(x)\right). \tag{36}$$

Proof. The basic idea is due to [18, 20]. Set $U_\delta = \{\vec{y} \in (Q(x,\delta))^m : \sum_{i=1}^m |x - y_i|^2 > \delta^2\}$. For a fixed point x and a cube $Q(x,\delta)$ centered at x with radius δ, it is clear that

$$\begin{aligned}
&\left|T^*\left(\vec{f}\right)(x)\right| \\
&\quad \le \sup_{\delta>0}\left(\int_0^\infty \left|\int_{U_\delta} K_t(x,\vec{y})\prod_{i=1}^m f_i(y_i)\,dy_i\right|^2 \frac{dt}{t}\right)^{1/2} \\
&\quad + \sup_{\delta>0}\left(\int_0^\infty \left|\int_{((Q(x,\delta))^m)^c} K_t(x,\vec{y})\prod_{i=1}^m f_i(y_i)\,dy_i\right|^2 \frac{dt}{t}\right)^{1/2}.
\end{aligned} \tag{37}$$

By using the size condition (7) and Minkowski's inequality, we get

$$\begin{aligned}
&\left(\int_0^\infty \left|\int_{U_\delta} K_t(x,\vec{y})\prod_{i=1}^m f_i(y_i)\,dy_i\right|^2 \frac{dt}{t}\right)^{1/2} \\
&\quad \le C\left|\int_{U_\delta}\left(\int_0^\infty |K_t(z,\vec{y})|^2 \frac{dt}{t}\right)^2 \prod_{i=1}^m f_i(y_i)\,dy_i\right| \\
&\quad \le C\left|\int_{U_\delta} \frac{A}{\left(\sum_{i=1}^m |y_i - x|\right)^{mn}} \prod_{i=1}^m f_i(y_i)\,dy_i\right| \\
&\quad \le C\prod_{i=1}^m \frac{1}{\delta^n}\int_{Q(x,\delta)} f_i(y_i)\,dy_i \le C\mathscr{M}\left(\vec{f}\right)(x).
\end{aligned} \tag{38}$$

We are ready to estimate the second term. Set $\vec{f}^0 = (f_1\chi_Q, \ldots, f_m\chi_Q)$. For any $z \in Q(x,\delta/2)$, we introduce an operator $\widetilde{T}_\delta$:

$$\begin{aligned}
&\widetilde{T}_\delta\left(\vec{f}\right)(z) \\
&\quad = \left(\int_0^\infty \left|\int_{((Q(x,\delta))^m)^c} K_t(z,\vec{y})\prod_{i=1}^m f_i(y_i)\,dy_i\right|^2 \frac{dt}{t}\right)^{1/2} \\
&\quad \le T\left(\vec{f}\right)(z) + T\left(\vec{f}^0\right)(z).
\end{aligned} \tag{39}$$

Let $R_{\mathscr{I}}$ be the sets in $(\mathbb{R}^n)^m$, where $\mathscr{I} := \{i_1, \ldots, i_l\} \subseteq \{1, \ldots, m\}$, such that for $\vec{y} = (y_1, \ldots, y_m) \in R_{\mathscr{I}}$ we have $i \in \mathscr{I}$ if and only if $|x - y_i| \le \delta$. Using the smooth condition (8), we obtain that

$$\begin{aligned}
&\left|\widetilde{T}_\delta\left(\vec{f}\right)(x) - \widetilde{T}_\delta\left(\vec{f}\right)(z)\right| \\
&\quad \le C\left(\int_0^\infty \left(\int_{((Q(x,\delta))^m)^c} |K_t(z,\vec{y}) - K_t(x,\vec{y})|\right.\right. \\
&\quad \left.\left.\cdot \prod_{i=1}^m |f_i(y_i)|\,dy_i\right)^2 \frac{dt}{t}\right)^{1/2} \\
&\quad \le C\int_{((Q(x,\delta))^m)^c}\left(\int_0^\infty |K_t(z,\vec{y}) - K_t(x,\vec{y})|^2 \frac{dt}{t}\right)^{1/2} \\
&\quad \cdot \prod_{i=1}^m |f_i(y_i)|\,dy_i \le C\sum_{\vec{y}\in R_{\mathscr{I}}}\int_{Q(x,\delta)^l}\prod_{i\in\mathscr{I}}\left|f_i^0(y_i)\right|dy_i \\
&\quad \cdot \sum_{k=1}^\infty \int_{(\mathscr{Q}_k)^{m-l}} \frac{A}{\left(\sum_{i\in\mathscr{I}^c}|z - y_i|\right)^{mn}}\omega\left(\frac{|z-x|}{\sum_{i\in\mathscr{I}^c}|z-y_i|}\right) \\
&\quad \cdot \prod_{i\in\mathscr{I}^c}\left|f_i^\infty(y_i)\right|dy_i \le C\sum_{\vec{y}\in R_{\mathscr{I}}}\int_{Q(x,\delta)^l}\prod_{i\in\mathscr{I}}\left|f_i^0(y_i)\right|dy_i \\
&\quad \cdot \sum_{k=1}^\infty \int_{(\mathscr{Q}_k)^{m-l}} \frac{\omega\left(2^{-k}\right)}{\left|2^kQ\right|^m}\prod_{i\in\mathscr{I}^c}\left|f_i^\infty(y_i)\right|dy_i \\
&\quad \le C\sum_{\vec{y}\in R_{\mathscr{I}}}\sum_{k=1}^\infty \omega\left(2^{-k}\right)\frac{1}{\left|2^kQ\right|^m}\prod_{i\in\mathscr{I}}\int_{Q(x,\delta)}\left|f_i^0(y_i)\right|dy_i \\
&\quad \cdot \prod_{i\in\mathscr{I}^c}\int_{2^kQ}\left|f_i^\infty(y_i)\right|dy_i \le C|\omega|_{\mathrm{Dini}(1)}\mathscr{M}\left(\vec{f}\right)(x),
\end{aligned} \tag{40}$$

where $\mathscr{Q}_k = (2^kQ) \setminus (2^{k-1}Q)$, for $k = 1, 2, \ldots, \infty$.

Thus, we obtain

$$\begin{aligned}
&\left(\int_0^\infty \left|\int_{((Q(x,\delta))^m)^c} K(x,\vec{y})\prod_{i=1}^m f_i(y_i)\,dy_i\right|^2 \frac{dt}{t}\right)^{1/2} \\
&\quad \le C\mathscr{M}\left(\vec{f}\right)(x) + \left|T\left(\vec{f}\right)(z)\right| + \left|T\left(\vec{f}^0\right)(z)\right|.
\end{aligned} \tag{41}$$

Raising the above inequality to the power η, integrating over $z \in Q(x,\delta/2)$, and dividing by $|Q|$, we conclude that

$$\begin{aligned}
&\left(\int_0^\infty \left|\int_{((Q(x,\delta))^m)^c} K(x,\vec{y})\prod_{i=1}^m f_i(y_i)\,dy_i\right|^2 \frac{dt}{t}\right)^{\eta/2} \\
&\quad \le C\left(\mathscr{M}\left(\vec{f}\right)(x)\right)^\eta + M\left(\left|T\left(\vec{f}\right)\right|^\eta\right)(x) \\
&\quad + \frac{1}{|Q|}\int_Q \left|T\left(\vec{f}^0\right)(z)\right|^\eta dz.
\end{aligned} \tag{42}$$

Next we estimate the last term in (42). By Theorem 4, we know that T is bounded from $L^1 \times \cdots \times L^1$ to $L^{1/m,\infty}$; then we can deduce that

$$\begin{aligned}\int_Q \left|T\left(\vec{f}^0\right)(z)\right|^\eta dz &= m\eta \int_0^\infty \lambda^{m\eta-1} \left|\left\{z \in Q : \left|T\left(\vec{f}^0\right)(z)\right|^{1/m} > \lambda\right\}\right| d\lambda \\ &\le m\eta \int_0^\infty \lambda^{m\eta-1} \cdot \min\left\{|Q|, \frac{c}{\lambda}\left(\prod_{j=1}^m \left\|f_j \chi_{Q(x,\delta)}\right\|_{L^1}\right)^{1/m}\right\} d\lambda.\end{aligned} \tag{43}$$

Letting $R = c(\prod_{j=1}^m \|f_j \chi_{Q(x,\delta)}\|_{L^1})^{1/m}$, we obtain that

$$\begin{aligned}\frac{1}{|Q|}\int_Q \left|T\left(\vec{f}^0\right)(z)\right|^\eta dz &\le \frac{C}{|Q|}\left(\int_0^{R/|Q|} \lambda^{m\eta-1}|Q|\, d\lambda + \int_{R/|Q|}^\infty \lambda^{m\eta-1} R\, d\lambda\right) \\ &\le CR^{m\eta}|Q|^{1-m\eta} \le C\left(\prod_{j=1}^m \frac{1}{|Q|}\int_Q \left|f_j(y_j)\right| dy_j\right)^\eta \\ &\le C\left(\mathscr{M}\left(\vec{f}\right)(x)\right)^\eta,\end{aligned} \tag{44}$$

where we have used the fact $m\eta < 1$ (it suffices to prove the lemma for η arbitrarily small). Finally, if we insert estimate (44) into (42) and raise to the power $1/\eta$, we obtain the desired estimate. This finishes the proof of Lemma 14. □

Proof of Theorem 6. Theorem 6 follows by using Lemmas 13 and 14. Using the pointwise estimate for T^* in Lemma 14, we obtain that

$$\begin{aligned}\left\|T^*\left(\vec{f}\right)\right\|_{L^p(\nu_{\vec{\omega}})} &\le C\left(\left\|M_\eta\left(T\left(\vec{f}\right)\right)\right\|_{L^p(\nu_{\vec{\omega}})} + \left\|\mathscr{M}\left(\vec{f}\right)\right\|_{L^p(\nu_{\vec{\omega}})}\right).\end{aligned} \tag{45}$$

Notice that $\nu_{\vec{\omega}} \in A_{mp}$ for all $\vec{\omega} \in A_{\vec{p}}$ (see [13, Theorem 3.6]). By Lemma 13, we have

$$\left\|M_\eta\left(T\left(\vec{f}\right)\right)\right\|_{L^p(\nu_{\vec{\omega}})} \le C\left\|T\left(\vec{f}\right)\right\|_{L^p(\nu_{\vec{\omega}})}. \tag{46}$$

Thus, we obtain the desired estimates by applying Theorem 5 and Lemma 12:

$$\left\|T^*\left(\vec{f}\right)\right\|_{L^p(\nu_{\vec{\omega}})} \le C\prod_{j=1}^m \left\|f_j\right\|_{L^{p_j}(\omega_j)}. \tag{47}$$

If we use Lemma 11, by using the same arguments, we can get the weak type estimates; we omit its proof here. □

3. Weighted Boundedness on Morrey Type Spaces

The classical Morrey space was first introduced by Morrey in [21] to study the local behavior of solutions to second order elliptic partial differential equations. Later, Komori and Shirai [22] introduced the weighted Morrey space $\mathscr{L}^{p,k}(\omega)$ for $1 \le p < \infty$ and investigated the boundedness of classical operators, including Hardy-Littlewood maximal operator, Calderón-Zygmund operator, and fractional integral operator. In order to deal with the multilinear case $m \ge 2$, Wang and Yi [23] extended the range $1 \le p < \infty$ to $0 < p < \infty$.

Motivated by the works on multilinear Calderón-Zygmund operators and multilinear square functions, as demonstrated in [4, 12, 15, 24–26], we are going to study the boundedness of multilinear square function of type $\omega(t)$ on weighted Morrey type spaces.

Let $0 < p < \infty$ and $0 < k < 1$ and let ω be a weighted function on $\mathbb{R}^n$. Then, the weighted Morrey space $\mathscr{L}^{p,k}$ is defined by

$$\mathscr{L}^{p,k} = \left\{f \in L^p_{\mathrm{loc}}(\mu) : \|f\|_{\mathscr{L}^{p,k}} < \infty\right\}, \tag{48}$$

where

$$\|f\|_{\mathscr{L}^{p,k}} = \sup_Q \left(\frac{1}{\omega(Q)^k}\int_Q |f(x)|^p \omega(x)\, dx\right)^{1/p}. \tag{49}$$

Furthermore, the weighted weak Morrey space $W\mathscr{L}^{p,k}$ is defined by

$$W\mathscr{L}^{p,k} = \left\{f \text{ measurable} : \|f\|_{W\mathscr{L}^{p,k}} < \infty\right\}, \tag{50}$$

where

$$\begin{aligned}&\|f\|_{W\mathscr{L}^{p,k}} \\ &\quad = \sup_Q \sup_{\lambda>0} \frac{1}{\omega(Q)^{k/p}}\lambda \cdot \left(\left\{x \in Q : |f(x)| > \lambda\right\}\right)^{1/p}.\end{aligned} \tag{51}$$

The main results in this section are the following Theorem.

Theorem 15. *Let T be a multilinear square function of type $\omega(t)$ and $\omega \in Dini(1)$. Suppose that $\vec{\omega} \in A_{\vec{p}}$ and $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$ with $1/p = 1/p_1 + \cdots + 1/p_m$. For $1/m < p < \infty$ and $0 < k < 1$, the following two weighted inequalities hold:*

(i) *If $1 < p_1, \ldots, p_m < \infty$, then*

$$\left\|T^*\left(\vec{f}\right)\right\|_{\mathscr{L}^{p,k}(\nu_{\vec{\omega}})} \le C\prod_{j=1}^m \left\|f_j\right\|_{\mathscr{L}^{p_j,k}(\omega_j)}. \tag{52}$$

(ii) *If $1 \le p_1, \ldots, p_m < \infty$ and $\min\{p_1, \ldots, p_m\} = 1$, then*

$$\left\|T^*\left(\vec{f}\right)\right\|_{W\mathscr{L}^{p,k}(\nu_{\vec{\omega}})} \le C\prod_{j=1}^m \left\|f_j\right\|_{\mathscr{L}^{p_j,k}(\omega_j)}. \tag{53}$$

Remark 16. Theorem 15 also holds with T^* replaced by T.

In order to prove Theorem 15, we will use the following lemmas.

Lemma 17 (see [23]). *Let $m \geq 2$, $p_1, \ldots, p_m \in [1, \infty)$, and $p \in (0, \infty)$ with $1/p = 1/p_1 + \cdots + 1/p_m$. Assume that $\omega_1, \ldots, \omega_m \in A_\infty$ and set $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$. Then, for any ball B, there exists a constant $C > 0$ such that*

$$\prod_{i=1}^m \left(\int_B \omega_i(x)\, dx \right)^{p/p_i} \leq C \int_B \nu_{\vec{\omega}}(x)\, dx. \tag{54}$$

Lemma 18 (see [27]). *Let $\omega \in A_p$ with $1 \leq p < \infty$. Then, for any ball B, there exists an absolute constant $C > 0$ such that*

$$\omega(2B) \leq C\omega(B). \tag{55}$$

Lemma 19 (see [27]). *Let $\omega \in A_\infty$. Then, for any ball B and all measurable subsets E of B, there exists $\delta > 0$ such that*

$$\frac{\omega(E)}{\omega(B)} \leq C \left(\frac{|E|}{|B|} \right)^\delta. \tag{56}$$

Now we are in the position to prove Theorem 15.

Proof. First, let us prove (i). For a fixed point $x_0 \in \mathbb{R}^n$ and a ball $B \ni x_0$, we split each f_i as $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i \chi_{2B}$ for $i = 1, \ldots, m$. Write

$$\prod_{i=1}^m f_i(y_i) = \prod_{i=1}^m f_i^0(y_i) + \sum_{\vec{\alpha}, \vec{\alpha} \neq \vec{0}} f_1^{\alpha_1}(y_1) \cdots f_m^{\alpha_m}(y_m), \tag{57}$$

where $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m)$, $\alpha_i = 0$ or ∞ for $i = 1, \ldots, m$.

Then, we have

$$\begin{aligned}
&\left(\frac{1}{\nu_{\vec{\omega}}(B)^{k/p}} \int_B \left| T^*(f_1, \ldots, f_m)(x) \right|^p \nu_{\vec{\omega}}(x)\, dx \right)^{1/p} \\
&\quad \leq C \frac{1}{\nu_{\vec{\omega}}(B)^{k/p}} \left(\int_B \left| T^*\left(f_1^0, \ldots, f_m^0\right)(x) \right|^p \right. \\
&\quad \left. \cdot \nu_{\vec{\omega}}(x)\, dx \right)^{1/p} \\
&\quad + \sum_{\vec{\alpha}, \vec{\alpha} \neq \vec{0}} \frac{1}{\nu_{\vec{\omega}}(B)^{k/p}} \left(\int_B \left| T^*\left(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m}\right)(x) \right|^p \right. \\
&\quad \left. \cdot \nu_{\vec{\omega}}(x)\, dx \right)^{1/p} = C \left(I_{\vec{0}} + \sum_{\vec{\alpha}, \vec{\alpha} \neq \vec{0}} I_{\vec{\alpha}} \right).
\end{aligned} \tag{58}$$

It was shown in [13, Theorem 3.6] that $\nu_{\vec{\omega}} \in A_{mp}$. This fact together with Theorem 6(i) and Lemmas 18 and 17 yields that

$$\begin{aligned}
I_{\vec{0}} &< \frac{1}{\nu_{\vec{\omega}}(B)^{k/p}} \prod_{i=1}^m \left(\int_B |f_i(x)|^{p_i} \omega_i(x)\, dx \right)^{1/p_i} \\
&\leq C \prod_{i=1}^m \|f_i\|_{\mathcal{L}^{p_i,k}(\omega_i)} \frac{\prod_{i=1}^m \omega_i(2B)^{k/p_i}}{\nu_{\vec{\omega}}(B)^{k/p}} \\
&\leq C \prod_{i=1}^m \|f_i\|_{\mathcal{L}^{p_i,k}(\omega_i)} \frac{\nu_{\vec{\omega}}(2B)^{k/p}}{\nu_{\vec{\omega}}(B)^{k/p}} \leq C \prod_{i=1}^m \|f_i\|_{\mathcal{L}^{p_i,k}(\omega_i)}.
\end{aligned} \tag{59}$$

We now estimate $I_{\vec{\alpha}}$ with $\vec{\alpha} = (\infty, \ldots, \infty)$. For any $x \in B$, by Minkowski's inequality, the size condition (7), and Lemma 17, we have

$$\begin{aligned}
\left| T^*\left(f_1^\infty, \ldots, f_m^\infty\right)(x) \right| &= \sup_{\delta > 0} \left(\int_0^\infty \left| \int_{\sum_{i=1}^m |x - y_i|^2 > \delta^2} K_{t,\delta}(x, y_1, \ldots, y_m) \prod_{i=1}^m f_i^\infty(y_i)\, d\vec{y} \right|^2 \frac{dt}{t} \right)^{1/2} \\
&\leq C \int_{\mathbb{R}^{nm}} \left(\int_0^\infty \left| K_t(x, \vec{y}) \right|^2 \frac{dt}{t} \right)^{1/2} \prod_{i=1}^m f_i^\infty(y_i)\, d\vec{y} \\
&\leq C \sum_{j=1}^\infty \int_{(2^{j+1}B \setminus 2^j B)^m} \frac{1}{\left(\sum_{i=1}^m |x - y_i|\right)^{mn}} \prod_{i=1}^m f_i(y_i)\, d\vec{y} \leq C \sum_{j=1}^\infty \prod_{i=1}^m \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f_i(y_i)|\, dy_i \\
&\leq C \sum_{j=1}^\infty \prod_{i=1}^m \frac{1}{|2^{j+1}B|} \left(\int_{2^{j+1}B} |f_i(y_i)|^{p_i} \omega_i(y_i)\, dy_i \right)^{1/p_i} \left(\int_{2^{j+1}B} \omega_i(y_i)^{1-p_i'}\, dy_i \right)^{1/p_i'} \\
&\leq C \sum_{j=1}^\infty \frac{1}{|2^{j+1}B|^m} \frac{\left|2^{j+1}B\right|^{1/p + \sum_{i=1}^m (1 - 1/p_i)}}{\nu_{\vec{\omega}}\left(2^{j+1}B\right)^{1/p}} \prod_{i=1}^m \left(\|f_i\|_{\mathcal{L}^{p_i,k}(\omega_i)} \omega_i\left(2^{j+1}B\right)^{k/p_i} \right) \\
&\leq C \prod_{i=1}^m \|f_i\|_{\mathcal{L}^{p_i,k}(\omega_i)} \sum_{j=1}^\infty \left(\frac{\prod_{i=1}^m \omega_i\left(2^{j+1}B\right)^{k/p_i}}{\nu_{\vec{\omega}}\left(2^{j+1}B\right)^{1/p}} \right) \leq C \prod_{i=1}^m \|f_i\|_{\mathcal{L}^{p_i,k}(\omega_i)} \sum_{j=1}^\infty \nu_{\vec{\omega}}\left(2^{j+1}B\right)^{(k-1)/p}.
\end{aligned} \tag{60}$$

Then, Lemma 19 implies that

$$\begin{aligned} I_{\vec{\alpha}} &\le C\nu_{\vec{\omega}}(B)^{(1-k)/p}\left|T^*\left(f_1^{\infty},\dots,f_m^{\infty}\right)(x)\right| \\ &\le C\prod_{i=1}^{m}\|f_i\|_{\mathscr{L}^{p_i,k}(w_i)}\sum_{j=1}^{\infty}\frac{\nu_{\vec{\omega}}(B)^{(1-k)/p}}{\nu_{\vec{\omega}}\left(2^{j+1}B\right)^{(1-k)/p}} \\ &\le C\prod_{i=1}^{m}\|f_i\|_{\mathscr{L}^{p_i,k}(w_i)}\sum_{j=1}^{\infty}\left(\frac{|B|}{\left|2^{j+1}B\right|}\right)^{\delta(1-k)/p} \\ &\le C\prod_{i=1}^{m}\|f_i\|_{\mathscr{L}^{p_i,k}(w_i)}. \end{aligned} \tag{61}$$

It remains for us to consider $I_{\vec{\alpha}}$ with $\vec{\alpha}=(\alpha_1,\dots,\alpha_m)$, $\alpha_i=0$ or ∞ for $i=1,\dots,m$. We may assume that $\alpha_1=\cdots=\alpha_l=\infty$ and $\alpha_{l+1}=\cdots=\alpha_m=0$. Minkowski's inequality and the size condition (7) imply that

$$\begin{aligned} &\left|T^*\left(f_1^{\infty},\dots,f_l^{\infty},f_{l+1}^{0},\dots,f_m^{0}\right)(x)\right| \\ &\quad\le C\int_{\mathbb{R}^{nm}}\left(\int_0^{\infty}\left|K_t(x,\vec{y})\right|^2\frac{dt}{t}\right)^{1/2} \\ &\quad\cdot\prod_{i=1}^{l}f_i^{\infty}(y_i)\prod_{i=l+1}^{m}f_i^{0}(y_i)\,d\vec{y} \\ &\quad\le C\int_{(\mathbb{R}^n\setminus 2B)^l}\int_{(2B)^{m-l}}\frac{\prod_{i=1}^{l}f_i^{\infty}(y_i)\prod_{i=l+1}^{m}f_i^{0}(y_i)}{\left(\sum_{i=1}^{m}|x-y_i|\right)^{mn}}d\vec{y} \\ &\quad\le C\prod_{i=l+1}^{m}\int_{2B}|f_i(y_i)|\,dy_i \\ &\quad\cdot\sum_{j=1}^{\infty}\frac{1}{\left|2^{j+1}B\right|^m}\int_{(2^{j+1}B\setminus 2^jB)^l}\prod_{i=1}^{l}f_i(y_i)\,dy_1\cdots dy_l \\ &\quad\le C\sum_{j=1}^{\infty}\prod_{i=1}^{m}\frac{1}{\left|2^{j+1}B\right|}\int_{2^{j+1}B}|f_i(y_i)|\,dy_i \\ &\quad\le C\prod_{i=1}^{m}\|f_i\|_{\mathscr{L}^{p_i,k}(w_i)}\sum_{j=1}^{\infty}\nu_{\vec{\omega}}\left(2^{j+1}B\right)^{(k-1)/p}. \end{aligned} \tag{62}$$

Together with Lemma 19, we obtain

$$\begin{aligned} &I_{\vec{\alpha}} \\ &\quad\le C\nu_{\vec{\omega}}(B)^{(1-k)/p}\left|T^*\left(f_1^{\infty},\dots,f_l^{\infty},f_{l+1}^{0},\dots,f_m^{0}\right)(x)\right| \\ &\quad\le C\prod_{i=1}^{m}\|f_i\|_{\mathscr{L}^{p_i,k}(w_i)}\sum_{j=1}^{\infty}\frac{\nu_{\vec{\omega}}(B)^{(1-k)/p}}{\nu_{\vec{\omega}}\left(2^{j+1}B\right)^{(1-k)/p}} \\ &\quad\le C\prod_{i=1}^{m}\|f_i\|_{\mathscr{L}^{p_i,k}(w_i)}\sum_{j=1}^{\infty}\left(\frac{|B|}{\left|2^{j+1}B\right|}\right)^{\delta(1-k)/p} \\ &\quad\le C\prod_{i=1}^{m}\|f_i\|_{\mathscr{L}^{p_i,k}(w_i)}. \end{aligned} \tag{63}$$

Combining the above estimates and then taking the supremum over all balls $B\in\mathbb{R}^n$, we complete the proof of Theorem 15(i).

We are now in a position to demonstrate (ii). For any $\lambda>0$, we have

$$\begin{aligned} &\nu_{\vec{\omega}}\left(\left\{x\in B:\left|T^*\left(f_1,\dots,f_m\right)(x)\right|>\lambda\right\}\right)^{1/p}\le C\nu_{\vec{\omega}}\left(\left\{x\right.\right. \\ &\quad\left.\left.\in B:\left|T^*\left(f_1^0,\dots,f_m^0\right)(x)\right|>\lambda\right\}\right)^{1/p} \\ &\quad+C\sum_{\vec{\alpha},\vec{\alpha}\ne\vec{0}}\nu_{\vec{\omega}}\left(\left\{x\in B:\left|T^*\left(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\right)(x)\right|\right.\right. \\ &\quad\left.\left.>\lambda\right\}\right)^{1/p}=C\left(P_{\vec{0}}+\sum_{\vec{\alpha},\vec{\alpha}\ne\vec{0}}P_{\vec{\alpha}}\right). \end{aligned} \tag{64}$$

Using Theorem 6(ii) and Lemmas 18 and 17, we obtain

$$\begin{aligned} P_{\vec{0}} &= \nu_{\vec{\omega}}\left(\left\{x\in B:\left|T^*\left(f_1^0,\dots,f_m^0\right)(x)\right|>\lambda\right\}\right)^{1/p} \\ &\le \frac{C\prod_{i=1}^{m}\left(\int_{2B}|f_i(x)|^{p_i}\omega_i(x)\,dx\right)^{1/p_i}}{\lambda} \\ &\le \frac{C\prod_{i=1}^{m}\omega_i(2B)^{k/p_i}\|f_i\|_{\mathscr{L}^{p_i,k}(w_i)}}{\lambda} \\ &\le \frac{C\nu_{\vec{\omega}}(2B)^{k/p}\prod_{i=1}^{m}\|f_i\|_{\mathscr{L}^{p_i,k}(w_i)}}{\lambda} \\ &\le \frac{C\nu_{\vec{\omega}}(B)^{k/p}\prod_{i=1}^{m}\|f_i\|_{\mathscr{L}^{p_i,k}(w_i)}}{\lambda}. \end{aligned} \tag{65}$$

We assume that $p_1=\cdots=p_l=1$ and $p_{l+1},\dots,p_m>1$. If $\vec{\alpha}\ne\vec{0}$, recall that in (60) and (62) we have proved the following fact:

$$\begin{aligned} &\left|T^*\left(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\right)(x)\right| \\ &\quad\le C\sum_{j=1}^{\infty}\prod_{i=1}^{m}\frac{1}{\left|2^{j+1}B\right|}\int_{2^{j+1}B}|f_i(y_i)|\,dy_i. \end{aligned} \tag{66}$$

Hence, we may obtain

$$\begin{aligned} &\left|T^*\left(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\right)(x)\right| \\ &\quad\le C\sum_{j=1}^{\infty}\prod_{i=1}^{l}\frac{1}{\left|2^{j+1}B\right|}\int_{2^{j+1}B}|f_i(y_i)|\,dy_i \\ &\quad\cdot\prod_{i=l+1}^{m}\frac{1}{\left|2^{j+1}B\right|}\int_{2^{j+1}B}|f_i(y_i)|\,dy_i \\ &\quad\le C\sum_{j=1}^{\infty}\prod_{i=1}^{l}\frac{1}{\left|2^{j+1}B\right|}\int_{2^{j+1}B}|f_i(y_i)|\,\omega_i(y_i)\,dy_i \\ &\quad\cdot\left(\inf_{y_i\in 2^{j+1}B}\omega_i(y_i)\right)^{-1} \\ &\quad\cdot\prod_{i=l+1}^{m}\frac{1}{\left|2^{j+1}B\right|}\left(\int_{2^{j+1}B}|f_i(y_i)|^{p_i}\omega_i(y_i)\,dy_i\right)^{1/p_i} \end{aligned}$$

$$\cdot \left(\int_{2^{j+1}B} \omega_i (y_i)^{1-p_i'} dy_i \right)^{1/p_i'} \leq C \prod_{i=1}^{m} \|f_i\|_{\mathscr{L}^{p_i,k}(w_i)}$$
$$\cdot \sum_{j=1}^{\infty} \nu_{\vec{\omega}} \left(2^{j+1}B\right)^{(k-1)/p} \leq C \prod_{i=1}^{m} \|f_i\|_{\mathscr{L}^{p_i,k}(w_i)}$$
$$\cdot \frac{1}{\nu_{\vec{\omega}}(B)^{(1-k)/p}} \cdot \sum_{j=1}^{\infty} \frac{\nu_{\vec{\omega}}(B)^{(1-k)/p}}{\nu_{\vec{\omega}}\left(2^{j+1}B\right)^{(k-1)/p}}$$
$$\leq C \prod_{i=1}^{m} \|f_i\|_{\mathscr{L}^{p_i,k}(w_i)} \cdot \frac{1}{\nu_{\vec{\omega}}(B)^{(1-k)/p}}$$
$$\cdot \sum_{j=1}^{\infty} \left(\frac{|B|}{|2^{j+1}B|} \right)^{\delta(1-k)/p} \leq C \prod_{i=1}^{m} \|f_i\|_{\mathscr{L}^{p_i,k}(w_i)}$$
$$\cdot \frac{1}{\nu_{\vec{\omega}}(B)^{(1-k)/p}}. \tag{67}$$

To prove Theorem 15(ii), we may assume that

$$\nu_{\vec{\omega}} \left(\left\{ x \in B : \left| T^* \left(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m} \right)(x) \right| > \lambda \right\} \right) > 0. \tag{68}$$

Otherwise, there is nothing needing to be proved.

Then, by the above estimates, we obtain that

$$\nu_{\vec{\omega}} \left(\left\{ x \in B : \left| T^* \left(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m} \right)(x) \right| > \lambda \right\} \right) \leq \frac{C \nu_{\vec{\omega}}(B) \prod_{i=1}^{m} \|f_i\|^p_{\mathscr{L}^{p_i,k}(w_i)}}{\lambda^p \cdot \nu_{\vec{\omega}}(B)^{1-k}}. \tag{69}$$

Therefore, we get

$$\frac{1}{\nu_{\vec{\omega}}(B)^{k/p}} \cdot \lambda \cdot \nu_{\vec{\omega}} \left(\left\{ x \in B : \left| T^* \left(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m} \right)(x) \right| > \lambda \right\} \right)^{1/p} \leq C \prod_{i=1}^{m} \|f_i\|_{\mathscr{L}^{p_i,k}(w_i)}. \tag{70}$$

This completes the proof of Theorem 15. □

Competing Interests

The authors declare that they have no competing interests.

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