Research Article

Optimal Bounds for Neuman Mean Using Arithmetic and Centroidal Means

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We present the best possible parameters $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ and $\alpha_3, \beta_3 \in (1/2, 1)$ such that the double inequalities $\alpha_1 A(a, b) + (1 - \alpha_1)C(a, b) < N_{QA}(a, b) < \beta_1 A(a, b) + (1 - \beta_1)C(a, b)$, $A^{\alpha_2}(a, b)C^{1-\alpha_2}(a, b) < N_{QA}(a, b) < A^{\beta_2}(a, b)C^{1-\beta_2}(a, b)$, and $C[\alpha_3 a + (1 - \alpha_3)b, \alpha_3 b + (1 - \alpha_3)a] < N_{QA}(a, b) < C[\beta_3 a + (1 - \beta_3)b, \beta_3 b + (1 - \beta_3)a]$ hold for all $a, b > 0$ with $a \neq b$ and give several sharp inequalities involving the hyperbolic and inverse hyperbolic functions. Here, $N(a, b)$, $A(a, b)$, $Q(a, b)$, and $C(a, b)$ are, respectively, the Neuman, arithmetic, quadratic, and centroidal means of a and b , and $N_{QA}(a, b) = N[Q(a, b), A(a, b)]$.

1. Introduction

Let $a, b > 0$ with $a \neq b$. Then, the Schwab-Borchardt mean $SB(a, b)$ [1–7] of a and b is given by

$$SB(a, b) = \begin{cases} \frac{\sqrt{b^2 - a^2}}{\arccos(a/b)}, & a < b, \\ \frac{\sqrt{a^2 - b^2}}{\cosh^{-1}(a/b)}, & a > b, \end{cases} \quad (1)$$

where $\cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1})$ is the inverse hyperbolic cosine functions.

It is well known that the Schwab-Borchardt mean $SB(a, b)$ is strictly increasing in both a and b , nonsymmetric, and homogeneous of degree 1 with respect to a and b . Many symmetric bivariate means are special cases of the Schwab-Borchardt mean. For example, $P(a, b) = (a - b)/[2\arcsin((a - b)/(a + b))] = SB[G(a, b), A(a, b)]$ is the first Seiffert mean, $T(a, b) = (a - b)/[2\arctan((a - b)/(a + b))] = SB[A(a, b), Q(a, b)]$ is the second Seiffert mean, $M(a, b) = (a - b)/[2\sinh^{-1}((a - b)/(a + b))] = SB[Q(a, b), A(a, b)]$ is the Neuman-Sándor mean, and $L(a, b) = (a - b)/[2\tanh^{-1}((a - b)/(a + b))] = SB[A(a, b), G(a, b)]$ is the logarithmic mean, where $\sinh^{-1}(x) = \log(x + \sqrt{1 + x^2})$ is the inverse hyperbolic sine function, $\tanh^{-1}(x) = \log((1 + x)/(1 - x))/2$ is the

inverse hyperbolic tangent function, $G(a, b) = \sqrt{ab}$, $A(a, b) = (a + b)/2$, and $Q(a, b) = \sqrt{(a^2 + b^2)/2}$ are, respectively, the geometric, arithmetic, and quadratic means of a and b , and the inequalities

$$H(a, b) < G(a, b) < L(a, b) < P(a, b) < A(a, b) < M(a, b) < T(a, b) < Q(a, b) \quad (2)$$

hold for all $a, b > 0$ with $a \neq b$ (see [1]), where $H(a, b) = 2ab/(a + b)$ is the harmonic mean of a and b .

Recently, the Schwab-Borchardt mean and its generated means have been the subject intensive research. In particular, many remarkable inequalities for these means can be found in the literature [1–5, 8–16].

Liu and Meng [17] proved that the double inequality

$$\alpha C(a, b) + (1 - \alpha)H(a, b) < T(a, b) < \beta C(a, b) + (1 - \beta)H(a, b) \quad (3)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq 3/\pi$ and $\beta \geq 1$, where

$$C(a, b) = \frac{2(a^2 + ab + b^2)}{3(a + b)} \quad (4)$$

is the centroidal mean of a and b .

Very recently, Neuman [18] introduced the Neuman mean

$$N(a, b) = \frac{1}{2} \left[a + \frac{b^2}{SB(a, b)} \right], \quad (5)$$

presented the explicit formula for $N_{QA}(a, b)$ as

$$\begin{aligned} N_{QA}(a, b) &\triangleq N[Q(a, b), A(a, b)] \\ &= \frac{1}{2} A(a, b) \left[\sqrt{1 + v^2} + \frac{\sinh^{-1}(v)}{v} \right], \end{aligned} \quad (6)$$

and proved that the double inequality

$$M(a, b) < N_{QA}(a, b) < T(a, b) \quad (7)$$

holds for all $a, b > 0$ with $a \neq b$, where $v = (a - b)/(a + b)$.

From (2), (3), and (7), we clearly see that the double inequality

$$A(a, b) < N_{QA}(a, b) < C(a, b) \quad (8)$$

holds for all $a, b > 0$ with $a \neq b$.

Let $J(x) = C[xa + (1 - x)b, xb + (1 - x)a]$. Then, it is not difficult to verify that the function $J(x)$ is continuous and strictly increasing on $[1/2, 1]$ for fixed $a, b > 0$ with $a \neq b$. Note that

$$J\left(\frac{1}{2}\right) = A(a, b) < N_{QA}(a, b) < C(a, b) = J(1) \quad (9)$$

for all $a, b > 0$ with $a \neq b$.

Motivated by inequalities (8) and (9), it is natural to ask what are best possible parameters $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ and $\alpha_3, \beta_3 \in (1/2, 1)$ such that the double inequalities

$$\begin{aligned} &\alpha_1 A(a, b) + (1 - \alpha_1) C(a, b) < N_{QA}(a, b) \\ &< \beta_1 A(a, b) + (1 - \beta_1) C(a, b), \\ &A^{\alpha_2}(a, b) C^{1-\alpha_2}(a, b) < N_{QA}(a, b) \\ &< A^{\beta_2}(a, b) C^{1-\beta_2}(a, b), \\ &C[\alpha_3 a + (1 - \alpha_3)b, \alpha_3 b + (1 - \alpha_3)a] < N_{QA}(a, b) \\ &< C[\beta_3 a + (1 - \beta_3)b, \beta_3 b + (1 - \beta_3)a] \end{aligned} \quad (10)$$

hold for all $a, b > 0$ with $a \neq b$? The main purpose of this paper is to answer these questions.

2. Lemmas

In order to prove our main results, we need several lemmas, which we present in this section.

Lemma 1 (see [19–21]). For $a, b \in \mathbb{R}$, let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and be differentiable on (a, b) , and let $g'(x) \neq 0$ on (a, b) . If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b) , then so are

$$\begin{aligned} &\frac{f(x) - f(a)}{g(x) - g(a)}, \\ &\frac{f(x) - f(b)}{g(x) - g(b)}. \end{aligned} \quad (11)$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2 (see [22–24]). Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ be two real power series converging on $\mathbb{R}$ with $b_n > 0$ for all $n \in \mathbb{N}$. Then, the following statements are true:

- (1) If the nonconstant sequence $\{a_n/b_n\}$ is increasing (decreasing) for all $n \in \mathbb{N}$, then the function $x \mapsto f(x)/g(x)$ is strictly increasing (decreasing) on $(0, \infty)$.
- (2) If there exists $m \geq 1$ such that the nonconstant sequence $\{a_n/b_n\}$ is increasing (decreasing) for $0 \leq n \leq m$ and decreasing (increasing) for $n \geq m$, then there exists $x_0 \in (0, \infty)$ such that the function $x \mapsto f(x)/g(x)$ is strictly increasing (decreasing) on $(0, x_0)$ and strictly decreasing (increasing) on (x_0, ∞) .

Lemma 3. The function

$$\begin{aligned} &f(x) \\ &= \frac{6x + 6x \cosh(2x) + (1/2) \sinh(4x) - 7 \sinh(2x)}{-4x + 4x \cosh(2x) + \sinh(4x) - 2 \sinh(2x)} \end{aligned} \quad (12)$$

is strictly increasing from $(0, \log(1 + \sqrt{2}))$ onto $(1/2, [3 \log(1 + \sqrt{2}) - \sqrt{2}]/[\log(1 + \sqrt{2}) + \sqrt{2}])$.

Proof. Making use of the power series formulas $\sinh(x) = \sum_{n=0}^{\infty} x^{2n+1}/(2n+1)!$ and $\cosh(x) = \sum_{n=0}^{\infty} x^{2n}/(2n)!$, we have

$$\begin{aligned} f(x) &= \frac{6x + 6x \sum_{n=0}^{\infty} (2^{2n}/(2n)!) x^{2n} + (1/2) \sum_{n=0}^{\infty} (4^{2n+1}/(2n+1)!) x^{2n+1} - 7 \sum_{n=0}^{\infty} (2^{2n+1}/(2n+1)!) x^{2n+1}}{-4x + 4x \sum_{n=0}^{\infty} (2^{2n}/(2n)!) x^{2n} + \sum_{n=0}^{\infty} (4^{2n+1}/(2n+1)!) x^{2n+1} - 2 \sum_{n=0}^{\infty} (2^{2n+1}/(2n+1)!) x^{2n+1}} \\ &= \frac{\sum_{n=0}^{\infty} (2^{2n+4} (2^{2n+1} + 3n + 1) / (2n+3)!) x^{2n}}{\sum_{n=0}^{\infty} (2^{2n+5} (2^{2n+1} + n + 1) / (2n+3)!) x^{2n}}. \end{aligned} \quad (13)$$

Let

$$\begin{aligned} c_n &= \sum_{n=0}^{\infty} \frac{2^{2n+4} (2^{2n+1} + 3n + 1)}{(2n + 3)!}, \\ d_n &= \sum_{n=0}^{\infty} \frac{2^{2n+5} (2^{2n+1} + n + 1)}{(2n + 3)!}. \end{aligned} \quad (14)$$

Then, we clearly see that

$$d_n > 0 \quad (15)$$

for all $n \geq 0$ and

$$\frac{c_{n+1}}{d_{n+1}} - \frac{c_n}{d_n} = \frac{1 + (1 - 3n) 2^{2n+1}}{(2^{2n+1} + n + 1)(2^{2n+3} + n + 2)}. \quad (16)$$

Let

$$e_n = 1 + (1 - 3n) 2^{2n+1}. \quad (17)$$

Then, simple computations lead to

$$\begin{aligned} e_0 &= 3 > 0, \\ e_n &< 1 - 2^{2n+2} < 0 \end{aligned} \quad (18)$$

for all $n \geq 1$.

It follows from (16)–(18) that the nonconstant sequence $\{c_n/d_n\}$ is increasing for $0 \leq n \leq 1$ and decreasing for $n \geq 1$. Then, from (13)–(15) and Lemma 2(2) we know that there exists $x_0 \in (0, \infty)$ such that $f(x)$ is strictly increasing on $(0, x_0)$ and strictly decreasing on (x_0, ∞) .

Differentiating $f(x)$ gives

$$\begin{aligned} f'(x) &= \frac{6 - 8 \cosh(2x) + 12x \sinh(2x) + 2 \cosh(4x)}{-4x + 4x \cosh(2x) + \sinh(4x) - 2 \sinh(2x)} \\ &\quad - \frac{[6x + 6x \cosh(2x) + (1/2) \sinh(4x) - 7 \sinh(2x)] [-4 + 8x \sinh(2x) + 4 \cosh(4x)]}{[-4x + 4x \cosh(2x) + \sinh(4x) - 2 \sinh(2x)]^2}, \\ f'[\log(1 + \sqrt{2})] &= \frac{6 - 24 + 24\sqrt{2} \log(1 + \sqrt{2}) + 34}{-4 \log(1 + \sqrt{2}) + 12 \log(1 + \sqrt{2}) + 12\sqrt{2} - 4\sqrt{2}} \\ &\quad - \frac{[6 \log(1 + \sqrt{2}) + 18 \log(1 + \sqrt{2}) + 6\sqrt{2} - 14\sqrt{2}] [-4 + 16\sqrt{2} \log(1 + \sqrt{2}) + 68]}{[-4 \log(1 + \sqrt{2}) + 12 \log(1 + \sqrt{2}) + 12\sqrt{2} - 4\sqrt{2}]^2} \\ &= \frac{10\sqrt{2} - 12 \log(1 + \sqrt{2}) - 3\sqrt{2} \log^2(1 + \sqrt{2})}{[\sqrt{2} + \log(1 + \sqrt{2})]^2}. \end{aligned} \quad (19)$$

Note that

$$\begin{aligned} 10\sqrt{2} - 12 \log(1 + \sqrt{2}) - 3\sqrt{2} \log^2(1 + \sqrt{2}) \\ = 0.2698 \dots > 0, \end{aligned} \quad (20)$$

$$\begin{aligned} f(0^+) &= \frac{c_0}{d_0} = \frac{1}{2}, \\ f[\log(1 + \sqrt{2})] &= \frac{3 \log(1 + \sqrt{2}) - \sqrt{2}}{\log(1 + \sqrt{2}) + \sqrt{2}}. \end{aligned} \quad (21)$$

It follows from (19) and (20) together with the piecewise monotonicity of $f(x)$ that $\log(1 + \sqrt{2}) < x_0$ and $f(x)$ is strictly increasing on $(0, \log(1 + \sqrt{2}))$.

Therefore, Lemma 3 follows from (21) and the monotonicity of $f(x)$ on the interval $(0, \log(1 + \sqrt{2}))$. $\square$

Lemma 4. *The function*

$$g(x) = \frac{\sinh(x) [2 - \cosh(x)] - x}{\sinh^3(x)} \quad (22)$$

is strictly increasing from $(0, \log(1 + \sqrt{2}))$ onto $(-1/3, 2 - \sqrt{2} - \log(1 + \sqrt{2}))$.

Proof. Making use of the power series formulas $\sinh(x) = \sum_{n=0}^{\infty} x^{2n+1}/(2n+1)!$ and $\cosh(x) = \sum_{n=0}^{\infty} x^{2n}/(2n)!$ together with the identity $4 \sinh^3(x) = \sinh(3x) - 3 \sinh(x)$, one has

$$\begin{aligned} g(x) &= \frac{2 \sinh(x) - (1/2) \sinh(2x) - x}{(1/4) [\sinh(3x) - 3 \sinh(x)]} = \frac{2 \sum_{n=0}^{\infty} (1/(2n+1)!) x^{2n+1} - (1/2) \sum_{n=0}^{\infty} (2^{2n+1}/(2n+1)!) x^{2n+1} - x}{(1/4) [\sum_{n=0}^{\infty} (3^{2n+1}/(2n+1)!) x^{2n+1} - 3 \sum_{n=0}^{\infty} (1/(2n+1)!) x^{2n+1}]} \\ &= \frac{\sum_{n=0}^{\infty} ((2 - 2^{2n+2})/(2n+3)!) x^{2n}}{\sum_{n=0}^{\infty} ((3^{2n+3} - 3)/(4 \times (2n+3)!)) x^{2n}}. \end{aligned} \quad (23)$$

Let

$$\begin{aligned} u_n &= \frac{2 - 2^{2n+2}}{(2n+3)!}, \\ v_n &= \frac{3^{2n+3} - 3}{4 \times (2n+3)!}. \end{aligned} \quad (24)$$

Then,

$$\frac{u_{n+1}}{v_{n+1}} - \frac{u_n}{v_n} = \frac{8 \left[(10 \times 2^{2n} - 8) 3^{2n+1} + 2^{2n+1} \right]}{(3^{2n+4} - 1)(3^{2n+2} - 1)} > 0, \quad (25)$$

$v_n > 0,$

for all $n \geq 0$. Note that

$$g(0^+) = \frac{u_0}{v_0} = -\frac{1}{3}, \quad (26)$$

$$g[\log(1 + \sqrt{2})] = 2 - \sqrt{2} - \log(1 + \sqrt{2}).$$

Therefore, Lemma 4 follows easily from (23)–(26) and Lemma 2(1). $\square$

3. Main Results

Theorem 5. *The double inequality*

$$\begin{aligned} \alpha_1 A(a, b) + (1 - \alpha_1) C(a, b) &< N_{QA}(a, b) \\ &< \beta_1 A(a, b) + (1 - \beta_1) C(a, b) \end{aligned} \quad (27)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \geq 4 - 3[\sqrt{2} + \log(1 + \sqrt{2})]/2 = 0.5566 \dots$ and $\beta_1 \leq 1/2$.

Proof. Since all the means $A(a, b)$, $N_{QA}(a, b)$, and $C(a, b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b$. Let $v = (a - b)/(a + b) \in (0, 1)$. Then, (4) leads to

$$C(a, b) = \left(1 + \frac{1}{3}v^2\right) A(a, b). \quad (28)$$

From (6) and (28), we clearly see that inequality (27) is equivalent to

$$\beta_1 < \frac{2v^2 - 3(\sqrt{1+v^2} + (\sinh^{-1}(v))/v) + 6}{2v^2} < \alpha_1. \quad (29)$$

Let $x = \sinh^{-1}(v)$. Then, $x \in (0, \log(1 + \sqrt{2}))$ and

$$\begin{aligned} &\frac{2v^2 - 3(\sqrt{1+v^2} + (\sinh^{-1}(v))/v) + 6}{2v^2} \\ &= \frac{\sinh(3x) - 3 \sinh(2x) + 9 \sinh(x) - 6x}{\sinh(3x) - 3 \sinh(x)} \\ &= 1 + \frac{3[\sinh(x)(2 - \cosh(x)) - x]}{2 \sinh^3(x)}. \end{aligned} \quad (30)$$

Therefore, Theorem 5 follows easily from (29) and (30) together with Lemma 4. $\square$

Theorem 6. *The double inequality*

$$\begin{aligned} A^{\alpha_2}(a, b) C^{1-\alpha_2}(a, b) &< N_{QA}(a, b) \\ &< A^{\beta_2}(a, b) C^{1-\beta_2}(a, b) \end{aligned} \quad (31)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \geq 1 - [\log(\sqrt{2} + \log(1 + \sqrt{2})) - \log 2]/[2 \log 2 - \log 3] = 0.5208 \dots$ and $\beta_2 \leq 1/2$.

Proof. Without loss of generality, we assume that $a > b$. Let $v = (a - b)/(a + b) \in (0, 1)$. Then, from (6) and (28) we clearly see that inequality (31) is equivalent to

$$\begin{aligned} &\beta_2 \\ &< \frac{\log(v^2 + 3) - \log[\sqrt{1+v^2} + (\sinh^{-1}(v))/v] - \log 3 + \log 2}{\log(v^2 + 3) - \log 3} \\ &< \alpha_2. \end{aligned} \quad (32)$$

Let $x = \sinh^{-1}(v)$. Then, $x \in (0, \log(1 + \sqrt{2}))$, and

$$\begin{aligned} &\frac{\log(v^2 + 3) - \log[\sqrt{1+v^2} + (\sinh^{-1}(v))/v] - \log 3 + \log 2}{\log(v^2 + 3) - \log 3} \\ &= \frac{\log[3 + \sinh^2(x)] - \log[x + \sinh(x) \cosh(x)] + \log[\sinh(x)] - \log 3 + \log 2}{\log[3 + \sinh^2(x)] - \log 3}. \end{aligned} \quad (33)$$

Let

$$G(x) = \frac{\log[3 + \sinh^2(x)] - \log[x + \sinh(x) \cosh(x)] + \log[\sinh(x)] - \log 3 + \log 2}{\log[3 + \sinh^2(x)] - \log 3}, \quad (34)$$

$$g_1(x) = \log[3 + \sinh^2(x)] - \log[x + \sinh(x) \cosh(x)] + \log[\sinh(x)] - \log 3 + \log 2, \quad (35)$$

$$g_2(x) = \log[3 + \sinh^2(x)] - \log 3. \quad (36)$$

Then, simple computations lead to

$$\begin{aligned} g_1(0^+) &= g_2(0^+) = 0, \\ \frac{g_1'(x)}{g_2'(x)} &= \frac{6x + 6x \cosh(2x) + (1/2) \sinh(4x) - 7 \sinh(2x)}{-4x + 4x \cosh(2x) + \sinh(4x) - 2 \sinh(2x)}. \end{aligned} \quad (37)$$

It follows from (34)–(37) and Lemmas 1 and 3 that the function

$$G(x) = \frac{g_1(x)}{g_2(x)} = \frac{g_1(x) - g_1(0^+)}{g_2(x) - g_2(0^+)} \quad (38)$$

is strictly increasing on the interval $(0, \log(1 + \sqrt{2}))$. Note that

$$\begin{aligned} G(0^+) &= \frac{g_1'(0^+)}{g_2'(0^+)} = \frac{c_0}{d_0} = \frac{1}{2}, \\ G[\log(1 + \sqrt{2})] &= 1 - \frac{\log[\sqrt{2} + \log(1 + \sqrt{2})] - \log 2}{2 \log 2 - \log 3}, \end{aligned} \quad (39)$$

where c_n and d_n are defined by (14).

Therefore, Theorem 6 follows easily from (32)–(34) and (39) together with the monotonicity of $G(x)$ on the interval $(0, \log(1 + \sqrt{2}))$. $\square$

Theorem 7. Let $\alpha_3, \beta_3 \in (1/2, 1)$. Then, the double inequality

$$\begin{aligned} C[\alpha_3 a + (1 - \alpha_3)b, \alpha_3 b + (1 - \alpha_3)a] &< N_{QA}(a, b) \\ &< C[\beta_3 a + (1 - \beta_3)b, \beta_3 b + (1 - \beta_3)a] \end{aligned} \quad (40)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq 1/2 + \sqrt{6[\sqrt{2} + \log(1 + \sqrt{2})] - 12}/4 = 0.8329 \dots$ and $\beta_3 \geq 1/2 + \sqrt{2}/4 = 0.8535 \dots$.

Proof. Without loss of generality, we assume that $a > b$. Let $v = (a - b)/(a + b) \in (0, 1)$ and $p \in (1/2, 1)$. Then, (4) and (6) lead to

$$\begin{aligned} C[pa + (1 - p)b, pb + (1 - p)a] - N_{QA}(a, b) &= \frac{1}{3} \\ &\cdot A(a, b) \left[3 + (2p - 1)^2 v^2 \right] - \frac{1}{2} A(a, b) \\ &\cdot \left[\sqrt{1 + v^2} + (\sinh^{-1}(v))/v \right] = A(a, b) \\ &\cdot \left[\frac{v(2 - \sqrt{1 + v^2}) - \sinh^{-1}(v)}{2v^3} + \frac{1}{3} (2p - 1)^2 \right] v^2. \end{aligned} \quad (41)$$

Let $x = \sinh^{-1}(v)$. Then, $x \in (0, \log(1 + \sqrt{2}))$ and

$$\begin{aligned} &\frac{v(2 - \sqrt{1 + v^2}) - \sinh^{-1}(v)}{2v^3} \\ &= \frac{\sinh(x)[2 - \cosh(x)] - x}{2 \sinh^3(x)} = \frac{1}{2} g(x), \end{aligned} \quad (42)$$

where $g(x)$ is defined by Lemma 4.

Therefore, Theorem 7 follows easily from (41) and (42) together with Lemma 4. $\square$

Let $a > b > 0$ and $t = (\log a - \log b)/2 > 0$. Then, we clearly see that

$$\begin{aligned} \frac{a - b}{a + b} &= \tanh(t), \\ \sqrt{1 + \left(\frac{a - b}{a + b}\right)^2} &= \frac{\cosh^{1/2}(2t)}{\cosh(t)}, \end{aligned} \quad (43)$$

and (4) and (6) lead to

$$\begin{aligned} C(a, b) &= \frac{4 \cosh^2(t) - 1}{3 \cosh^2(t)} A(a, b), \\ C[\alpha a + (1 - \alpha)b, \alpha b + (1 - \alpha)a] &= \frac{2(\alpha^2 - \alpha + 1) \cosh(2t) + (1 + 2\alpha - 2\alpha^2)}{3 \cosh^2(t)} \\ &\cdot A(a, b), \\ N_{QA}(a, b) &= \frac{1}{2} \left[\frac{\cosh^{1/2}(2t)}{\cosh(t)} + \frac{\sinh^{-1}(\tanh(t))}{\tanh(t)} \right] \\ &\cdot A(a, b). \end{aligned} \quad (44)$$

From (44) and Theorems 5–7, we get Corollary 8 immediately.

Corollary 8. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ and $\alpha_3, \beta_3 \in (1/2, 1)$. Then, the double inequalities

$$\begin{aligned} &\frac{8 - 2\alpha_1}{3} - \frac{2(1 - \alpha_1)}{3 \cosh^2(t)} \\ &< \frac{\cosh^{1/2}(2t)}{\cosh(t)} + \frac{\sinh^{-1}(\tanh(t))}{\tanh(t)} \\ &< \frac{8 - 2\beta_1}{3} - \frac{2(1 - \beta_1)}{3 \cosh^2(t)}, \end{aligned}$$

$$\begin{aligned}
& 2 \left(\frac{4}{3} - \frac{1}{3 \cosh^2(t)} \right)^{1-\alpha_2} \\
& < \frac{\cosh^{1/2}(2t)}{\cosh(t)} + \frac{\sinh^{-1}(\tanh(t))}{\tanh(t)} \\
& < 2 \left(\frac{4}{3} - \frac{1}{3 \cosh^2(t)} \right)^{1-\beta_2}, \\
& \frac{4(\alpha_3^2 - \alpha_3 + 1) \cosh(2t) + 2(1 + 2\alpha_3 - 2\alpha_3^2)}{3 \cosh^2(t)} \\
& < \frac{\cosh^{1/2}(2t)}{\cosh(t)} + \frac{\sinh^{-1}(\tanh(t))}{\tanh(t)} \\
& < \frac{4(\beta_3^2 - \beta_3 + 1) \cosh(2t) + 2(1 + 2\beta_3 - 2\beta_3^2)}{3 \cosh^2(t)} \quad (45)
\end{aligned}$$

hold for all $t > 0$ if and only if $\alpha_1 \geq 4 - 3[\sqrt{2} + \log(1 + \sqrt{2})]/2$, $\beta_1 \leq 1/2$, $\alpha_2 \geq 1 - [\log(\sqrt{2} + \log(1 + \sqrt{2})) - \log 2]/[2 \log 2 - \log 3]$, $\beta_2 \leq 1/2$, $\alpha_3 \leq 1/2 + \sqrt{6[\sqrt{2} + \log(1 + \sqrt{2})] - 12}/4$, and $\beta_3 \geq 1/2 + \sqrt{2}/4$.

Let $x = \tanh(t) \in (0, 1)$. Then, we clearly see that

$$\begin{aligned}
\cosh(t) &= \frac{1}{\sqrt{1-x^2}}, \\
\cosh^{1/2}(2t) &= \sqrt{\frac{1+x^2}{1-x^2}}, \quad (46)
\end{aligned}$$

Corollary 9. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ and $\alpha_3, \beta_3 \in (1/2, 1)$. Then, the double inequalities

$$\begin{aligned}
& 2 + \frac{2(1-\alpha_1)}{3}x^2 - \sqrt{1+x^2} < \frac{\sinh^{-1}(x)}{x} < 2 \\
& + \frac{2(1-\beta_1)}{3}x^2 - \sqrt{1+x^2}, \\
& 2 \left(1 + \frac{x^2}{3} \right)^{1-\alpha_2} - \sqrt{1+x^2} < \frac{\sinh^{-1}(x)}{x} \\
& < 2 \left(1 + \frac{x^2}{3} \right)^{1-\beta_2} - \sqrt{1+x^2}, \\
& \frac{4(\alpha_3^2 - \alpha_3 + 1)}{3}(1+x^2) \\
& + \frac{2(1+2\alpha_3-2\alpha_3^2)}{3}(1-x^2) - \sqrt{1+x^2} \\
& < \frac{\sinh^{-1}(x)}{x} < \frac{4(\beta_3^2 - \beta_3 + 1)}{3}(1+x^2) \\
& + \frac{2(1+2\beta_3-2\beta_3^2)}{3}(1-x^2) - \sqrt{1+x^2} \quad (47)
\end{aligned}$$

hold for all $x \in (0, 1)$ if and only if $\alpha_1 \geq 4 - 3[\sqrt{2} + \log(1 + \sqrt{2})]/2$, $\beta_1 \leq 1/2$, $\alpha_2 \geq 1 - [\log(\sqrt{2} + \log(1 + \sqrt{2})) - \log 2]/[2 \log 2 - \log 3]$, $\beta_2 \leq 1/2$, $\alpha_3 \leq 1/2 + \sqrt{6[\sqrt{2} + \log(1 + \sqrt{2})] - 12}/4$, and $\beta_3 \geq 1/2 + \sqrt{2}/4$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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