**Research Article**

**Convergence Theorems for Bregman K-Mappings and Mixed Equilibrium Problems in Reflexive Banach Spaces**

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We introduce a new mixed equilibrium problem with a relaxed monotone mapping in a reflexive Banach space and prove the existence of solution of the equilibrium problem. Using Bregman distance, we introduce the concept of Bregman K-mapping for a finite family of Bregman quasiasymptotically nonexpansive mappings and show the fixed point set of the Bregman K-mapping is the set of common fixed points of \( \{ T_i \} _{i=1} ^ n \). Using the Bregman K-mapping, we introduce an iterative sequence for finding a common point in the set of common fixed points of the finite family of Bregman quasiasymptotically nonexpansive mappings and the set of solutions of some mixed equilibrium problems. Strong convergence of the iterative sequence is proved. Our results generalise and improve many recent results in the literature.

**1. Introduction**

Let \( E \) be a real Banach space and let \( E^* \) be the dual of \( E \). Let \( C \) be a nonempty closed and convex subset of \( E \). A mapping \( T : C \to C \) is called nonexpansive if \( \| Tx - Ty \| \leq \| x - y \| \ \forall x, y \in C \). A point \( x \in C \) is a fixed point of \( T \) if \( Tx = x \). We denote by \( F(T) \) the fixed point set of \( T \); that is, \( F(T) = \{ x \in C : Tx = x \} \). Let \( g : C \times C \to \mathbb{R} \) be a bifunction. The equilibrium problem with respect to \( g \) and \( C \) in the sense of Blum and Oettli [1] is to find \( z \in C \) such that

\[
g(z, y) \geq 0 \quad \forall y \in C. \tag{1}
\]

The set of solutions of equilibrium problem is denoted by \( EP(g) \); that is,

\[
EP(g) = \{ z \in C : g(z, y) \geq 0 \ \forall y \in C \}. \tag{2}
\]

In order to solve equilibrium problem (1), the bifunction \( g \) is usually assumed to satisfy the following conditions:

(C1) \( g(x, x) = 0 \) for all \( x, y \in C \);

(C2) \( g \) is monotone; that is, \( g(x, y) + g(y, x) \leq 0 \) for all \( x, y \in C \);

(C3) for all \( x, y, z \in C \), \( \limsup_{t \to 0} g(tz + (1 - t)x, y) \leq g(x, y) \);

(C4) for all \( x \in C \), \( g(x, \cdot) \) is convex and lower semicontinuous.

Fang and Huang [2] introduced the concept of relaxed \( \eta - \alpha \) monotone mapping for solving mixed equilibrium problems. A mapping \( A : C \to E^* \) is said to be relaxed \( \eta - \alpha \) monotone (see also [3]) if there exist a mapping \( \eta : C \times C \to \mathbb{R} \) and a function \( \alpha : E \to \mathbb{R} \) such that

\[
\langle A(z) + \eta(z, y), y \rangle + \alpha(y) - \alpha(z) \geq 0 \quad \forall y \in C, \tag{3}
\]

Particularly if \( \eta(x, y) = x - y \ \forall x, y \in C \) and \( \alpha(z) = k\| z \|^p \), where \( p > 1 \) and \( k > 1 \) are two constants, then \( A \) is called \( p \) monotone; see [4, 5].

Fang and Huang [2] proved that under some suitable assumptions, the following variational inequality is solvable: find \( z \in C \) such that

\[
\langle Az, \eta(y, z) \rangle + \psi(y) - \psi(z) \geq 0 \quad \forall y \in C, \tag{4}
\]
where \( \psi \) is a function from \( C \) to \( \mathbb{R} \cup \{\infty\} \). They also proved that the following inequality is equivalent to variational inequality (4): find \( z \in C \) such that

\[
\langle Ax, \eta(y, z) \rangle + \psi(y) - \psi(z) \geq a(y - z) \quad \forall y \in C. \tag{5}
\]

The mixed equilibrium problem (see [6, 7]) is to find \( z \in C \) such that

\[
g(z, y) + \langle Az, \eta(y, z) \rangle + \psi(y) - \psi(z) \geq 0 \quad \forall y \in C. \tag{6}
\]

We denote the set of solutions of mixed equilibrium problem (6) by \( \text{EP}(g, A) \). It is easily seen that if \( g(z, y) = 0 \) \( \forall z, y \in C \), then mixed equilibrium problem (6) reduces to variational inequality (4). In the case of \( A = 0 \) and \( \psi = 0 \), then, \( \text{EP}(g, A) \) coincides with \( \text{EP}(g) \).

Equilibrium problems and mixed equilibrium problems have been used as tools for solving problems arising from linear and nonlinear programming, optimization problems, variational inequalities, fixed point problems, and also problems in physics, economics, engineering, and so forth (see, e.g., [1, 6–15] and the references therein).

Let \( J : E \to 2^{E^*} \) be the normalised duality mapping defined by

\[
J(x) = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\},
\]

where \( \langle \cdot, \cdot \rangle \) denotes the generalised duality pairing. It is well known that if \( E \) is smooth, strictly convex, and reflexive, then \( J \) is single-valued, one-to-one, and onto.

Let \( f : E \to (-\infty, +\infty] \) be a convex function. We denote by \( dom f \) the domain of \( f \); that is, \( dom f = \{ x \in E : f(x) < +\infty \} \). The function \( f \) is said to be coercive if \( \lim_{\|x\| \to +\infty} f(x) = +\infty \). \( f \) is said to be strongly coercive if \( \lim_{\|x\| \to +\infty} (f(x)/\|x\|) = +\infty \).

For any convex function \( f : E \to (-\infty, +\infty] \), the Fenchel conjugate of \( f \) is the function \( f^* : E^* \to (-\infty, +\infty] \) defined by

\[
f^*(x^*) = \sup \{ \langle x^*, x \rangle - f(x) : x \in E \}.
\]

The subdifferential of \( f \) is a mapping \( \partial f : E \to E^* \) defined by

\[
\partial f(x) = \left\{ x^* \in E^* : f(y) \geq f(x) + \langle x^*, y - x \rangle \quad \forall y \in E \right\} \tag{9}
\]

\[\forall x \in E.\]

It is well known that (see [16]) \( x^* \in \partial f(x) \) if and only if \( f(x) + f^*(x^*) = \langle x^*, x \rangle \) for all \( x \in E \). It is also known that if \( f : E \to (-\infty, +\infty] \) is a proper, convex, and lower semicontinuous function, then \( f^* : E^* \to (-\infty, +\infty] \) is a proper, convex, and weak*-lower semicontinuous function; see, for example, [17].

For any convex function \( f : E \to (-\infty, +\infty] \), let \( x \in \text{dom } f \) and \( y \in E \). The right-hand derivative of \( f \) at \( x \) in the direction \( y \) is defined by

\[
f^r(x, y) = \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t}. \tag{10}
\]

The function \( f \) is said to be Gâteaux differentiable at \( x \) if \( \lim_{t \to 0} ((f(x + ty) - f(x))/t) \) exists for any \( y \in E \). In this case \( f^r(x, y) \) coincides with \( \nabla f(x) \), the value of the gradient \( \nabla f \) of \( f \) at \( x \). \( f \) is said to be Gâteaux differentiable at each \( x \in \text{dom } f \). If the limit in (10) is attained uniformly in \( \|y\| = 1 \), then \( f \) is said to be Fréchet differentiable at \( x \). \( f \) is said to be uniformly Fréchet differentiable on \( C \) if the limit in (10) is attained uniformly for every \( x \in C \) and \( \|y\| = 1 \). We know that if \( f \) is uniformly Fréchet differentiable on bounded subset of \( E \), then \( f \) is uniformly continuous on bounded set of \( E \) (see, e.g., [18]).

The function \( f \) is said to be essentially smooth if \( \partial f \) is both bounded and single-valued on its domain. It is called essentially strictly convex if \( (\partial f)^{-1} \) is locally bounded on its domain and \( f \) is strictly convex on every convex subset of \( \text{dom } \partial f \). \( f \) is said to be a Legendre function if it is both essentially smooth and essentially strictly convex. When the subdifferential of \( f \) is single-valued, it coincides with the gradient; that is, \( \partial f = \nabla f \); see, for example, [19].

For a Legendre function \( f \), the following properties are well known:

(i) \( f \) is essentially smooth if and only if \( f^* \) is essentially strictly convex; see [16];

(ii) \( (\partial f)^{-1} = \partial f^* \); see [20];

(iii) \( f \) is Legendre if and only if \( f^* \) is Legendre function; see [16];

(iv) if \( f \) is Legendre function, then \( \nabla f \) is bijection satisfying \( \nabla f = (\nabla f^*)^{-1} \), \( \text{ran } \nabla f = \text{dom } f^* = \text{int dom } f^* \), and \( \text{ran } \nabla f^* = \text{dom } \nabla f = \text{int dom } f \); see [16].

If \( E \) is smooth and strictly convex, the function \( f(x) = (1/p)\|x\|^p, 1 < p < \infty \), is Legendre function; see, for example, [21]. In this case \( \nabla f = f_p, 1 < p < \infty \). In particular if \( E = H \) is a Hilbert space we have \( \nabla f = I \), the identity mapping.

Let \( f : E \to \mathbb{R} \) be a convex and Gâteaux differentiable function. The function \( D_f : E \times E \to \mathbb{R} \) defined by

\[
D_f(x, y) = f(x) - f(y) - \langle \nabla f(y), x - y \rangle
\]

\[\forall x, y \in E.\tag{11}\]

is called Bregman distance corresponding to \( f \); see [22, 23]. It follows from the strict convexity of \( f \) that \( D_f(x, y) \geq 0 \) \( \forall x, y \in E \) and \( D_f(x, y) = 0 \) if and only if \( x = y \); see [24].

Bregman projection with respect to \( f \) of \( x \in E \) onto the nonempty closed convex subset \( C \) of \( E \) is the unique vector \( P_C^f(x) \in C \) satisfying

\[
D_f(P_C^f(x), x) = \inf \{D_f(y, x) : y \in C \}. \tag{12}
\]

Remark 1. If \( E \) is smooth and strictly convex Banach space and \( f(x) = \|x\|^2 \forall x \in E \), then we have \( \nabla f(x) = 2f(x) \forall x \in E \) and hence \( D_f(x, y) = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2 = (\phi(x, y)) \forall x, y \in E \) which is the Lyapunov function introduced by Alber [25] and the Bregman projection \( P_C^f(x) \) reduces to the generalised projection \( P_C(x) \) which is defined by

\[
\phi(P_C(x), x) = \inf \{\phi(y, x) : y \in C \}. \tag{13}
\]
If \( E = H \), a Hilbert space, then the Bregman projection \( P^*_C(x) \) reduces to the metric projection \( P_C(x) \) of \( H \) onto \( C \).

Observing (11), we have

\[
D_f(z, x) - D_f(z, y) = D_f(y, x) + \langle \nabla f(y) - \nabla f(z), z - y \rangle \quad \forall x, y, z \in E,
\]

which is called the three-point identity.

Let \( f: E \to (-\infty, +\infty] \) be a convex, Legendre, and Gâteaux differentiable function. Following [23, 25] we make use of the function \( V_f : E \times E^* \to [0, +\infty) \) associated with \( f \) defined by

\[
V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*) \quad \forall x \in E, \quad x^* \in E^*.
\]

Then \( V_f \) is nonnegative and \( V_f(x, x^*) = D_f(x, \nabla f^*(x^*)) \) \( \forall x \in E \) and \( x^* \in E^* \). Also from definition (15), it is obvious that \( D_f(x, y) = V_f(x, \nabla f(y)) \) and \( V_f \) is convex in the second variable. Therefore for \( t \in (0, 1) \) and \( x, y \in E \), we have

\[
D_f(z, \nabla f^*(t\nabla f(x) + (1 - t)\nabla f(y))) \leq tD_f(z, x) + (1 - t)D_f(z, y).
\]

Moreover by subdifferential inequality [26], we have

\[
V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x \rangle \leq V_f(x, x^* + y^*) \quad \forall x \in E, \quad x^* \in E^*.
\]

Recall that a mapping \( T: C \to C \) is said to be \( \phi \)-quasi nonexpansive if \( \Phi(T) \neq \emptyset \) and \( \Phi(p, Tx) \leq \Phi(p, x) \) \( \forall x \in C \) and \( p \in F(T) \). \( T \) is \( \phi \)-quasiasymptotically nonexpansive if \( F(T) \neq \emptyset \) and there exists a real sequence \( \{v_n\} \subset [0, \infty) \) such that \( v_n \to 0 \) as \( n \to \infty \) and \( \Phi(p, T^n x) \leq (1 + v_n)\Phi(p, x) \) \( \forall x \in C \) and \( p \in F(T) \). \( T \) is called Bregman quasi nonexpansive if \( F(T) \neq \emptyset \) and \( D_f(p, Tx) \leq D_f(p, x) \) \( \forall x \in C \) and \( p \in F(T) \). \( T \) is Bregman quasiasymptotically nonexpansive if \( F(T) \neq \emptyset \) and there exists a real sequence \( \{v_n\} \subset [0, \infty) \) such that \( v_n \to 0 \) as \( n \to \infty \) and \( D_f(p, T^n x) \leq (1 + v_n)D_f(p, x) \) \( \forall x \in C \) and \( p \in F(T) \). \( T \) is said to be closed if for any sequence \( \{x_n\} \subset C \) with \( x_n \to x \) and \( T x_n \to y \), \( T x = y \).

It is worth mentioning that several iterative schemes have been constructed and proposed for finding points which solve fixed point problems and mixed equilibrium problems with relaxed monotone mappings in various settings. In 2010 Wang et al. [7] introduced the following iterative scheme for finding a common element of the set of solutions of a mixed equilibrium problem with relaxed monotone mapping and the set of fixed points of nonexpansive mappings in Hilbert spaces:

\[
x_1 \in C \text{ chosen arbitrarily},
\]

\[
\Phi(u_n, y) + \langle Tu_n, \eta(y, u_n) \rangle + \frac{1}{\lambda_n} \langle y - u_n, u_n - x_n \rangle \geq 0 \quad \forall y \in C,
\]

\[
y_n = \alpha_n x_n + (1 - \alpha_n) \beta_n S x_n + (1 - \alpha_n)(1 - \beta_n) u_n,
\]

\[
C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\},
\]

\[
D_n = \bigcap_{j=1}^n C_j,
\]

\[
x_{n+1} = P_{D_n} x_1, \quad n \geq 1,
\]

where \( T \) is a relaxed \( \eta \)-monotone mapping and \( S : C \to C \) is a nonexpansive mapping. Under some mild conditions on the three control sequences \( \{\alpha_n\}, \{\beta_n\}, \text{ and } \{\lambda_n\} \), they obtained strong convergence of scheme (18) to common solution of mixed equilibrium problems and fixed point of nonexpansive mapping.

Recently, Chen et al. [27] introduced a new mixed equilibrium problem with the relaxed monotone mapping in uniformly convex and uniformly smooth Banach spaces and proved the existence of solutions of the mixed equilibrium problem. They also proposed the following iterative scheme to find the common element of the set of solutions of the mixed equilibrium problem and the set of fixed points of a quasi-\( \phi \)-nonexpansive mapping:

\[
x_1 = x \in C \text{ chosen arbitrarily},
\]

\[
y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S x_n),
\]

\[
u_n \in C \text{ such that}
\]

\[
\theta(u_n, y) + \langle Au_n, \eta(y, u_n) \rangle + f(y) - f(u_n)
\]

\[
+ \frac{1}{r_n} \langle y - u_n, u_n - f(y) \rangle \geq 0 \quad \forall y \in C,
\]

\[
C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\},
\]

\[
D_n = \bigcap_{j=1}^n C_j,
\]

\[
x_{n+1} = \Pi_{D_n} x_1, \quad n \geq 1,
\]

where \( S \) is a quasi-\( \phi \)-nonexpansive mapping from \( C \) into itself and \( J \) is the normalised duality mapping. Under some assumptions on the parameter sequences \( \{\alpha_n\} \) and \( \{r_n\} \), they obtained strong convergence of scheme (19) to common solution of mixed equilibrium problems and fixed point of nonexpansive mapping.

Motivated and inspired by the above results, in this paper we introduce and prove the existence of solutions of the mixed equilibrium problem with relaxed monotone
mapping in reflexive Banach spaces. Using Bregman distance, we introduce the concept of Bregman K-mapping of a finite family of Bregman quasiamproximally nonexpansive mappings and propose an iterative sequence for finding a common element of the set of fixed points of a finite family of Bregman quasiamproximally nonexpansive mappings and the set of solutions of mixed equilibrium problem.

2. Preliminaries

Let \( f : E \to (-\infty, +\infty] \) be a convex and Gâteaux differentiable function. The modulus of total convexity of \( f \) at \( x \in \text{int} \, \text{dom} \, f \) is the function \( \nu_f(x, t) : [0, +\infty) \to [0, +\infty) \) defined by

\[
\nu_f(x, t) = \inf \{ D_f(y, x) : y \in \text{dom} f, \| y - x \| = t \}. \tag{20}
\]

The function \( f \) is totally convex at \( x \) if \( \nu_f(x, t) > 0 \) for all \( t > 0 \) and \( f \) is totally convex if it is totally convex at each point \( x \in \text{dom} \, f \). Let \( B \) be a bounded subset of \( E \). For \( t > 0 \), define a functional on \( B \), \( \nu_f : \text{int} \, \text{dom} \, f \times [0, +\infty) \to [0, +\infty) \) defined by

\[
\nu_f(B, t) = \inf \{ \nu_f(x, t) : x \in B \cup \text{dom} f \}. \tag{21}
\]

\( f \) is totally convex on bounded set \( B \) if \( \nu_f(B, t) > 0 \) for any bounded subset \( D \) of \( E \) and \( t > 0 \), where \( \nu_f(., t) \) is the total convexity of the function \( f \) on the set \( B \).

Let \( B_r = \{ x \in E : \| x \| \leq r \} \), for all \( r > 0 \) and \( S = \{ x \in E : \| x \| = 1 \} \). The function \( f \) is bounded if \( f(B_r) \) is bounded for all \( r > 0 \) and \( f \) is uniformly convex on bounded subsets of \( E \) [28] if the function \( \rho : [0, +\infty) \to [0, +\infty) \) defined by

\[
\rho_f(t) = \inf_{x,y \in B_r, \| x-y \|=t} \frac{af(x) + (1-a)f(y) - f(ax + (1-a)y)}{a(1-a)} \tag{22}
\]

satisfies

\[
\rho_f(t) > 0 \quad \forall r, t > 0, \tag{23}
\]

where \( \rho_f \) is called the gauge of uniform convexity of \( f \).

The gauge of uniform smoothness of \( f \) is the function \( \sigma_f : [0, +\infty) \to [0, +\infty) \) defined by

\[
\sigma_r(t) = \sup_{x \in B_r, y \in S, a \in (0,1)} \frac{af(x + (1-a)ty) + (1-a)f(x - ta(y) - f(x))}{a(1-a)}. \tag{24}
\]

Lemma 6 (see [32]). If \( x \in \text{dom} f \), then the following statements are equivalent:

(i) the function \( f \) is totally convex at \( x \);

(ii) for any sequence \( \{y_n\} \subset \text{dom} f \)

\[
\lim_{n \to \infty} D_f(y_n, x) = 0 \quad \text{implies} \quad \lim_{n \to \infty} \| y_n - x \| = 0. \tag{25}
\]

Recall that a function \( f : E \to (-\infty, +\infty] \) is called sequentially consistent [29] if for any two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( E \) such that \( x_n \) is bounded,

\[
\lim_{n \to \infty} D_f(y_n, x_n) = 0 \quad \text{implies} \quad \lim_{n \to \infty} \| y_n - x_n \| = 0. \tag{26}
\]

Lemma 7 (see [24]). The function \( f : E \to (-\infty, +\infty] \) is totally convex on bounded sets if and only if \( f \) is sequentially consistent.

Lemma 8 (see [26]). Let \( f : E \to (-\infty, +\infty] \) be a Legendre function such that \( \nabla f \) is bounded on bounded subsets of \( E^* \).

Let \( x \in E \). If the sequence \( \{D_f(x, x_n)\} \) is bounded, then the sequence \( \{x_n\} \) is bounded.
Lemma 9 (see [33]). Let $r > 0$ be a constant and let $f : E \to \mathbb{R}$ be a uniformly convex function on bounded subsets of $E$. Then for any $x, y \in B_r$ and $\alpha \in (0, 1)$,
\begin{equation}
    f(\alpha x + (1 - \alpha) y) \leq \alpha f(x) + (1 - \alpha) f(y) - \alpha (1 - \alpha) \rho_r(\|x - y\|),
\end{equation}
where $\rho_r$ is the gauge of the uniform convexity of $f$.

Lemma 10 (see [34]). Let $f : E \to (-\infty, +\infty]$ be a uniformly Fréchet differentiable function and bounded on bounded subsets of $E$. Then $\nabla f$ is uniformly continuous on bounded subsets of $E$. Hence $\nabla f$ is uniformly convex on bounded subsets of $E$.

Lemma 11 (see [28]). Let $f : E \to (-\infty, +\infty]$ be a convex function. If $\nabla f$ is uniformly continuous on bounded subsets of $E$, then $f^* : E^* \to (-\infty, +\infty]$ is uniformly convex on bounded subsets of $E^*$.

Lemma 12 (see [28]). Let $f : E \to (-\infty, +\infty]$ be a convex function which is bounded on bounded subsets of $E$. Then the following assertions are equivalent:

(i) $f$ is strongly coercive and uniformly convex on bounded subsets of $E$;

(ii) $f^*$ is Fréchet differentiable and $\nabla f^*$ is uniformly norm-to-norm continuous on bounded subsets of $\text{dom} f^* = E^*$.

Lemma 13 (see [29]). Let $C$ be a nonempty closed and convex subset of $E$. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Let $x \in E$. Then
\begin{enumerate}
    \item[(i)] $z = P_C^f(x)$ if and only if $(\nabla f(x) - \nabla f(y), y - z) \leq 0$ for all $y \in C$;
    \item[(ii)] $D_f(y, P_C^f(x)) + D_f(P_C^f(x), y) \leq D_f(x, y) \forall y \in C$.
\end{enumerate}

Lemma 14 (see [35]). Let $E$ be a reflexive Banach space and $C$ be a nonempty closed and convex subset of $E$. Let $f : E \to (-\infty, +\infty]$ be a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex which is bounded on bounded subsets of $E$. Let $T : C \to C$ be a closed and Bregman quasiasymptotically nonexpansive mapping. If $F(T) \neq \emptyset$, then it is closed and convex.

Lemma 15. Let $E$ be a reflexive Banach space and $C$ be a nonempty closed and convex subset of $E$. Let $f : E \to (-\infty, +\infty]$ be a Legendre, uniformly Fréchet differentiable, strongly coercive, and totally convex function on bounded subsets of $E$. Then the Bregman projection $P_C^f : E \to C$ is continuous.

Proof. Let $\{x_n\}$ be a sequence in $E$ such that $x_n \to x$ as $n \to \infty$. Let $x' = P_C^f x$ and $x'' = P_C^f x$. By Lemma 13(ii), we have
\begin{equation}
    D_f(y, x') + D_f(x'', x_n) \leq D_f(x, x_n), \quad \forall y \in C. \tag{28}
\end{equation}
From inequality (28), we have
\begin{equation}
    D_f(y, x') \leq D_f(y, x_n) \quad \forall y \in C. \tag{29}
\end{equation}
Since $\{D_f(y, x_n)\}$ converges, it is bounded and using the above inequality it follows that $D_f(y, x'_n)$ is bounded. The function $f$ is strongly coercive and totally convex which is bounded on bounded subsets of $E$; therefore in view of Lemma 12 $\nabla f^*$ is uniformly norm-to-norm continuous on bounded subsets of $\text{dom} f^* = E^*$ and consequently $\nabla f^*$ is bounded. Hence by Lemma 8 we obtain that $\{x'_n\}$ is bounded.

Since $x' = P_C^f x$ we have $D_f(x', x) \leq D_f(x'_n, x)$. Therefore
\begin{equation}
    D_f(x'_n, x_n) - D_f(x'_n, x) \geq D_f(x'_n, x) - D_f(x'_n, x_n) = f(x) - f(x_n) + \langle \nabla f(x_n), x_n - x' \rangle \tag{30}
\end{equation}
\begin{equation}
    + \langle \nabla f(x), x - x' \rangle.
\end{equation}
Since $f$ is uniformly Fréchet differentiable on bounded subsets of $E$, it follows that $f$ is uniformly continuous on bounded subsets of $E$ (see, e.g., [18]). Thus, taking lim inf as $n \to \infty$ of both sides of the above inequality, we obtain
\begin{equation}
    \liminf_{n \to \infty} D_f(x'_n, x_n) - D_f(x', x) \geq 0. \tag{31}
\end{equation}
Now let $\epsilon > 0$. Using (28), we have
\begin{equation}
    D_f(x', x'_n) \leq D_f(x', x_n) - D_f(x'_n, x_n) \leq f(x) - f(x_n) \tag{32}
\end{equation}
\begin{equation}
    + \langle \nabla f(x), x'_n - x \rangle + \langle \nabla f(x), x_n - x \rangle + \epsilon \quad \forall n \geq N_0,
\end{equation}
where $N_0$ is some natural number. As $n \to \infty$ and $\epsilon \to 0$, we obtain
\begin{equation}
    D_f(x', x'_n) \to 0. \tag{33}
\end{equation}
By total convexity of $f$, we get $x'_n \to x'$ as $n \to \infty$. This completes the proof. \qed

3. Main Results

Lemma 16. Let $C$ be a nonempty, closed, and convex subset of a reflexive Banach space $E$ with the dual $E^*$. Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. Let $A : C \to E^*$ be $\eta$-hemiconcave and relaxed $\eta - \alpha$ monotone mapping and $y : C \times C \to \mathbb{R}$ be a bifunction satisfying (C1) and (C4). Let $\psi : C \times C \to \mathbb{R}$ be proper, convex, and lower semicontinuous. For $r > 0$ and $x \in E$, suppose the following conditions hold:
\begin{enumerate}
    \item[(i)] $\eta(z, z) = 0 \quad \forall z \in C$;
    \item[(ii)] $\langle A(u, \eta(u, v)) \rangle$ is convex for fixed $u, v \in C$.
\end{enumerate}
Then problems (34) and (35) are equivalent.
Find \( z \in C \) such that
\[
g(z, y) + \langle Az, \eta(y, z) \rangle + \psi(y) - \psi(z) + \frac{1}{r} \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0 \quad \forall y \in C. \tag{34}
\]

Find \( z \in C \) such that
\[
g(z, y) + \langle Ay, \eta(y, z) \rangle + \psi(y) - \psi(z) + \frac{1}{r} \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq \alpha(y - z) \quad \forall y \in C. \tag{35}
\]

**Proof.** Suppose (34) holds. Let \( z \) be a solution of (34); then
\[
g(z, y) + \langle Az, \eta(y, z) \rangle + \psi(y) - \psi(z) + \frac{1}{r} \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0 \quad \forall y \in C. \tag{36}
\]

Since \( A \) is relaxed \( \eta - \alpha \) monotone, we obtain
\[
g(z, y) + \langle Ay, \eta(y, z) \rangle + \psi(y) - \psi(z) + \frac{1}{r} \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq g(z, y) + \langle Az, \eta(y, z) \rangle + \psi(y) - \psi(z) + \frac{1}{r} \langle \nabla f(z) - \nabla f(x), y - z \rangle + \alpha(y - z), \quad \forall y \in C. \tag{37}
\]

Hence by (C1), (C4), (i), and (ii), we obtain
\[
t^\alpha(y - z) \leq g(z, (1 - t)z + ty) + \langle Ay, \eta((1 - t)z + ty), y \rangle + \psi((1 - t)z + ty) - \psi(z) + \frac{1}{r} \langle \nabla f(z) - \nabla f(x), ((1 - t)z + ty) - z \rangle \tag{39}
\]

Thus,
\[
0 \leq g(z, y) + \langle Az, \eta(y, z) \rangle + \psi(y) - \psi(z) + \frac{1}{r} \langle \nabla f(z) - \nabla f(x), y - z \rangle = g(z, y) + \langle A((1 - t)z + ty), \eta(y, z) \rangle + \psi(y) - \psi(z) + \frac{1}{r} \langle \nabla f(z) - \nabla f(x), y - z \rangle. \tag{40}
\]

Since \( A \) is \( \eta \)-hemicontinuous and \( p > 1 \), by allowing \( t \to 0^+ \) we obtain
\[
0 \leq g(z, y) + \langle Az, \eta(y, z) \rangle + \psi(y) - \psi(z) + \frac{1}{r} \langle \nabla f(z) - \nabla f(x), y - z \rangle. \tag{41}
\]

This shows that \( z \) is a solution of (34).

**Lemma 17.** Let \( E \) be a reflexive Banach space with the dual \( E^* \) and let \( C \) be a nonempty closed, convex, and bounded subset of \( E \). Let \( f : E \to (-\infty, +\infty] \) be a Legendre function and uniformly Fréchet differentiable on bounded subsets of \( E \). Let \( A : C \to E^* \) be \( \eta \)-hemicontinuous and relaxed \( \eta - \alpha \) monotone mapping and \( g : C \times C \to \mathbb{R} \) be a bifunction satisfying (C1) and (C4). Let \( \psi : C \to \mathbb{R} \) be proper, convex lower semicontinuous. For \( r > 0 \) and \( x \in E \), suppose
\begin{align*}
(\text{i}) \quad & \eta(x, x) = 0 \text{ for all } x \in C, \\
(\text{ii}) \quad & \eta(z, y) + \eta(y, z) = 0 \text{ for all } y \in C, \\
(\text{iii}) \quad & \langle Au, \eta(v, v) \rangle \text{ is convex and lower semicontinuous for } \text{fixed } u, v \in C, \\
(\text{iv}) \quad & \alpha : E \to \mathbb{R} \text{ is weakly lower semicontinuous.}
\end{align*}
Then there exists \( z \in C \) such that

\[
g(z, y) + \langle Az, \eta(y, z) \rangle + \psi(y) - \psi(z) + \frac{1}{r} \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0 \quad \forall y \in C.
\]

Proof. Define two set-valued mappings \( H, W : C \to 2^E \) as follows:

\[
H(y) = \left\{ z \in C : g(z, y) + \langle Az, \eta(y, z) \rangle + \psi(y) - \psi(z) + \frac{1}{r} \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0 \right\} \quad \forall y \in C,
\]

\[
W(y) = \left\{ z \in C : g(z, y) + \langle Ay, \eta(y, z) \rangle + \psi(y) - \psi(z) + \frac{1}{r} \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq \alpha(y - z) \right\} \quad \forall y \in C.
\]

We claim \( H \) is a KKM mapping. By contradiction suppose then there do not exist \( \{y_1, y_2, \ldots, y_n\} \subseteq C \) such that \( \sum_{i=1}^{n} t_i = 1 \) and \( y = \sum_{i=1}^{n} t_i y_i \notin \bigcup_{i=1}^{n} H(y_i) \). This implies

\[
g(y, y_i) + \langle Ay, \eta(y_i, y) \rangle + \psi(y_i) - \psi(y) + \frac{1}{r} \langle \nabla f(y) - \nabla f(x), y - y_i \rangle < 0 \quad \forall i = 1, 2, \ldots, n.
\]

It follows that

\[
0 = g(y, y) + \langle Ay, \eta(y, y) \rangle = g\left(y, \sum_{i=1}^{n} t_i y_i\right)
\]

\[
+ \langle Ay, \eta\left(\sum_{i=1}^{n} t_i y_i, y\right)\rangle \leq \sum_{i=1}^{n} t_i g(y, y_i)
\]

\[
+ \sum_{i=1}^{n} t_i \langle Ay, \eta(y_i, y) \rangle < \sum_{i=1}^{n} t_i \left( \psi(y) - \psi(y_i) \right)
\]

\[
+ \frac{1}{r} \langle \nabla f(y) - \nabla f(x), y - y_i \rangle = \psi(y)
\]

\[
- \sum_{i=1}^{n} t_i \psi(y_i) \leq \psi(y) - \psi\left(\sum_{i=1}^{n} t_i y_i\right) = 0,
\]

which is a contradiction. Thus \( H \) is a KKM mapping.

Next we show that \( y \in C \) \( H(y) \subseteq W(y) \).

Let \( z \in H(y) \). Then

\[
g(z, y) + \langle Az, \eta(y, z) \rangle + \psi(y) - \psi(z) + \frac{1}{r} \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0.
\]

Since \( A \) is relaxed \( \eta - \alpha \) monotone, we have

\[
g(z, y) + \langle Ay, \eta(y, z) \rangle + \psi(y) - \psi(z) + \frac{1}{r} \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq g(z, y)
\]

\[
+ \langle Az, \eta(y, z) \rangle + \psi(y) - \psi(z) + \frac{1}{r} \langle \nabla f(z) - \nabla f(x), y - z \rangle + \alpha(y - z)
\]

\[
\geq \alpha(y - z).
\]

Showing that \( z \in W(y) \) for all \( y \in C \). By Remark 4 it follows that \( W \) is a KKM mapping.

We claim also that \( W(y) \) is closed in the weak topology of \( E \). Let \( y \in C \) and \( \bar{z} \) be the weak closure point of \( W(y) \). Since \( E \) is reflexive, there exists a sequence \( \{z_n\} \subseteq W(y) \) such that \( z_n \rightharpoonup \bar{z} \in C \) as \( n \to \infty \). Observe that

\[
g(z_n, y) + \langle Ay, \eta(z_n, y) \rangle + \psi(y) - \psi(z_n)
\]

\[
+ \frac{1}{r} \langle \nabla f(z_n) - \nabla f(x), y - z_n \rangle \geq \alpha(y - z_n)
\]

is equivalent to

\[
\frac{1}{r} \langle \nabla f(z_n) - \nabla f(x), y - z_n \rangle \geq g(y, z_n) + \langle Ay, \eta(z_n, y) \rangle + \psi(z_n) - \psi(y)
\]

\[
+ \alpha(y - z_n).
\]

By (iii) and (iv) and taking \( \liminf \) as \( n \to \infty \) of both sides of (49) we obtain

\[
\frac{1}{r} \langle \nabla f(\bar{z}) - \nabla f(x), y - \bar{z} \rangle \geq g(y, \bar{z}) + \langle Ay, \eta(\bar{z}, y) \rangle + \psi(\bar{z}) - \psi(y)
\]

\[
+ \alpha(y - \bar{z}).
\]

That is, \( \bar{z} \in W(y) \) \( \forall y \in C \). This implies \( W(y) \) is weakly closed for all \( y \in C \). Since \( C \) is weakly compact, then \( W(y) \) is weakly compact in \( C \) for all \( y \in C \).

It is clear that the solution sets of problem (34) and (35) are \( \bigcap_{y \in C} H(y) \) and \( \bigcap_{y \in C} W(y) \). Using Lemmas 16 and 5 we obtain

\[
\bigcap_{y \in C} H(y) = \bigcap_{y \in C} W(y) \neq \emptyset.
\]

Hence there exists \( z \in C \) such that

\[
g(z, y) + \langle Az, \eta(y, z) \rangle + \psi(y) - \psi(z)
\]

\[
+ \frac{1}{r} \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0 \quad \forall y \in C.
\]

This completes the proof.
Lemma 18. Let $E$ be a reflexive Banach space with the dual $E^*$ and let $C$ be a nonempty closed, convex, and bounded subset of $E$. Let $f : E \to (-\infty, +\infty]$ be Legendre and Gâteaux differentiable function. Let $A : C \to E^*$ be $\eta$-hemicontractive and relaxed $\eta - \alpha$ monotone mapping and $g : C \times C \to \mathbb{R}$ be a bifunction satisfying (C1), (C2), and (C4). Let $\psi : C \to \mathbb{R}$ be proper, convex, and lower semicontinuous. For $r > 0$ and $x \in E$, define a map $T_r : E \to 2^E$ by

$$T_r (x) = \left\{ z \in C : g (z, y) + \langle Ay, \eta (yz) \rangle + \psi (y) - \psi (z) + \frac{1}{r} \langle \nabla f (z) - \nabla f (x), y - z \rangle \geq 0 \; \forall y \in C \right\}.$$  

(53)

Assume that

(i) $\eta (z, y) + \eta (y, z) = 0 \; \forall z, y \in C$;
(ii) $\langle Au, \eta_i (v) \rangle$ is convex and lower semicontinuous for fixed $u, v \in C$;
(iii) $\alpha : E \to \mathbb{R}$ is weakly lower semicontinuous;
(iv) $\alpha (x - y) + \alpha (y - z) \geq 0 \; \forall x, y \in C$.

Then

1. $T_r$ is single-valued;
2. $T_r$ is a Bregman firmly nonexpansive type mapping; that is,

$$\langle \nabla f (x) - \nabla f (y), x - T_r x \rangle \leq \langle \nabla f (x) - \nabla f (y), T_r x - T_r y \rangle \quad \forall x, y \in C;$$

(54)

3. $F(T_r) = EP(g, A)$;
4. $T_r$ is a Bregman quasi nonexpansive satisfying

$$D_j (u, T_r x) + D_j (T_r x, x) \leq D_j (u, x);$$

(55)

5. $EP(g, A)$ is closed and convex.

Proof. First we show that $T_r$ is single-valued. Let $z_1, z_2 \in T_r x$; then

$$g (z_1, z_2) + \langle \nabla z_1, \eta (z_2, z_1) \rangle + \psi (z_2) - \psi (z_1)$$

$$+ \frac{1}{r} \langle \nabla f (z_1) - \nabla f (x), z_2 - z_1 \rangle \geq 0,$$

$$g (z_2, z_1) + \langle \nabla z_2, \eta (z_1, z_2) \rangle + \psi (z_1) - \psi (z_2)$$

$$+ \frac{1}{r} \langle \nabla f (z_2) - \nabla f (x), z_1 - z_2 \rangle \geq 0.$$  

(56)

By using (C2), adding (56) yields

$$\langle \nabla z_1, \eta (z_2, z_1) \rangle + \langle \nabla z_2, \eta (z_1, z_2) \rangle$$

$$+ \frac{1}{r} \langle \nabla f (z_1) - \nabla f (z_2), z_2 - z_1 \rangle \geq 0.$$  

(57)

By (i) we have

$$\langle Az_1 - Az_2, \eta (z_2, z_1) \rangle$$

$$+ \frac{1}{r} \langle \nabla f (z_1) - \nabla f (z_2), z_2 - z_1 \rangle \geq 0.$$  

(58)

Since $A$ is relaxed $\eta - \alpha$ monotone, we obtain

$$\frac{1}{r} \langle \nabla f (z_1) - \nabla f (z_2), z_2 - z_1 \rangle$$

$$\geq \langle Az_2 - Az_1, \eta (z_2, z_1) \rangle \geq \alpha (z_2 - z_1).$$

Thus

$$\langle \nabla f (z_1) - \nabla f (z_2), z_2 - z_1 \rangle \geq r \alpha (z_2 - z_1).$$

(59)

Interchanging $z_1$ and $z_2$ in (60), we have

$$\langle \nabla f (z_2) - \nabla f (z_1), z_1 - z_2 \rangle \geq r \alpha (z_1 - z_2).$$

(61)

Adding (60) and (61), we have

$$\langle \nabla f (z_1) - \nabla f (z_2), z_2 - z_1 \rangle$$

$$+ \langle \nabla f (z_2) - \nabla f (z_1), z_1 - z_2 \rangle$$

$$\geq r (\alpha (z_2 - z_1) + \alpha (z_1 - z_2)).$$

(62)

Hence

$$2 \langle \nabla f (z_1) - \nabla f (z_2), z_2 - z_1 \rangle$$

$$\geq r (\alpha (z_2 - z_1) + \alpha (z_1 - z_2)).$$

(63)

By (iv), we have

$$\langle \nabla f (z_1) - \nabla f (z_2), z_2 - z_1 \rangle \geq 0.$$  

(64)

Thus,

$$\langle \nabla f (z_2) - \nabla f (z_1), z_2 - z_1 \rangle \leq 0.$$  

(65)

Since $f$ is convex and Gâteaux differentiable we have

$$\langle \nabla f (z_2) - \nabla f (z_1), z_2 - z_1 \rangle \geq 0.$$  

(66)

By (65) and (66) we obtain

$$\langle \nabla f (z_2) - \nabla f (z_1), z_2 - z_1 \rangle = 0.$$  

(67)

Since $f$ is Legendre function, then $z_1 = z_2$.

Next we show that $T_r$ is Bregman firmly nonexpansive type. Let $x, y \in C$; then

$$g (T_r x, T_r y) + \langle AT_r x, \eta (T_r y, T_r x) \rangle + \psi (T_r y) - \psi (T_r x)$$

$$+ \frac{1}{r} \langle \nabla f (T_r x) - \nabla f (x), T_r y - T_r x \rangle \geq 0,$$

$$g (T_r y, T_r x) + \langle AT_r y, \eta (T_r x, T_r y) \rangle + \psi (T_r x) - \psi (T_r y)$$

$$+ \frac{1}{r} \langle \nabla f (T_r y) - \nabla f (y), T_r x - T_r y \rangle \geq 0.$$  

(68)
Adding (68), using (i) and (C2), we obtain
\[
\langle AT_x - AT_y, \eta(T_y, T_x) \rangle + \frac{1}{r} \langle \nabla f(T_y) \rangle - \nabla f(x) + \nabla f(y), T_y - T_x \rangle \geq 0,
\]
so that
\[
\frac{1}{r} \langle \nabla f(T_x) \rangle - \nabla f(x), T_y \rangle - T_x \rangle \geq \langle AT_y - AT_x, \eta(T_y, T_x) \rangle.
\]
Since \( A \) is relaxed \( \eta - \alpha \) monotone and \( r > 0 \), we have
\[
\langle \nabla f(T_x) - \nabla f(T_y), T_y - T_x \rangle \geq r \alpha \langle T_y - T_x, \rangle.
\]
Also interchanging the roles of \( x \) and \( y \) in (71) and applying (iv), we have
\[
2 \langle \nabla f(T_x) \rangle - \nabla f(T_y), T_x - T_y \rangle \geq 0.
\]
Hence,
\[
\langle \nabla f(T_x) \rangle - \nabla f(T_y), T_x - T_y \rangle \leq \langle \nabla f(x) \rangle - \nabla f(y), T_x - T_y \rangle,
\]
showing that \( T_1^r \) is Bregman firmly nonexpansive type. We now show that \( F(T_1) = \text{EP}(g, A) \). Indeed,
\[
u \in F(T_1) \iff u \in T_1 u \iff g(u, y) + \langle Au, \eta(y, u) \rangle + \psi(y) - \psi(u)
\]
\[
+ \frac{1}{r} \langle \nabla f(u) \rangle - \nabla f(u), y - u \rangle \geq 0 \quad \forall y \in C \iff
\]
\[
g(u, y) + \langle Au, \eta(y, u) \rangle + \psi(y) - \psi(u) \geq 0
\]
\[
\forall y \in C \iff u \in \text{EP}(g, A).
\]
Next we prove that \( T_i \) is Bregman quasi nonexpansive mapping. Since
\[
D_f(T_i, T_j) + D_f(T_j, T_i)
\]
\[
\leq D_f(T_i, T_j) + D_f(T_j, T_i) - D_f(T_i, T_j) + D_f(T_j, T_i)
\]
we have
\[
\frac{1}{r} \langle \nabla f(T_y) \rangle - \nabla f(x), T_y - x \rangle \geq D_f(T_y, T_x) + D_f(T_x, T_y).
\]
Let \( u = y \in F(T_i) \); then from (76) we obtain
\[
D_f(u, T_x) \leq D_f(u, y).
\]
This shows that \( T_i \) is Bregman quasi nonexpansive mapping which is Bregman quasisymptotically nonexpansive. Also from (75), we have
\[
D_f(T_i, T_j) + D_f(T_j, T_i) + D_f(T_i, T_j)
\]
\[
+ D_f(T_j, T_i) \leq D_f(T_i, T_j) + D_f(T_j, T_i).
\]
As \( u = y \in F(T_i) \), we obtain
\[
D_f(u, T_x) \leq D_f(u, x).
\]
Lastly, using (3), (4), and Lemma 14 we obtain that \( \text{EP}(g, A) \) is closed and convex.

Definition 19. Let \( C \) be a nonempty, closed, and convex subset of a real Banach space \( E \). Let \( \{T_i\}_{i=1}^N \) be a finite family of Bregman quasisymptotically nonexpansive mappings. For any \( n \in \mathbb{N} \), define a mapping \( K_n : C \to C \) as follows:
\[
S_n,0 x = x
\]
\[
S_n,1 x = P^n_C(\nabla f^*(\alpha_n \nabla f(T_n^a) + (1 - \alpha_n) \nabla f(x)))
\]
\[
S_n,2 x = P^n_C(\nabla f^*(\alpha_n \nabla f(T_n^a S_n,1 x)) + (1 - \alpha_n) \nabla f(S_n,1 x))
\]
\[
S_n,3 x = P^n_C(\nabla f^*(\alpha_n \nabla f(T_n^a S_n,2 x)) + (1 - \alpha_n) \nabla f(S_n,2 x))
\]
\[
\vdots
\]
\[
S_n,N-1 x = P^n_C(\nabla f^*(\alpha_{N-1} \nabla f(T_n^a S_{n,N-2} x)) + (1 - \alpha_{N-1}) \nabla f(S_{n,N-2} x))
\]
\[
K_n x = S_n,x = P^n_C(\nabla f^*(\alpha_n \nabla f(T_n^a S_{n,N-1} x)) + (1 - \alpha_n) \nabla f(S_{n,N-1} x))
\]
Such a mapping \( K_n \) is called the Bregman \( K \)-mapping generated by \( T_1, T_2, T_3, \ldots, T_N \) and \( \alpha_{ij} \in (0, 1) \), \( i = 1, 2, 3, \ldots, N \).

Using the above definition, we have the following Lemma.

Lemma 20. Let \( E \) be a reflexive Banach space with the dual \( E^* \) and let \( C \) be a nonempty, closed, convex, and bounded subset of \( E \). Let \( f : E \to (-\infty, +\infty] \) be strongly coercive, Legendre, uniformly Fréchet differentiable, and totally convex function which is bounded on bounded subsets of \( E \). Let \( \{T_i\}_{i=1}^N \) be a finite
family of continuous Bregman quasiasymptotically nonexpansive mappings of $C$ into itself such that $\inf_{i=1}^N F(T_i) \neq 0$. Let $\{\alpha_{nj}\}$ be a real sequence in $(0, 1)$ such that $\lim \inf_{i=1}^N \alpha_{nj} > 0 \ \forall i \in \{1, 2, 3, \ldots, N\}$. Let $K_n$ be Bregman $K$-mapping generated by $T_1, T_2, T_3, \ldots, T_N$ in (80). Then

(i) $D_f(x^*, K_n x) \leq (1 + t_n) D_f(x^*, x)$ for all $x^* \in F(K_n)$ and $x \in C$, where $t_n \rightarrow 0$ as $n \rightarrow \infty$;

(ii) $F(K_n) = \bigcap_{i=1}^N F(T_i)$;

(iii) $K_n$ is a closed mapping.

Proof. Let $x^* \in F(K_n)$ and $x \in C$. Then by Lemma 13 (ii) and inequality (16) we have

$$D_f(x^*, K_n x) = D_f(x^*, P_C^f(\nabla f^* (\alpha_{nN} \nabla f (T_n^a S_{n-1,N-1} x) + (1 - \alpha_{nN}) \nabla f (S_{n-1,N-1} x)))$$

$$\leq D_f(x^*, \nabla f^* (\alpha_{nN} \nabla f (T_n^a S_{n-1,N-1} x) + (1 - \alpha_{nN}) \nabla f (S_{n-1,N-1} x)))$$

$$\leq \alpha_{nN} D_f(x^*, T_n^a S_{n-1,N-1} x) + (1 - \alpha_{nN}) D_f(x^*, S_{n-1,N-1} x)$$

$$\leq \alpha_{nN} (1 + \nu_{nN}) D_f(x^*, S_{n-1,N-1} x) + (1 - \alpha_{nN})$$

$$\cdot D_f(x^*, S_{n-1,N-1} x)$$

$$= (1 + \alpha_{nN} \nu_{nN}) D_f(x^*, S_{n-1,N-1} x)$$

$$= (1 + \alpha_{nN} \nu_{nN}) D_f(x^*, \nabla f^* (\alpha_{nN} \nabla f (T_n^a S_{n-1,N-1} x) + (1 - \alpha_{nN}) \nabla f (S_{n-1,N-1} x)))$$

$$\leq (1 + \alpha_{nN} \nu_{nN}) [\alpha_{nN-1} D_f(x^*, T_n^a S_{n-1,N-1} x) + (1 - \alpha_{nN}) D_f(x^*, S_{n-2,N-2} x)]$$

$$\leq (1 + \alpha_{nN} \nu_{nN}) [\alpha_{nN-1} (1 + \nu_{nN-1}) D_f(x^*, S_{n-2,N-2} x) + (1 - \alpha_{nN})]$$

$$\cdot D_f(x^*, S_{n-2,N-2} x)$$

$$\vdots$$

$$\leq (1 + \alpha_{nN} \nu_{nN}) (1 + \alpha_{nN-1} \nu_{nN-1}) \cdots (1 + \nu_{n2} \nu_{n2})$$

$$\cdot D_f(x^*, S_{n1,n1} x).$$

Hence,

$$D_f(x^*, K_n x) \leq (1 + \alpha_{nN} \nu_{nN})$$

$$(1 + \alpha_{nN-1} \nu_{nN-1}) \cdots (1 + \alpha_{n2} \nu_{n2})$$

$$\cdot D_f(x^*, \nabla f^* (\alpha_{nN} \nabla f (T_n^a x) + (1 - \alpha_{nN}) \nabla f (x)))$$

$$\leq (1 + \alpha_{nN} \nu_{nN})$$

$$(1 + \alpha_{nN-1} \nu_{nN-1}) \cdots (1 + \alpha_{n1} \nu_{n1}) D_f(x^*, x)$$

$$= \prod_{i=1}^N (1 + \alpha_{n1} \nu_{n1}) D_f(x^*, x).$$

Observe that $\alpha_{n1} \nu_{n1} \rightarrow 0$ as $n \rightarrow \infty$ for all $i \in \{1, 2, 3, \ldots, N\}$. Let $\prod_{i=1}^N (1 + \alpha_{n1} \nu_{n1}) = (1 + t_n)$; then $t_n \rightarrow 0$ as $n \rightarrow \infty$ and

$$D_f(x^*, K_n x) \leq (1 + t_n) D_f(x^*, x).$$

Next we show that $F(K_n) = \bigcap_{i=1}^N F(T_i)$. It is obvious that $\bigcap_{i=1}^N F(T_i) \subseteq F(K_n)$. Now let $z \in F(K_n)$ and $x^*$ be any point in $\bigcap_{i=1}^N F(T_i)$. Let $r_i = \sup \{\|\nabla f(z)\|, \|\nabla f(S_{n1,N} z)\|, \|\nabla f(T_n^a S_{n1,N} z)\| \}$ for $1 \leq i \leq N$.

$$D_f(x^*, z) = D_f(x^*, K_n z) = D_f(x^*, $$

$$P_C^f(\nabla f^* (\alpha_{nN} \nabla f (T_n^a S_{n1,N} z)$$

$$+ (1 - \alpha_{nN}) \nabla f (S_{n1,N} z))) \leq D_f(x^*,$$

$$\nabla f^* (\alpha_{nN} \nabla f (T_n^a S_{n1,N} z)$$

$$+ (1 - \alpha_{nN}) \nabla f (S_{n1,N} z))) = V_f(x^*, \alpha_{nN} \nabla f (T_n^a S_{n1,N} z)$$

$$+ (1 - \alpha_{nN}) \nabla f (S_{n1,N} z))$$

$$\cdot (S_{n1,N} z)) + \alpha_{nN} \nabla f (T_n^a S_{n1,N} z)$$

$$+ \alpha_{nN} \nabla f (T_n^a S_{n1,N} z)$$

$$+ (1 - \alpha_{nN}) \nabla f (S_{n1,N} z))$$

$$\cdot (S_{n1,N} z).$$

Since $f$ is uniformly Fréchet differentiable function which is bounded on bounded subsets of $E$, then by Lemma 10 $\nabla f$ is uniformly continuous on bounded subsets and consequently from Lemma 11 $f^*$ is uniformly convex. Therefore in view of Lemma 9 we have

$$D_f(x^*, z) \leq f(x^*) - \langle x^*, \alpha_{nN} \nabla f (T_n^a S_{n1,N} z)$$

$$+ (1 - \alpha_{nN}) \nabla f (S_{n1,N} z)$$

$$+ \alpha_{nN} f^* (\nabla f (T_n^a S_{n1,N} z)) + (1 - \alpha_{nN})$$

$$\cdot f^* (\nabla f (S_{n1,N} z)) - \alpha_{nN} (1 - \alpha_{nN})$$

$$\cdot p_{r_1} (\|\nabla f (T_n^a S_{n1,N} z) - \nabla f (S_{n1,N} z)\|)$$

$$\cdot (T_n^a S_{n1,N} z) - \nabla f (S_{n1,N} z))$$

$$\cdot (S_{n1,N} z).$$


\begin{align*}
&= \alpha_{n,N} (f(x^*) - \langle x^*, \nabla f (T^n_S S_{n-1} z) \rangle) \\
&+ f^* (\nabla f (T^n_S S_{n-1} z))) + (1 - \alpha_{n,N}) (f (x^*) \\
&- \langle x^*, \nabla f (S_{n-1} z) \rangle) + f^* (\nabla f (S_{n-1} z))) \\
&- \alpha_{n,N} (1 - \alpha_{n,N}) \\
&\cdot \rho_{1}^* (\| \nabla f (T^n_S S_{n-1} z) - \nabla f (S_{n-1} z) \|) \\
&= \alpha_{n,N} V_f (x^*, \nabla f (T^n_S S_{n-1} z)) + (1 - \alpha_{n,N}) \\
&\cdot V_f (x^*, \nabla f (S_{n-1} z)) - \alpha_{n,N} (1 - \alpha_{n,N}) \\
&\cdot \rho_{1}^* (\| \nabla f (T^n_S S_{n-1} z) - \nabla f (S_{n-1} z) \|) \\
&= \alpha_{n,N} D_f (x^*, T^n_S S_{n-1} z) + (1 - \alpha_{n,N}) \\
&\cdot D_f (x^*, S_{n-1} z) - \alpha_{n,N} (1 - \alpha_{n,N}) \\
&\cdot \rho_{1}^* (\| \nabla f (T^n_S S_{n-1} z) - \nabla f (S_{n-1} z) \|) \leq \alpha_{n,N} (1 + v_{n,N}) D_f (x^*, S_{n-1} z) + (1 - \alpha_{n,N}) \\
&\cdot D_f (x^*, S_{n-1} z) - \alpha_{n,N} (1 - \alpha_{n,N}) \\
&\cdot \rho_{1}^* (\| \nabla f (T^n_S S_{n-1} z) - \nabla f (S_{n-1} z) \|) = (1 + \alpha_{n,N} v_{n,N}) D_f (x^*, S_{n-1} z) - \alpha_{n,N} (1 - \alpha_{n,N}) \\
&\cdot \rho_{1}^* (\| \nabla f (T^n_S S_{n-1} z) - \nabla f (S_{n-1} z) \|).
\end{align*}

(85)

Continuing in this fashion we obtain

\begin{align*}
D_f (x^*, z) &\leq (1 + \alpha_{n,N} v_{n,N}) \\
&\cdot (1 + \alpha_{n,N-1} v_{n,N-1}) \cdots (1 + \alpha_{n,1} v_{n,1}) D_f (x^*, z) \\
&- (1 + \alpha_{n,N} v_{n,N}) \\
&\cdot (1 + \alpha_{n,N-1} v_{n,N-1}) \cdots (1 + \alpha_{n,2} v_{n,2}) \alpha_{n,1} (1 - \alpha_{n,1}) \\
&\cdot \rho_{1}^* (\| \nabla f (T^n_{1} z) - \nabla f (z) \|) - (1 + \alpha_{n,N} v_{n,N}) \\
&\cdot (1 + \alpha_{n,N-1} v_{n,N-1}) \cdots (1 + \alpha_{n,3} v_{n,3}) \alpha_{n,2} (1 - \alpha_{n,2}) \\
&\cdot \rho_{1}^* (\| \nabla f (T^n_{2} S_{n-1} z) - \nabla f (S_{n-1} z) \|) \\
&- (1 + \alpha_{n,N} v_{n,N}) \\
&\cdot (1 + \alpha_{n,N-1} v_{n,N-1}) \cdots (1 + \alpha_{n,4} v_{n,4}) \alpha_{n,3} (1 - \alpha_{n,3}) \\
&\cdot \rho_{1}^* (\| \nabla f (T^n_{3} S_{n-1} z) - \nabla f (S_{n-1} z) \|) \\
&\vdots \\
&- (1 + \alpha_{n,N} v_{n,N}) \alpha_{n,N-1} (1 - \alpha_{n,N-1}) \\
&\cdot \rho_{1}^* ((\| \nabla f (T^n_{N-1} S_{n-2} S_{n-N} z) - \nabla f (S_{n-2} S_{n-N} z) \|) \\
&- \alpha_{n,N} (1 - \alpha_{n,N}) \\
&\cdot \rho_{1}^* (\| \nabla f (T^n_{N-1} S_{n-2} S_{n-N} z) - \nabla f (S_{n-2} S_{n-N} z) \|).
\end{align*}

(86)

From (86), we have

\begin{align*}
(1 + \alpha_{n,N} v_{n,N}) (1 + \alpha_{n,N-1} v_{n,N-1}) \cdots (1 + \alpha_{n,2} v_{n,2}) \\
&\cdot \alpha_{n,1} (1 - \alpha_{n,1}) \rho_{1}^* (\| \nabla f (T^n_{1} z) - \nabla f (z) \|) \\
&\leq (1 + \alpha_{n,N} v_{n,N}) \\
&\cdot (1 + \alpha_{n,N-1} v_{n,N-1}) \cdots (1 + \alpha_{n,1} v_{n,1}) D_f (x^*, z) \\
&\leq D_f (x^*, z).
\end{align*}

(87)

Since \( \lim \inf \alpha_{n,i} (1 - \alpha_{n,i}) > 0 \), we have

\[
\lim_{n \to \infty} \alpha_{n,i} (1 - \alpha_{n,i}) = 0.
\]

(88)

By the property of \( \rho_{1}^* \), we obtain

\[
\lim_{n \to \infty} \| \nabla f (T^n_{1} z) - \nabla f (z) \| = 0.
\]

(89)

In similar fashion and assuming \( \lim \inf \alpha_{n,i} (1 - \alpha_{n,i}) > 0 \) \( \forall i \in \{2, 3, \ldots, N\} \) we have

\[
\lim_{n \to \infty} \| \nabla f (T^n_{2} S_{n-2} z) - \nabla f (S_{n-2} z) \| = \ldots = \lim_{n \to \infty} \| \nabla f (T^n_{N} S_{n-N} z) - \nabla f (S_{n-N} z) \|. \]

(90)

Since by Lemma 12 \( \nabla f^* \) is uniformly continuous, we obtain from (89) that \( T^n_{1} z \to z \) as \( n \to \infty \). By our assumption, \( T^n_{1} \) is continuous and so \( T^n_{1} z = z \), that is, \( z \in F(T^n_{1}) \). Also from (90), we obtain

\[
T^n_{2} S_{n-2} z = S_{n-1} z \\
T^n_{3} S_{n-3} z = S_{n-2} z \\
\vdots \\
T^n_{N} S_{n-N} z = S_{n-N-1} z.
\]

(91)

On the other hand

\begin{align*}
D_f (z, S_{n-1} z) &\leq D_f (z, \nabla f^* (\alpha_{n,i} v_{n,i} (\nabla f (T^n_{1} x) + (1 - \alpha_{n,i}) \nabla f (x)))) \\
&\leq \alpha_{n,1} D_f (z, T^n_{1} x) + (1 - \alpha_{n,1}) D_f (z, z).
\end{align*}

(92)
Since $T^n_z z = z$ we get

$$D_f(z, S_{n,1} z) = 0$$
and so $S_{n,1} z = z$. (93)

From (91) and (93), we have $z \in F(S_{n,1})$ and $z \in F(T_i)$. Applying the same argument we can show that $z \in F(T_i)$ for $i = 3, 4, \ldots, N$. Thus, it follows that $z \in \bigcap_{i=1}^{N} F(T_i)$.

Next we show that $K_n$ is closed.

From (80)

$$S_{n,1} x = P_f^f (\nabla f^* (\alpha_{n,1} \nabla f (T_i^n x) + (1 - \alpha_{n,1}) \nabla f (x))) .$$

Let $\{x_{m, j}\}$ be a sequence in $C$ such that $x_{m, j} \rightarrow x$ and $S_{n,1} x_{m, j} \rightarrow y$ as $m \rightarrow \infty$. Since $T_i$ is continuous for each $i = 1, 2, 3, \ldots, N$, $\nabla f, \nabla f^*$ are uniformly continuous and applying Lemma 15 we have

$$S_{n,1} x_m \rightarrow S_{n,1} x \quad \text{as} \quad m \rightarrow \infty.$$ (95)

By uniqueness of limit, we get $S_{n,1} x = y$ showing that $S_{n,1}$ is closed. Using (95), we have that $S_{n,1}$ is closed. Continuing in this way we obtain that $K_n$ is closed.$\blacksquare$

Now we prove the strong convergence theorems.

**Theorem 21.** Let $E$ be a reflexive Banach space with the dual $E^*$ and let $C$ be a nonempty, closed, convex, and bounded subset of $E$. Let $f : E \rightarrow (-\infty, +\infty]$ be a strongly coercive, Legendre, uniformly Fréchet differentiable, and totally convex function which is bounded on bounded subsets of $E$. For each $j = 1, 2, 3, \ldots, m$, let $A_j : C \rightarrow E^*$ be a bounded, hemicontinuous and relaxed $\eta$-monotone mappings and $g_j : C \times C \rightarrow \mathbb{R}$ be bifunctions satisfying (C1)–(C4). Let $\psi_j : C \rightarrow \mathbb{R}$ be proper, convex, and lower semicontinuous mappings. Let $\{T_i^n\}_{i=1}^{N}$ be a finite family of continuous Bregman quasiasymptotically nonexpansive mappings of $C$ into itself such that $\mathcal{F} = \bigcap_{j=1}^{j} \{T_i^n\} \cap \{EP(g_j, A_j) \} \neq \emptyset$. Let $K_n$ be the Bregman K-mapping generated by $T_1, T_2, T_3, \ldots, T_N$ in (80). Assume that the conditions of Lemma 18 and the following condition are satisfied:

(v) For all $x, y, z, w \in C$ one has

$$\limsup_{t \to 0^-} \langle A z, \eta (x, t y + (1 - t) w) \rangle \leq \langle A z, \eta (x, w) \rangle .$$ (96)

Let $\{x_n\}$ be iteratively defined as follows:

$$x_0 = x \in C \quad \text{chosen arbitrarily},$$
$$C_{1,j} = C = C_0,$$
$$y_n = \nabla f^* \left( \beta_n \nabla f (x_n) + (1 - \beta_n) \nabla f (K_n x_n) \right),$$
$$u_{n,j} \in C \quad \text{such that}$$
$$g_j \left( u_{n,j}, y \right) + \left\langle A \mu_{n,j}, \eta \left( y, u_{n,j} \right) \right\rangle + \psi_j (y)$$

$$\geq 0 \quad \forall y \in C,$$ (97)

where $\{\beta_n\}$ is a real sequence in $(0, 1)$ satisfying $\liminf_{n \rightarrow \infty} \beta_n (1 - \beta_n) > 0, \{r_n\} \subset [a, \infty)$ for some $a > 0$ and $\theta_n = (1 - \beta_n) r_n \sup_{p \in \mathcal{F}} D_f(p, x_n)$.

Then $\{x_n\}$ converges strongly to $u = P_{\mathcal{F}} x_0$.

**Proof.** It follows from Lemmas 20(ii) and 14 that $F(K_n)$ is closed and convex. On the other hand by Lemma 18(5) $EP(g_j, A_j)$ is closed and convex for each $j = 1, 2, 3, \ldots, m$; consequently $\mathcal{F}$ is closed and convex.

Next we prove $C_n$ is closed and convex. The proof is by induction.

For any fixed $j$, $C_{1,j} = C$ is closed and convex which implies $C_1$ is closed and convex.

Now suppose $C_{k,j}$ is closed and convex for some $k \in \mathbb{N}$. Observe that

$$D_f \left( x_n, u_{n,j} \right) \leq D_f \left( x_n, u_{n,j+1} \right) + \theta_n$$

$$\leq \left\langle \nabla f (x_n) - \nabla f \left( u_{n,j+1} \right), z \right\rangle$$

$$\leq \left\langle f (u_{n,j}) - f (x_n) + \left\langle \nabla f (u_{n,j}) - \nabla f (x_n), u_{n,j} \right\rangle \right\rangle + \left\langle \nabla f (x_n) - x_n - u_{n,j} \right\rangle + \theta_n .$$ (98)

It follows that $C_{k+1,j}$ is closed and convex and so $C_{k+1,j}$ is closed and convex. Hence $C_n$ is closed and convex for all $n \geq 0$. This implies that the iterative sequence is well defined.

Next we show that $\mathcal{F} \subset C_n \forall n \geq 0$. Obviously $\mathcal{F} \subset C_{k,j}$ for some $k \in \mathbb{N}$ and $\forall j = 1, 2, \ldots, m$. Let $x^* \in \mathcal{F}$; then $x^* \in C_{k,j} \forall j = 1, 2, \ldots, m$ and so $x^* \in C_k$.

Observe from scheme (97) that $u_{n,j} \in T^-_{i,n} y_n$. Therefore by Lemma 18(4) we have

$$D_f \left( x^*, u_{n,j} \right) = D_f \left( x^*, T^-_{i,n} y_n \right) \leq D_f \left( x^*, y_n \right).$$ (99)

But

$$D_f \left( x^*, y_n \right)$$
$$= D_f \left( x^*, \nabla f^* (\beta_n \nabla f (x_n) + (1 - \beta_n) \nabla f (K_n x_n)) \right)$$
$$= V_f \left( x^*, \beta_n \nabla f (x_n) + (1 - \beta_n) \nabla f (K_n x_n) \right)$$
$$= f (x^*) - \left\langle x^*, \beta_n \nabla f (x_n) + (1 - \beta_n) \nabla f (K_n x_n) \right\rangle$$
$$+ f^* \left( \beta_n \nabla f (x_n) + (1 - \beta_n) \nabla f (K_n x_n) \right).$$
\[ \leq f(x^*) - \beta_n (x^*, \nabla f(x_n)) - (1 - \beta_n) \langle x^*, \nabla f(K_n x_n) \rangle + \beta_n f^*(\nabla f(x_n)) + (1 - \beta_n) D_f(x^*, K_n x_n) - \beta_n (1 - \beta_n) \rho^* \frac{\|\nabla f(x_n) - \nabla f(K_n x_n)\|}{r_2}. \]  

(112)

Hence
\[ D_f(x_m, x_n) \to 0 \quad \text{as } n, m \to \infty. \]  

(108)

It follows that
\[ \|x_m - x_n\| \to 0 \quad \text{as } n, m \to \infty. \]  

(109)

This shows that \( \{x_n\} \) is Cauchy sequence in \( E \). Since \( E \) is reflexive and \( C \) is weakly closed, then there exists \( u \in C \) such that
\[ \lim_{n \to \infty} \|x_n - u\| = 0. \]  

(110)

Observe that \( x_{n+1} \in C_{n+1} = \bigcap_{j=1}^m C_{n+1, j}, \forall j \in \{1, 2, \ldots, m\} \). Hence we obtain
\[ D_f(x_{n+1}, u_{n,j}) \leq D_f(x_{n+1}, x_n) + \theta_n. \]  

(111)

By (105) and the fact that \( \theta_n \to 0 \) as \( n \to \infty \), we get
\[ \lim_{n \to \infty} D_f(x_{n+1}, u_{n,j}) = 0 \quad \forall j \in \{1, 2, \ldots, m\}, \]  

and so
\[ \lim_{n \to \infty} \|u_{n,j} - x_n\| = 0, \quad \forall j \in \{1, 2, \ldots, m\}. \]  

(113)

Now
\[ \|u_{n,j} - x_n\| \leq \|u_{n,j} - x_{n+1}\| + \|x_{n+1} - x_n\|. \]  

(114)

Therefore using (106) and (113) we obtain
\[ \lim_{n \to \infty} \|u_{n,j} - x_n\| = 0, \quad \forall j \in \{1, 2, \ldots, m\}. \]  

(115)

Also by (110) and (115), it follows that
\[ \lim_{n \to \infty} \|u_{n,j} - u\| = 0, \quad \forall j \in \{1, 2, \ldots, m\}. \]  

(116)

Let \( r_2 = \sup\{\|\nabla f(x_n)\|, \|\nabla f(K_n x_n)\|\} \). In view of Lemma 9, we have
\[
D_f(x^*, y_n) = D_f(x^*, \nabla f(x_n) + (1 - \beta_n) \nabla f(K_n x_n)) \]
\[
= V_f(x^*, \beta_n \nabla f(x_n) + (1 - \beta_n) \nabla f(K_n x_n)) + f^*(\beta_n \nabla f(x_n) + (1 - \beta_n) \nabla f(K_n x_n)) \]
\[
+ f^*(\beta_n \nabla f(x_n) + (1 - \beta_n) \nabla f(K_n x_n)) \]
\[
\leq \beta_n V_f(x^*, \nabla f(x_n)) + (1 - \beta_n) V_f(x^*, \nabla f(K_n x_n)) \]
\[
- \beta_n (1 - \beta_n) \rho^* \frac{\|\nabla f(x_n) - \nabla f(K_n x_n)\|}{r_2}. \]  

(117)
From (99), we have
\[
D_f (x^*, u_{n,j}) \leq \beta_n D_f (x^*, x_n) + (1 - \beta_n) D_f (x^*, K_n x_n)
- \beta_n (1 - \beta_n) \rho^* \left( \| \nabla f (x_n) - \nabla f (K_n x_n) \| \right).
\]
(118)
Therefore by Lemma 20(i), we have
\[
D_f (x^*, u_{n,j}) \
\leq \beta_n D_f (x^*, x_n) + (1 - \beta_n) D_f (x^*, K_n x_n) \
- \beta_n (1 - \beta_n) \rho^* \left( \| \nabla f (x_n) - \nabla f (K_n x_n) \| \right) \
\leq D_f (x^*, x_n) + (1 - \beta_n) t_n D_f (x^*, x_n) \
- \beta_n (1 - \beta_n) \rho^* \left( \| \nabla f (x_n) - \nabla f (K_n x_n) \| \right) \
\leq D_f (x^*, x_n) + (1 - \beta_n) t_n \sup_{p \in E} D_f (p, x_n) \
- \beta_n (1 - \beta_n) \rho^* \left( \| \nabla f (x_n) - \nabla f (K_n x_n) \| \right) \
= D_f (x^*, x_n) + \theta_n \sup_{p \in E} D_f (p, x_n) \
- \beta_n (1 - \beta_n) \rho^* \left( \| \nabla f (x_n) - \nabla f (K_n x_n) \| \right).
\]
(119)
Observe that
\[
D_f (x^*, x_n) - D_f (x^*, u_{n,j}) = f (u_{n,j}) - f (x_n) + \langle \nabla f (u_{n,j}), x^* - u_{n,j} \rangle \
- \langle \nabla f (x_n), x^* - x_n \rangle \
= f (u_{n,j}) - f (x_n) + \langle \nabla f (u_{n,j}), x^* - x_n \rangle \
+ \langle \nabla f (u_{n,j}), x_n - u_{n,j} \rangle - \langle \nabla f (x_n), x^* - x_n \rangle \
= f (u_{n,j}) - f (x_n) \
+ \langle \nabla f (u_{n,j}) - \nabla f (x_n), x^* - x_n \rangle \
+ \langle \nabla f (u_{n,j}), x_n - u_{n,j} \rangle.
\]
Therefore
\[
\left| D_f (x^*, x_n) - D_f (x^*, u_{n,j}) \right| \
\leq \left| f (u_{n,j}) - f (x_n) \right| \
+ \left| \langle \nabla f (u_{n,j}) - \nabla f (x_n), x^* - x_n \rangle \right| \
+ \left| \langle \nabla f (u_{n,j}), x_n - u_{n,j} \rangle \right| \
\leq \left| f (u_{n,j}) - f (x_n) \right| \
+ \| \nabla f (u_{n,j}) - \nabla f (x_n) \| \| x^* - x_n \| \
+ \| f (u_{n,j}) \| \| x_n - u_{n,j} \|.
\]
(121)
From (115), we obtain
\[
\left| D_f (x^*, x_n) - D_f (x^*, u_{n,j}) \right| \rightarrow 0 \\
\text{as } n \rightarrow \infty \forall j \in \{1, 2, \ldots, m\}.
\]
(122)
By (122) and \( \liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0 \), it follows from (119) that
\[
\lim_{n \to \infty} \rho^* \left( \| \nabla f (x_n) - \nabla f (K_n x_n) \| \right) = 0.
\]
(123)
From the property of \( \rho^* \), we deduce \( \lim_{n \to \infty} \| \nabla f (x_n) - \nabla f (K_n x_n) \| = 0 \). By uniform continuity of \( \nabla f^* \) on bounded subsets of \( E \), we obtain
\[
\lim_{n \to \infty} \| x_n - K_n x_n \| = 0.
\]
(124)
On the other hand
\[
\| K_n x_n - u \| \leq \| K_n x_n - x_n \| + \| x_n - u \|.
\]
(125)
From (110) and (124), we have
\[
\lim_{n \to \infty} \| K_n x_n - u \| = 0.
\]
(126)
Now using (110), (126), and Lemma 20(iii), we obtain \( K_n u = u \forall n \in \mathbb{N} \). This implies \( u \in F(K_n) = \bigcap_{i=1}^{N} F(T_i) \).

Next we show \( u \in EP(g_j, A_j) \).

From scheme (97)
\[
\| \nabla f (x_n) - \nabla f (y_n) \| = (1 - \beta_n) \| \nabla f (x_n) - \nabla f (K_n x_n) \|.
\]
(127)
Therefore by (124) and uniform continuity of \( \nabla f \), we have
\[
\lim_{n \to \infty} \| \nabla f (x_n) - \nabla f (y_n) \| = 0.
\]
(128)
As \( \nabla f^* \) is uniformly continuous, we obtain
\[
\lim_{n \to \infty} \| x_n - y_n \| = 0.
\]
(129)
From (115) and (129), we have
\[
\lim_{n \to \infty} \| u_{n,j} - y_n \| = 0 \quad \forall j = 1, 2, \ldots, m.
\]
(130)
Again since \( \nabla f \) is uniformly continuous and \( r_n > a \), we get
\[
\lim_{n \to \infty} \frac{\| \nabla f (u_{n,j}) - \nabla f (y_n) \|}{r_n} = 0 \quad \forall j = 1, 2, \ldots, m.
\]
(131)
From scheme (97)
\[
g_j (u_{n,j}, y) + \left< A_j u_{n,j}, \eta (y, u_{n,j}) \right> + \psi_j (y) \\
- \psi_j (u_{n,j}) \\
+ \frac{1}{r_n} \left< \nabla f (u_{n,j}) - \nabla f (y_n), y - u_{n,j} \right> \geq 0
\]
\( \forall y \in C. \)
Using (C2) and Lemma 17(ii), it follows that
\[
\frac{1}{r_n} \left\| \nabla f(u_{n,j}) - \nabla f(y_n) \right\| u_{n,j} - y \geq 0
\]
\[
\geq \left\langle A_j u_{n,j}, \eta (u_{n,j}, y) \right\rangle + \psi_j(u_{n,j}) - \psi_j(y) - g_j(u_{n,j}, y) \quad \forall y \in C
\]
\[
\geq \left\langle A_j u_{n,j}, \eta (u_{n,j}, y) \right\rangle + \psi_j(u_{n,j}) - \psi_j(y) + g_j(y, u_{n,j}) \quad \forall y \in C.
\]
Using (116) and (131) and taking \( \lim \inf \) as \( n \to \infty \) of the above inequality, we get
\[
0 \geq \left\langle A_j u, \eta (u, y_j) \right\rangle + \psi_j(u) - \psi_j(y) + g_j(y, u) \quad \forall y \in C \forall j = 1, 2, \ldots, m.
\]
(144)

Now for any \( t \in (0, 1) \) and \( y \in C \), let \( y_t = ty + (1 - t)u \). Then \( y_t \in C \) and so
\[
0 \geq \left\langle A_j u, \eta (u, y_t) \right\rangle + \psi_j(u) - \psi_j(y_t) + g_j(y_t, u)
\]
\[
\forall j = 1, 2, \ldots, m.
\]
(135)

Therefore by (C1), (C4), Lemma 18(i), (ii), and (135), we have
\[
0 = g_j(y_t, y) + \left\langle A_j u, \eta (u, y_t) \right\rangle + \psi_j(y_t)
\]
\[
- \psi_j(y_t) = g_j(y_t, ty + (1 - t)u)
\]
\[
+ \left\langle A_j u, \eta (ty + (1 - t)u, y_t) \right\rangle + \psi_j(ty + (1 - t)u) + (1 - t)u - \psi_j(y_t) \leq t \left( g_j(y_t, y) \right)
\]
\[
+ \left\langle A_j u, \eta (y_t, y) \right\rangle + \psi_j(y_t) - \psi_j(y_t) + (1 - t)
\]
\[
\cdot \left[ g_j(y, u) + \left\langle A_j u, \eta (u, y) \right\rangle + \psi_j(u) - \psi_j(y_t) \right]\]
\[
- \psi_j(y_t) \leq t \left[ g_j(y, u) + \left\langle A_j u, \eta (y, y_t) \right\rangle + \psi_j(u) - \psi_j(y_t) \right] + (1 - t)
\]
\[
\forall j \geq 1, 2, \ldots, m.
\]
(136)

That is,
\[
g_j(y_t, y) + \left\langle A_j u, \eta (y, y_t) \right\rangle + \psi_j(y_t) - \psi_j(y_t) \geq 0.
\]
(137)

Since \( y_t = ty + (1 - t)u \), we have
\[
g_j(ty + (1 - t)u, y) + \left\langle A_j u, \eta (y, ty + (1 - t)u) \right\rangle + \psi_j(y) - \psi_j(y_t) \geq 0.
\]
(138)

From (C3), (v), and lower semicontinuity of \( \psi \), we have by allowing \( t \to 0^+ \)
\[
g_j(u, y) + \left\langle A_j u, \eta (y, u) \right\rangle + \psi_j(y) - \psi_j(u) \geq 0 \quad \forall y \in C.
\]
(139)

This shows that \( u \in \text{EP}(g_j, A_j) \).

Lastly we show \( u = P_{C^*} x_0 \).

From \( x_n = P_{C^*} x_0 \) and Lemma 13(i) we have
\[
\langle \nabla f(x_0) - \nabla f(x_n), x_n - z \rangle \geq 0 \quad \forall z \in C_n.
\]
(140)

Since \( C \subset C_n \), this implies that
\[
\langle \nabla f(x_0) - \nabla f(x_n), x_n - h \rangle \geq 0 \quad \forall h \in C.
\]
(141)

Letting \( n \to \infty \) in (141), we obtain
\[
\langle \nabla f(x_0) - \nabla f(x_n), x_n - h \rangle \geq 0 \quad \forall h \in C.
\]
(142)

Again by Lemma 13(i) we have \( u = P_{C^*} x_0 \). This completes the proof. \( \square \)

If \( N = 1 \) and \( m = 1 \), in Theorem 21 then we have the following corollary.

**Corollary 22.** Let \( E \) be a reflexive Banach space with the dual \( E^* \) and let \( C \) be a nonempty, closed, convex, and bounded subset of \( E \). Let \( f : E \to (-\infty, +\infty] \) be a strongly coercive, Legendre, uniformly Fréchet differentiable, and totally convex function which is bounded on bounded subsets of \( E \). Let \( A : C \to E^* \) be \( \eta \)-hemicontinuous and relaxed \( \eta \)-monotone mapping and \( g : C \times C \to \mathbb{R} \) be bifunctions satisfying (C1)-(C4). Let \( \psi : C \to \mathbb{R} \) be a proper, convex, and lower semicontinuous mapping. Let \( T \) be a continuous Bregman quasiasymptotically nonexpansive mapping of \( C \) into itself such that \( \mathcal{F} = F(T) \cap \text{EP}(g, A) \neq \emptyset \).

Assume that the conditions of Lemma 18 and the following condition are satisfied:

(v) For all \( x, y, z, w \in C \) one has
\[
\limsup_{t \to 0^+} \langle Az, \eta (x, ty + (1 - t)w) \rangle \leq \langle Az, \eta (x, w) \rangle.
\]
(143)

Let \( \{x_n\} \) be iteratively defined as follows:
\[
x_0 = x \in C \quad \text{chosen arbitrarily},
\]
\[
y_n = \nabla f^* (\beta_n \nabla f(x_n) + (1 - \beta_n) \nabla f(T^n x_n)),
\]
\[
u_n \in C \quad \text{such that}
\]
\[
g(u_n, y) + \left\langle Au_n, \eta (y, u_n) \right\rangle + \psi(y) - \psi(u_n)
\]
\[
+ \frac{1}{r_n} \left\langle \nabla f(u_n) - \nabla f(y_n), y - u_n \right\rangle \geq 0 \quad \forall y \in C,
\]
\[
C_{n+1} = \{z \in C_n : D_f(z, u_n) \leq D_f(z, x_n) + \theta_n\},
\]
\[
x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 0,
\]
where \( \{ \beta_n \} \) is a real sequence in \((0,1)\) satisfying
\[
\liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0 ,
\]
and \( \theta_n = (1 - \beta_n) \tau_n \sup_{p \in \mathbb{R}} D_j(p, x_n) \). Then \( \{ x_n \} \) converges strongly to \( u = p_{x_0}^l \).

Setting \( A_j = 0 \) \( \forall j = 1, 2, \ldots, m \) in Theorem 21 then we obtain the following result.

**Corollary 23.** Let \( E \) be a reflexive Banach space with the dual \( E^* \) and let \( C \) be a nonempty, closed, convex, and bounded subset of \( E \). Let \( f : E \to (\infty, +\infty) \) be a strongly coercive, Legendre, uniformly Fréchet differentiable, and totally convex function which is bounded on bounded subsets \( \mathcal{F} = \{ F_{\beta_n} \}_{n \in \mathbb{N}} \). Assume that \( \mathcal{F} \neq \emptyset \). Let \( K_n \) be the Bregman K-mapping generated by \( T_1, T_2, \ldots, T_N \) in (80). Let \( \{ x_n \} \) be iteratively defined as follows:

\[
x_0 = x \in C \quad \text{chosen arbitrarily},
\]

\[
C_{1,j} = C = C_0
\]

\[
y_n = \nabla f^* (\beta_n \nabla f (x_n) + (1 - \beta_n) \nabla f (K_n x_n) ),
\]

\[
u_{n,j} \in C \quad \text{such that}
\]

\[
g_j (u_{n,j}, y) + \psi_j (y) - \psi_j (u_{n,j})
\]

\[
\geq 0
\]

\[
\forall y \in \mathcal{F}
\]

\[
C_{n+1,j} = \left\{ z \in C_n : D_f (z, u_{n,j}) \leq D_f (z, x_n) + \theta_n \right\}
\]

\[
C_{m+1} = \bigcap_{j=1}^{m} C_{n+1,j}
\]

\[
x_{n+1} = P_{C_{m+1}}^l x_0 , \quad n \geq 0,
\]

where \( \{ \beta_n \} \) is a real sequence in \((0,1)\) satisfying
\[
\liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0 , \quad \tau_n \in [a, \infty) \quad \text{for some} \quad a > 0
\]

and \( \theta_n = (1 - \beta_n) \tau_n \sup_{p \in \mathbb{R}} D_j(p, x_n) \). Then \( \{ x_n \} \) converges strongly to \( u = p_{x_0}^l \).

If \( A_j = 0 \) and \( \psi_j = 0 \) \( \forall j = 1, 2, \ldots, m \) in Theorem 21 then we obtain the following result.

**Corollary 24.** Let \( E \) be a reflexive Banach space with the dual \( E^* \) and let \( C \) be a nonempty, closed, convex, and bounded subset of \( E \). Let \( f : E \to (-\infty, +\infty) \) be a strongly coercive, Legendre, uniformly Fréchet differentiable, and totally convex function which is bounded on bounded subsets \( \mathcal{F} = \{ F_{\beta_n} \}_{n \in \mathbb{N}} \). Assume that \( \mathcal{F} \neq \emptyset \). Let \( K_n \) be the Bregman K-mapping generated by \( T_1, T_2, \ldots, T_N \) in (80). Assume that the conditions of Lemma 18 and the following condition are satisfied:

\[
\text{(v)} \quad \text{For all} \quad x, y, z, w \in C \quad \text{one has}
\]

\[
\limsup_{t \to 0^+} \langle Az, \eta (x, ty + (1-t)w) \rangle \leq \langle Az, \eta (x, w) \rangle.
\]
Let \( \{x_n\} \) be iteratively defined as follows:
\[
x_0 = x \in C \text{ chosen arbitrarily,}
\]
\[
C_{1,j} = C = C_0
\]
\[
y_n = J^{-1} \left( \beta_n J(x_n) + (1 - \beta_n) J(K_n x_n) \right),
\]
\[
u_{n,j} \in C \text{ such that}
\]
\[
g_j (u_{n,j}, y) + \left( A_j u_{n,j}, \eta(y, u_{n,j}) \right) + \psi_j(y)
- \psi_j(u_{n,j}) + \frac{1}{r_n} \left( J(u_{n,j}) - J(y_n), y - u_{n,j} \right) \geq 0 \quad (148)
\]

\[
C_{n+1,j} = \{ z \in C_n : \phi(z, u_{n,j}) \leq \phi(z, x_n) + \theta_n \},
\]
\[
C_{n+1} = \bigcap_{j=1}^m C_{n+1,j},
\]
\[
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 0,
\]
where \( \{\beta_n\} \) is a real sequence in \( (0, 1) \) satisfying
\[
\liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0, \quad \{r_n\} \subset [a, \infty) \text{ for some } a > 0
\]
and \( \theta_n = (1 - \beta_n) r_n \sup_{p \in \mathbb{R}} D J(p, x_n). \) Then \( \{x_n\} \) converges
strongly to \( u = P_{\mathbb{F} X_0}. \)

**Remark 26.** Our theorems and corollaries generalise the
main theorem of Chen et al. [27] in the following senses:

(1) For the structure of Banach spaces, we extend the
duality mapping to more general case: that is, a
Legendre, strongly coercive, uniformly Fréchet differ-
entiable, and totally convex function.

(2) For the mapping, we consider Bregman quasiasympto-
totically nonexpansive mappings which contain Breg-
man quasi nonexpansive mappings as a special case
which itself is generalisation of quasi-\( \phi \)-nonexpansive
mappings.

(3) In Chen et al. [27] the authors considered mixed
equilibrium problems while in this paper a system of
equilibrium problems is considered.

**Competing Interests**
The authors declare that they have no competing interests.

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