Research Article
Matrix Quasinorms Induced by Maximal and Minimal Vector Norms

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Received 6 July 2016; Accepted 28 September 2016

1. Introduction

The standard Euclidean norm in \( \mathbb{R}^n \) (or \( \mathbb{C}^n \)) is

\[
\|x\| := \sqrt{\sum_{j=1}^{n} |x_j|^2},
\]

where \( x = (x_1, \ldots, x_n) \) \( \in \mathbb{R}^n \) (or \( \mathbb{C}^n \)). We could easily extend this vector norm to matrices, just by taking a matrix \( Z = (z_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \) as a vector in \( \mathbb{R}^{mn} \) (or \( \mathbb{C}^{mn} \)). This natural extension is called the Frobenius norm \( \|Z\|_F \) (also called Hilbert-Schmidt norm or Schur norm) defined by

\[
\|Z\|_F = \sqrt{\sum_{j=1}^{n} \sum_{k=1}^{m} |z_{jk}|^2} = \sqrt{\text{tr}(ZZ^T)}.
\]

In \( \mathbb{C}^n \), there are another two well-known norms \( L(\cdot) \) and \( N^*(\cdot) \), called the maximal norm and the minimal norm introduced by Siciak [1] and Hahn-Pflug [2], respectively. For \( z \in \mathbb{C}^n \), the explicit forms of \( L(z) \) and \( N^*(z) \) are given by

\[
L(z) = \sqrt{\|z\|^2 + \|z\|^4 - |z \cdot z|^2},
\]

\[
N^*(z) = \sqrt{\frac{1}{2} \left( \|z\|^2 + |z \cdot z| \right)}.
\]

where \( z \cdot w = \sum_{j=1}^{n} z_j w_j \) for \( z, w \in \mathbb{C}^n \).

It is known that \( L(z) \) and \( N^*(z) \) have the following properties [1–3]:

(i) \( L(\cdot) \) and \( N^*(\cdot) \) are norms.

(ii) \( N^*(z) = L(z) = \|z\| \) for all \( z \in \mathbb{R}^n \).

(iii) \( N^*(z) \leq \|z\| \leq L(z) \) for all \( z \in \mathbb{C}^n \).

(iv) They are dual in the sense that \( |z \cdot w| \leq L(z)N^*(w) \) for all \( z, w \in \mathbb{C}^n \).

In fact, \( L(z) \) and \( N^*(z) \) are maximal and minimal norms satisfying (ii) and (iii). The Bergman kernel for the ball \( \mathcal{B}_z = \{ z \in \mathbb{C}^n : N^*_z(z) < 1 \} \) was computed explicitly in [4]. Moreover, recently many papers deal with function theoretic problems on \( \mathcal{B}_z \) in [5–9].

In this paper, similarly to the Frobenius norm, we extend \( L(\cdot) \) and \( N^*(\cdot) \) to complex matrices satisfying (ii), (iii), and (iv). At first, we show that these two extensions are quasinorms (see Theorem 4). The second result is the duality of \( L(\cdot) \) and \( N^*(\cdot) \) like Hölder inequality (see Theorem 5). Also we construct \( N_p(Z) \) for \( p \geq 1 \). If \( p = 1 \), then \( N_1(Z) = N^*(Z) \) and \( \lim_{p \to 1} N_p(Z) = L(Z) \). Finally, we proved the dual relation between \( N_p \) and \( N_q \) when \( 1/p + 1/q = 1 \) (see Corollary 6 and Theorem 7).
2. Statements of Main Results

In 1981, Siciak [1] found the existence of complex maximal extension of $\|\cdot\|$ called the Lie norm $L(z)$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. The maximal norm $L(z)$ satisfies

(i) $L(x) = \|x\|$ for $x \in \mathbb{R}^n$,

(ii) $N(z) \leq L(z)$ for any complex norm $N(z)$ with $N(x) = \|x\|$ for $x \in \mathbb{R}^n$.

In addition, the explicit form of $L(z)$ is known as (3). In fact, the ball $\{z \in \mathbb{C}^n : L(z) < 1\}$ is called the Lie ball which is a classical bounded symmetric domain of type IV.

For the minimal complex extension, it is known that such a norm does not exist, since there is a sequence of complex extensions of $\|\cdot\|$ which converges to 0 at certain points. One can see a counterexample in [2]. In 1988, Hahn and Pflug [2] constructed the complex minimal extension of $\|\cdot\|$ called the minimal norm $N^*(z)$ in a slightly different sense as follows:

(i) $N^*(x) = \|x\|$ for $x \in \mathbb{R}^n$,

(ii) $N^*(z) \leq N(z)$ for any complex norm $N(z)$ with $N(x) = \|x\|$ for $x \in \mathbb{R}^n$ and $N(z) \leq \|z\|$ for $z \in \mathbb{C}^n$.

Moreover, the explicit form of $N^*(z)$ is known as (4). The Bergman kernel for the ball $\{z \in \mathbb{C}^n : N^*(z) < 1\}$ was computed explicitly in [4].

In [3], Morimoto and Fujita proved the following relation between $L(z)$ and $N^*(z)$ when $z \in \mathbb{C}^n$.

Proposition 1 (see [3]). Let $L(z)$ and $N^*(z)$ be norms defined as in (3) and (4) for $z \in \mathbb{C}^n$. Then, for $z, w \in \mathbb{C}^n$,

(i) $|z \cdot w| \leq L(z)N^*(w)$,

(ii) $L(z) = \sup \{ |z \cdot w| : N^*(w) = 1 \}$.

Now we deal with the generalization of $N^*(\cdot)$ and $L(\cdot)$ when $z$ is a complex matrix. In 2002, Youssfi [9] introduced the natural generalization of $N^*(\cdot)$ from complex vectors to complex matrices. Let $\mathcal{M}$ be the set of all $m \times n$ complex matrices. For $Z = (z_{jk})_{1 \leq j \leq m, 1 \leq k \leq n} \in \mathcal{M}$, we define the row vectors $Z^{(j)}$ by

$$Z^{(j)} = (z_{j1}, z_{j2}, \ldots, z_{jn}),$$

where $j = 1, 2, \ldots, m$. The Bergman kernel and the Szegő kernel for the ball $\{|Z| \in \mathcal{M} : N^*(Z) < 1\}$ have been computed by using proper holomorphic liftings (see [9]).

Definition 2. For $Z = (z_{jk})_{1 \leq j \leq m, 1 \leq k \leq n} \in \mathcal{M}$, we define $N^*(Z)$ by

$$N^*(Z) = \sqrt{\frac{1}{2} \sum_{j=1}^{m} \left( \|Z^{(j)}\|^2 + |Z^{(j)} \cdot Z^{(j)}| \right)}$$

$$= \sqrt{\frac{1}{2} \left( \sum_{j=1}^{m} \sum_{k=1}^{n} |z_{jk}|^2 + \sum_{j=1}^{m} \sum_{k=1}^{n} z_{jk}^2 \right)}.$$  

Similarly, we define $L(Z)$ and $M(Z)$ for $Z \in \mathcal{M}$ as follows.

Definition 3. For $Z = (z_{jk})_{1 \leq j \leq m, 1 \leq k \leq n} \in \mathcal{M}$, we define $L(Z)$ and $M(Z)$ by

$$L(Z) = \sqrt{\sum_{j=1}^{m} \left( \|Z^{(j)}\|^2 + \left( \sum_{j=1}^{m} |Z^{(j)} \cdot Z^{(j)}| \right)^2 \right)}$$

$$= \sqrt{\sum_{j=1}^{m} \sum_{k=1}^{n} |z_{jk}|^2 + \left( \sum_{j=1}^{m} \sum_{k=1}^{n} z_{jk}^2 \right)^2},$$

$$M(Z) = \sqrt{\sum_{j=1}^{m} \left( \|Z^{(j)}\|^2 - \left( \sum_{j=1}^{m} |Z^{(j)} \cdot Z^{(j)}| \right)^2 \right)}$$

$$= \sqrt{\sum_{j=1}^{m} \sum_{k=1}^{n} |z_{jk}|^2 - \left( \sum_{j=1}^{m} \sum_{k=1}^{n} z_{jk}^2 \right)^2}.$$  

If $m = 1$, then $L(Z)$ and $N^*(Z)$ are the Lie norm and the minimal norm in $\mathbb{C}^n$, respectively, and if $n = 1$, then $L(Z) = N^*(Z)$ is the Euclidean norm in $\mathbb{C}^m$. Moreover, it is easily proved that the inequality

$$\frac{1}{\sqrt{2}} \|Z\|_F \leq N^*(Z) \leq \|Z\|_F \leq L(Z) \leq \sqrt{2} \|Z\|_F$$

holds, where the Frobenius norm $\|Z\|_F$ is defined as in (2).

At first we prove that extensions $L(Z)$ and $N^*(Z)$ of these norms to complex matrices are quasinorms of $\mathcal{M}$.

Theorem 4. For $Z \in \mathcal{M}$, $L(Z)$ and $N^*(Z)$ are quasinorms. Precisely, for $Z, W \in \mathcal{M}$ we have

(i) $L(Z + W) \leq \sqrt{m} L(Z) + L(W)$,

(ii) $N^*(Z + W) \leq \sqrt{m} N^*(Z) + N^*(W)$.

The generalization of Proposition 1(i) will be proved as follows.

Theorem 5. For $Z, W \in \mathcal{M}$, we have

$$m \sum_{j=1}^{m} \sum_{k=1}^{n} |z_{jk} w_{jk}| \leq L(Z) N^*(W).$$

We also generalize Proposition 1(ii) for complex matrices as follows.
Corollary 6. For $Z \in \mathcal{M}$, we have

(i) $\min_{1 \leq j \leq m} L(Z^{(j)})$

\[ \leq \sup \left\{ \sum_{j=1}^{m} |Z^{(j)} \cdot W^{(j)}| : 1 \leq \sum_{j=1}^{m} N^*(W^{(j)}) \leq \sqrt{m} \right\} \]

\[ \leq \sqrt{m} L(Z), \quad (10) \]

(ii) $\min_{1 \leq j \leq m} L(Z^{(j)})$

\[ \leq \sup \left\{ \sum_{j=1}^{m} |Z^{(j)} \cdot W^{(j)}| : \sum_{j=1}^{m} N^*(W^{(j)}) = 1 \right\} \]

\[ \leq L(Z). \]

For $p \geq 1$ and $Z \in \mathcal{M}$, we define

\[ N_p(Z) = \left\{ \frac{1}{2} \left( \frac{m}{m} L(Z^{(j)})^p + M(Z^{(j)})^p \right) \right\}^{1/p}, \quad (11) \]

where

\[ M(Z^{(j)}) = \sqrt{\|Z^{(j)}\|^2 - \sqrt{\|Z^{(j)}\|^2 - \|Z^{(j)} \cdot Z^{(j)}\|^2}}. \quad (12) \]

One can easily see that $N_1(Z) = N^*(Z)$ and $\lim_{p \to \infty} N_p(Z) = L(Z)$. Then, we finally proved the following.

Theorem 7. For $Z \in \mathcal{M}$, we have

(i) $\min_{1 \leq j \leq m} N_p(Z^{(j)})$

\[ \leq \sup \left\{ \sum_{j=1}^{m} |Z^{(j)} \cdot W^{(j)}| : 1 \leq \sum_{j=1}^{m} N_q(W^{(j)}) \leq m^{1/p} \right\} \]

\[ \leq m^{1/p} \sup \left\{ \sum_{j=1}^{m} N_q(W^{(j)}) : 1 \leq \sum_{j=1}^{m} N_q(W^{(j)}) \leq m \right\}, \quad (13) \]

(ii) $\min_{1 \leq j \leq m} N_p(Z^{(j)})$

\[ \leq \sup \left\{ \sum_{j=1}^{m} |Z^{(j)} \cdot W^{(j)}| : \sum_{j=1}^{m} N_q(W^{(j)}) = 1 \right\} \]

\[ \leq N_p(Z), \quad (14) \]

where $1/p + 1/q = 1$.

Remark 8. If $m = 1$, then Corollary 6 and Theorem 7 are identical to previous results proved by Morimoto and Fujita [3].

3. Proofs

Throughout this section, it is convenient to define

\[ a_j = \sum_{k=1}^{n} |x_{jk}|, \]

\[ b_j = \sum_{k=1}^{n} x_{jk}^2, \]

\[ c_j = \sum_{k=1}^{n} w_{jk}^2, \]

\[ d_j = \sum_{k=1}^{n} w_{jk}^2, \]

\[ e_j = \sum_{k=1}^{n} (x_{jk} + w_{jk})^2, \]

\[ f_j = \sum_{k=1}^{n} (x_{jk} + w_{jk})^2. \quad (15) \]

Note that $a_j \geq b_j \geq 0, c_j \geq d_j \geq 0,$ and $e_j \geq f_j \geq 0$.

Lemma 9. Assume that $x_1, \ldots, x_m, y_1, \ldots, y_m$ satisfy $x_j \geq y_j \geq 0$ for all $j = 1, \ldots, m$. Then, we have

(i) $\sum_{j=1}^{m} \sqrt{x_j^2 - y_j^2} \leq \sum_{j=1}^{m} \sqrt{\left( \frac{1}{m} \sum_{j=1}^{m} x_j \right)^2 - \left( \frac{1}{m} \sum_{j=1}^{m} y_j \right)^2}$,

(ii) $\sum_{j=1}^{m} x_j + \sqrt{\left( \frac{1}{m} \sum_{j=1}^{m} x_j \right)^2 - \left( \frac{1}{m} \sum_{j=1}^{m} y_j \right)^2} \leq \sum_{j=1}^{m} \sqrt{x_j^2 - y_j^2}.

Proof. (i) If we apply Cauchy-Schwarz inequality, then we have

\[ \sum_{j=1}^{m} \sqrt{x_j^2 - y_j^2} = \sum_{j=1}^{m} \sqrt{x_j + y_j \sqrt{x_j - y_j}} \]

\[ \leq \left\{ \sum_{j=1}^{m} (x_j + y_j) \right\} \sum_{j=1}^{m} (x_j - y_j). \quad (17) \]
(ii) Note that
\[
\left(\sum_{j=1}^{m} x_j + \sqrt{x_j^2 - y_j^2}\right)^2
- \left(\sum_{j=1}^{m} x_j + \sqrt{x_j^2 - y_j^2}\right)^2
= 2\sum_{j=1}^{m} x_j + \sqrt{x_j^2 - y_j^2} \left(\sum_{j=1}^{m} x_j + \sqrt{x_j^2 - y_j^2}\right)
+ \sum_{j=1}^{m} \sqrt{x_j^2 - y_j^2} - \left(\sum_{j=1}^{m} x_j + \sqrt{x_j^2 - y_j^2}\right).
\]
The last term is estimated as follows:
\[
\sqrt{\sum_{j=1}^{m} x_j^2 - \left(\sum_{j=1}^{m} y_j\right)^2}
= \sqrt{\sum_{j=1}^{m} (x_j^2 - y_j^2) + 2\sum_{j \neq l} (x_j x_l - y_j y_l)}
\leq \sum_{j=1}^{m} \sqrt{x_j^2 - y_j^2} + \sqrt{2\sum_{j \neq l} (x_j x_l - y_j y_l)}.
\]
It follows that
\[
\left(\sum_{j=1}^{m} \sqrt{x_j^2 + \sqrt{x_j^2 - y_j^2}}\right)^2
- \left(\sum_{j=1}^{m} \sqrt{x_j^2 - y_j^2}\right)^2
\geq 2\sum_{j \neq l} \sqrt{x_j x_l - y_j y_l} \geq \sqrt{2\sum_{j \neq l} (x_j x_l - y_j y_l)}
\geq (2 - \sqrt{2}) \sum_{j \neq l} x_j x_l > 0,
\]

since \(y_j y_l \geq 0\) for all \(j, l\). The proof of (ii) is finished.

3.1. Proof of Theorem 4(i). For \(j = 1, 2, \ldots, m\), we define the row vectors \(Z^{(j)}\) and \(W^{(j)}\) by
\[
Z^{(j)} = (z_{j1}, \ldots, z_{jn}),
\]
\[
W^{(j)} = (w_{j1}, \ldots, w_{jn}).
\]
Since \(L(z)\) is a norm with \(z \in \mathbb{C}^n\), for each \(j = 1, 2, \ldots, m\), we have
\[
L\left(Z^{(j)} + W^{(j)}\right) \leq L\left(Z^{(j)}\right) + L\left(W^{(j)}\right),
\]
so that
\[
\sum_{j=1}^{m} L\left(Z^{(j)} + W^{(j)}\right) \leq \sum_{j=1}^{m} L\left(Z^{(j)}\right) + \sum_{j=1}^{m} L\left(W^{(j)}\right).
\]
Now we will obtain the upper bound of
\[
\sum_{j=1}^{m} L\left(Z^{(j)}\right) + \sum_{j=1}^{m} L\left(W^{(j)}\right).
\]
By Cauchy-Schwarz inequality and Lemma 9(i), we have
\[
\sum_{j=1}^{m} L\left(Z^{(j)}\right)
= \sum_{j=1}^{m} \sqrt{\sum_{k=1}^{n} |z_{jk}|^2 + \left(\sum_{k=1}^{n} |z_{jk}|^2\right) - \sum_{k=1}^{n} |z_{jk}|^2}
\leq \sqrt{m} \sum_{j=1}^{m} \sqrt{a_j + \sqrt{a_j^2 - b_j^2}}
\leq \sqrt{m} \sum_{j=1}^{m} \sqrt{a_j + \left(\sum_{j=1}^{m} a_j\right)^2 - \left(\sum_{j=1}^{m} b_j\right)^2}
= \sqrt{m} L\left(Z\right).
\]
Similarly, we have
\[
\sum_{j=1}^{m} L\left(W^{(j)}\right) \leq \sqrt{m} L\left(W\right).
\]
Thus, we see that
\[
\sum_{j=1}^{m} L\left(Z^{(j)}\right) + \sum_{j=1}^{m} L\left(W^{(j)}\right) \leq \sqrt{m} \left(L\left(Z\right) + L\left(W\right)\right).
\]
By Lemma 9(ii), we have
\[
L\left(Z + W\right) = \sqrt{\sum_{j=1}^{m} e_j + \sqrt{\left(\sum_{j=1}^{m} e_j\right)^2 - \left(\sum_{j=1}^{m} f_j\right)^2}}
\leq \sqrt{\sum_{j=1}^{m} e_j + \sqrt{e_j^2 - f_j^2}}
= \sqrt{m} L\left(Z + W^{(j)}\right).
\]
Combining (24), (28), and (29), we obtain that \(L(\cdot)\) is a quasinorm.
3.2. Proof of Theorem 4(ii). Now we show that $N^*(\cdot)$ is a quasinorm. Note that

$$N^*(Z + W) = \sqrt{\sum_{j=1}^{m} \left( e_j + f_j \right)^2} < \sum_{j=1}^{m} \frac{1}{\sqrt{2}} \left( e_j + f_j \right) \leq \sum_{j=1}^{m} N^*((Z + W)^{(j)})$$

Since $N^*(Z)$ is a norm when $m = 1$ in [3], we have

$$\sum_{j=1}^{m} N^*((Z + W)^{(j)}) \leq \sum_{j=1}^{m} N^*((Z)^{(j)}) + \sum_{j=1}^{m} N^*((W)^{(j)}).$$

(31)

By Cauchy-Schwarz inequality,

$$\sum_{j=1}^{m} N^*(Z^{(j)}) = \sum_{j=1}^{m} \sqrt{\frac{1}{2} \sum_{k=1}^{m} \left( a_j + b_j \right)^2} \leq \sqrt{m} \sum_{j=1}^{m} \sqrt{\frac{1}{2} \sum_{k=1}^{m} \left( a_j + b_j \right)^2} = \sqrt{m} N^*(Z)$$

and similarly

$$\sum_{j=1}^{m} N^*(W^{(j)}) \leq \sqrt{m} N^*(W).$$

(33)

Combining the above inequalities, we obtain that $N^*(\cdot)$ is also a quasinorm.

3.3. Proof of Theorem 5. We will prove that Proposition 1(i) holds also for the matrices. By Proposition 1(i), we have

$$\left| \sum_{k=1}^{n} z_{jk} w_{kj} \right| \leq L(\mathbf{Z}^{(j)}) N^*(\mathbf{Z}^{(j)})$$

$$= \sqrt{a_j + \sqrt{a_j^2 - b_j^2} \times \frac{1}{2} (c_j + d_j)}$$

for each $j = 1, \ldots, m$. By Cauchy-Schwarz inequality and Lemma 9(i), we have

$$\sum_{j=1}^{m} \sum_{k=1}^{n} z_{jk} w_{kj} \leq \sum_{j=1}^{m} \sqrt{a_j + \sqrt{a_j^2 - b_j^2} \times \frac{1}{2} (c_j + d_j)} \leq \sqrt{\sum_{j=1}^{m} \left( a_j + \sqrt{a_j^2 - b_j^2} \right)^2} \times \frac{1}{2} (c_j + d_j)$$

$$\leq \sqrt{\frac{m}{2} \sum_{j=1}^{m} a_j^2} - \frac{m}{2} \left( \sum_{j=1}^{m} b_j \right)^2$$

$$\times \frac{1}{2} \left( \sum_{j=1}^{m} c_j + \sum_{j=1}^{m} d_j \right) = L(Z) N^*(W).$$

(35)

3.4. Proof of Corollary 6. Note that

$$N^*(W) \leq \sum_{j=1}^{m} N^*(W^{(j)}) \leq \sqrt{m} N^*(W).$$

(36)

If $1 \leq \sum_{j=1}^{m} N^*(W^{(j)}) \leq \sqrt{m}$, then $N^*(W) \leq \sqrt{m}$ by (36). So, by Theorem 5,

$$\sum_{j=1}^{m} |Z^{(j)} \cdot W^{(j)}| \leq \sqrt{m} L(Z),$$

(37)

so that

$$\sup \left\{ \sum_{j=1}^{m} |Z^{(j)} \cdot W^{(j)}| : 1 \leq \sum_{j=1}^{m} N^*(W^{(j)}) \leq \sqrt{m} \right\} \leq \sqrt{m} L(Z).$$

(38)

On the other hand, we have

$$\sup \left\{ \sum_{j=1}^{m} |Z^{(j)} \cdot W^{(j)}| : 1 \leq \sum_{j=1}^{m} N^*(W^{(j)}) \leq \sqrt{m} \right\} = \sup \left\{ \sum_{j=1}^{m} |Z^{(j)} \cdot \left( \frac{W}{k_j} \right)^{(j)}| : 1 \leq \sum_{j=1}^{m} k_j \leq \sqrt{m} \right\}$$

$$\leq \sqrt{m} N^* \left( \left( \frac{W}{k_j} \right)^{(j)} \right) = 1,$$

where $k_j = N^*(W^{(j)})$ for $1 \leq j \leq m$. By Proposition 1(ii), the last term is greater than or equal to

$$\inf \left\{ \sum_{j=1}^{m} k_j L(Z^{(j)}) : 1 \leq \sum_{j=1}^{m} k_j \leq \sqrt{m} \right\} \geq \min_{1 \leq j \leq m} L(Z^{(j)}) \sum_{j=1}^{m} k_j \geq \min_{1 \leq j \leq m} L(Z^{(j)}) \sum_{j=1}^{m} k_j$$

(40)

It follows that

$$\min_{1 \leq j \leq m} L(Z^{(j)}) \leq \sup \left\{ \sum_{j=1}^{m} |Z^{(j)} \cdot W^{(j)}| : 1 \leq \sum_{j=1}^{m} k_j \leq \sqrt{m} \right\} \leq \sqrt{m} L(Z),$$

(41)

The proof of (i) of Corollary 6 is finished. The proof of (ii) is similar to that of (i).

3.5. Proof of Theorem 7. We use the following inequalities for vectors.
Proposition 10 (see [3]). For \( z, w \in \mathbb{C}^n \), we have

\[
\begin{align*}
& (i) \quad 2|z \cdot w| \leq L(z)L(w) + M(z)M(w) \\
& (ii) \quad N_p(z) = \sup \{|z \cdot w| : N_q(w) = 1\}, \text{ where } 1/p + 1/q = 1.
\end{align*}
\]

By Proposition 10(i), we have

\[
2 \sum_{j=1}^{m} |Z^{(j)} \cdot W^{(j)}| \leq \sum_{j=1}^{m} L(Z^{(j)})L(W^{(j)}) + M(Z^{(j)})M(W^{(j)})
\]

\[
\cdot M(W^{(j)}) \leq \left( \sum_{j=1}^{m} L(Z^{(j)})^{p} + M(Z^{(j)})^{p} \right)^{1/p}
\]

\[
\cdot \left( \sum_{j=1}^{m} L(W^{(j)})^{q} + M(W^{(j)})^{q} \right)^{1/q},
\]

where \( 1/p + 1/q = 1 \). From 2 = \( 2^{1/p + 1/q} \), we obtain

\[
\sum_{j=1}^{m} |Z^{(j)} \cdot W^{(j)}| \leq N_p(Z)N_q(W).
\]

(43)

The Hölder inequality tells us that if \( x_j \geq 0 \) for all \( 1 \leq j \leq m \), then

\[
\left( \sum_{j=1}^{m} x_j \right)^{1/q} \leq \sum_{j=1}^{m} x_j^{1/q} \leq m^{1/p} \left( \sum_{j=1}^{m} x_j \right)^{1/q}.
\]

(44)

If we substitute \( x_j = (1/2)(L(W^{(j)})^{p} + M(W^{(j)})) \) in (44), then we obtain

\[
N_q(W) \leq \sum_{j=1}^{m} N_q(W^{(j)}) \leq m^{1/p}N_q(W).
\]

(45)

Now we can prove Theorem 7 similarly to the proof of Corollary 6. From (43) and (45), if \( 1 \leq \sum_{j=1}^{m} N_q(W^{(j)}) \leq m^{1/p} \), then

\[
\sum_{j=1}^{m} |Z^{(j)} \cdot W^{(j)}| \leq m^{1/p}N_p(Z).
\]

(46)

Similarly to the proof of Corollary 6, we write \( k_j = N_q(W^{(j)}) \) for \( 1 \leq j \leq m \). Then, using (45) and Proposition 10(ii), we have

\[
\sup \left\{ \sum_{j=1}^{m} |Z^{(j)} \cdot W^{(j)}| : 1 \leq \sum_{j=1}^{m} N_q(W^{(j)}) \leq m^{1/p} \right\}
\]

\[
= \sup \left\{ \sum_{j=1}^{m} k_j |Z^{(j)} \cdot \left( \frac{W}{k_j} \right)^{(j)}| : 1 \leq \sum_{j=1}^{m} k_j \right\}
\]

\[
\leq m^{1/p}N_q \left( \left( \frac{W}{k_j} \right)^{(j)} \right) = 1
\]

\[
\geq \inf \left\{ \sum_{j=1}^{m} k_j N_p \left( Z^{(j)} \right) : 1 \leq \sum_{j=1}^{m} k_j \leq \sqrt{m} \right\}
\]

\[
\geq \min_{1 \leq j \leq m} N_p \left( Z^{(j)} \right) \sum_{j=1}^{m} k_j \geq \min N_p \left( Z^{(j)} \right).
\]

(47)

It follows that

\[
\min_{1 \leq j \leq m} N_p \left( Z^{(j)} \right) \leq \sup \left\{ \sum_{j=1}^{m} |Z^{(j)} \cdot W^{(j)}| : 1 \right\}
\]

\[
\leq \sum_{j=1}^{m} N_q(W^{(j)}) \leq m^{1/p}.
\]

(48)

From (46) and (48), we complete the proof of (i) of Theorem 7. The proof of (ii) is similar to that of (i).

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by the National Research Foundation of Korea (NRF-2015R1D1A1A01060295).

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