Research Article

Uniform Attractors for Nonclassical Diffusion Equations with Memory

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We introduce a new method (or technique), asymptotic contractive method, to verify uniform asymptotic compactness of a family of processes. After that, the existence and the structure of a compact uniform attractor for the nonautonomous nonclassical diffusion equation with fading memory are proved under the following conditions: the nonlinearity f satisfies the polynomial growth of arbitrary order and the time-dependent forcing term g is only translation-bounded in $L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$.

1. Introduction

In this paper, we consider dynamical behavior of solutions for the following nonclassical diffusion equation with a fading memory term:

$$\begin{aligned} u_t(t) - \Delta u_t(t) - \Delta u(t) - \int_0^\infty k(s) \Delta u(t-s) ds \\ + f(u(t)) = g(t), \quad \text{in } \Omega \times (\tau, \infty). \end{aligned} \quad (1)$$

The problem is supplemented with the boundary condition,

$$u(x, t)|_{\partial\Omega} = 0, \quad \forall t \in \mathbb{R}, \quad (2)$$

and initial condition,

$$u(x, t) = u_\tau(x, t), \quad t \leq \tau, \quad (3)$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$ and $g(t)$ is a given external time-dependent forcing.

Concerning the memory kernel $\mu(s) = -k'(s)$, as in [1–6], we assume the following hypotheses:

$$\begin{aligned} \mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \\ \mu(s) \geq 0, \quad \mu'(s) \leq 0 \quad \forall s \in \mathbb{R}^+, \end{aligned} \quad (4)$$

and there is a constant $\delta > 0$, such that

$$\mu'(s) + \delta\mu(s) \leq 0 \quad \forall s \in \mathbb{R}^+. \quad (5)$$

From (4) and (5), we get

$$\mu(\infty) = \lim_{s \rightarrow +\infty} \mu(s) = 0. \quad (6)$$

We also denote

$$m_0 := \int_0^\infty \mu(s) ds < \infty. \quad (7)$$

As Wang and Zhong [5], we introduce the past history of u , that is,

$$\eta^t = \eta^t(x, s) := \int_0^s u(x, t-r) dr, \quad s \in \mathbb{R}^+, \quad (8)$$

as a new variable of the system, which will be ruled by a supplementary equation. Denote

$$\begin{aligned} \eta^t_t &= \frac{\partial}{\partial t} \eta^t, \\ \eta^t_s &= \frac{\partial}{\partial s} \eta^t, \end{aligned} \quad (9)$$

and (1) transforms into the following system:

$$\begin{aligned} u_t(t) - \Delta u_t(t) - \Delta u(t) - \int_0^\infty \mu(s) \Delta \eta^t(s) ds \\ + f(u(t)) = g(t), \\ \eta^t_t = -\eta^t_s + u, \end{aligned} \quad (10)$$

with initial-boundary conditions

$$\begin{aligned} u(x, t)|_{\partial\Omega} &= 0, \\ \eta^t(x, s)|_{\partial\Omega \times \mathbb{R}^+} &= 0, \\ t &\geq \tau, \\ u(x, \tau) &= u_\tau(x, \tau), \quad x \in \Omega, \\ \eta^\tau &= \eta^\tau(x, s) = \int_0^s u_\tau(x, \tau - r) dr, \\ (x, s) &\in \Omega \times \mathbb{R}^+. \end{aligned} \quad (11)$$

The past history $u_\tau(\tau - s)$ of the variable u satisfies the condition as follows: there exist positive constants $\mathfrak{R}$ and $\varrho \leq \delta$ (from (5)), such that

$$\int_0^\infty e^{-\varrho s} \|u_\tau(\tau - s)\|_0^2 ds \leq \mathfrak{R}. \quad (12)$$

The nonlinearity $f \in \mathcal{C}^1$ fulfills $f(0) = 0$, along with arbitrary order polynomial growth restriction

$$\gamma_1 |s|^p - \beta_1 \leq f(s) \leq \gamma_2 |s|^p + \beta_2, \quad p \geq 2, \quad (13)$$

and the dissipation condition

$$f'(s) \geq -l, \quad (14)$$

where γ_i, β_i ($i = 1, 2$) and l are positive constants.

This equation appears as an extension of the usual nonclassical diffusion equation in fluid mechanics, solid mechanics, and heat conduction theory (see [7–10]). Equation (1) with a one-time derivative appearing in the highest order term is called pseudoparabolic or Sobolev-Galpern equations [11–13]. Aifantis [7] proposed a general frame for establishing the equations. For certain classes of materials such as polymer and high-viscosity liquids, the diffusive process is nontrivially influenced by the past history of u , which is represented in (1) by the convolution term against a suitable memory kernel characterizing the diffusive species [14].

Proving the existence of uniform attractors for (1) may be a hard task, mainly due to the fact that the nonlinearity satisfies arbitrary polynomial growth condition instead of critical, so Sobolev compact embedding theorems are no longer useful. The asymptotic compactness of solutions cannot be obtained by the usual method (used, e.g., in [2, 5, 6, 15, 16]).

When μ is a Dirac measure at some fixed time instant or when it vanishes, (1) reduces to the usual nonclassical diffusion equation, which has been investigated extensively by many authors, especially about the asymptotic behavior of solutions; see, for example, [16–22] and the references therein. In [21], the author has proved the existence of global attractor in $H_0^1(\Omega)$, when $g(x) \in L^2(\Omega)$ under the assumptions that f satisfies critical exponent growth condition corresponding to $N = 3$ and some additional condition for nonlinearity, which essentially requires that the nonlinearity is subcritical. Recently, the authors in [23] obtained the existence of a global attractor for $g(x) \in H^{-1}(\Omega)$ only under critical nonlinearity, and when $g(x, t) \in L_b^2(\mathbb{R}, L^2(\Omega))$, they proved the

asymptotic regularity and the existence of (nonautonomous) exponential attractor. The asymptotic behavior of solutions of this equation has received considerably less attention in the literature under the assumption that the nonlinearity satisfies arbitrary polynomial growth condition. In the ordinary case for some recent results on this equations the reader can refer to Sun et al. [24] and Anh and Toan [25, 26]. Hereafter in [23] the authors mention that these are some mistakes in the coauthor's earlier paper [24]. In [25], they proved that the nonautonomous dynamical system generated by this class of solutions has a pullback attractor. In [27], they proved the existence of global attractor in $H^1(\mathbb{R}^N)$ with the initial data $u_0 \in H^2(\mathbb{R}^N)$ and $g \in L^2(\mathbb{R}^N)$.

For convenience, hereafter let $|u|$ be the modular (or absolute value) of u and let $|\cdot|_p$ be the norm of $L^p(\Omega)$ ($p \geq 1$). Let $\|\cdot\|_0 = |\nabla \cdot|_2$ be the norm of $\mathcal{V}$. Denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$ and let $\|\cdot\|_{H^{-1}}$ be the norm of $H^{-1}(\Omega)$.

C denotes any positive constant which may be different from line to line even in the same line.

Let $L_\mu^2(\mathbb{R}^+; \mathcal{V})$ be the Hilbert spaces of functions $\varphi : \mathbb{R}^+ \rightarrow \mathcal{V}$, endowed with the inner product and norm, respectively. Consider

$$\langle \varphi, \psi \rangle_{\mu, \mathcal{V}} = \int_0^\infty \mu(s) \langle \varphi(s), \psi(s) \rangle_{\mathcal{V}} ds, \quad (15)$$

$$\|\varphi\|_{\mu, \mathcal{V}}^2 = \int_0^\infty \mu(s) \|\varphi(s)\|_0^2 ds.$$

We also define the product space $\mathcal{M}_1$:

$$\mathcal{M}_1 = \mathcal{V} \times L_\mu^2(\mathbb{R}^+; \mathcal{V}), \quad (16)$$

endowed with the norm

$$\|z\|_{\mathcal{M}_1}^2 = \|(u, \eta^t)\|_{\mathcal{M}_1}^2 = \frac{1}{2} (|u|_2^2 + \|u\|_0^2 + \|\eta^t\|_{\mu, \mathcal{V}}^2). \quad (17)$$

For convenience, we first show the preliminary result as follows.

Lemma 1 (see [5]). *Let $I = [0, T]$, $\forall T > 0$ and let the memory kernel $\mu(s)$ satisfy (4) and (5). Then for any $\eta^t \in C(I; L_\mu^2(\mathbb{R}^+; \mathcal{V}))$ the following estimate*

$$\langle \eta^t, \eta_s^t \rangle \geq \frac{\delta}{2} \|\eta^t\|_{\mu, \mathcal{V}}^2 \quad (18)$$

holds, where δ is from (5).

For the time-dependent forcing $g(x, t)$, we assume the following hypotheses:

$$g \in L_{w, \text{loc}}^2(\mathbb{R}; L^2(\Omega)). \quad (19)$$

Here $L_{w, \text{loc}}^2(\mathbb{R}; L^2(\Omega))$ is the space $L_{\text{loc}}^2(\mathbb{R}; L^2(\Omega))$ under the local weak convergence topology. Recall that a sequence $\{g_n\}$

converges to g as $n \rightarrow \infty$ in $L^2_{w,loc}(\mathbb{R}; L^2(\Omega))$ if and only if

$$\lim_{n \rightarrow \infty} \int_{t_1}^{t_2} \int_{\Omega} v(x, s) (g_n(x, s) - g(x, s)) ds = 0 \quad (20)$$

for all $[t_1, t_2] \subset \mathbb{R}$ and every $v \in L^2(t_1, t_2; L^2(\Omega))$ holds.

Let $g_0 \in L^2_b(\mathbb{R}; L^2(\Omega))$ (translation-bounded in $L^2_{w,loc}(\mathbb{R}; L^2(\Omega))$), and introduce a set of functions obtained by time-translation in g_0 :

$$\Sigma_0 = \{g_0(x, t + h) : h \in \mathbb{R}\}. \quad (21)$$

We define the hull of g_0 , denoted as $\Sigma = \mathcal{H}(g_0)$, as the closure of Σ_0 with respect to the local weak convergence topology of $L^2_{loc}(\mathbb{R}; L^2(\Omega))$. If $g \in \Sigma = \mathcal{H}(g_0)$, then $g \in L^2_b(\mathbb{R}; L^2(\Omega))$; that is,

$$\|g\|_{L^2_b}^2 = \sup_{t \in \mathbb{R}} \int_t^{t+1} |g(s)|_2^2 ds < \infty, \quad (22)$$

where $\|\cdot\|_{L^2_b}$ means the norm of $L^2_b(\mathbb{R}; L^2(\Omega))$.

Let $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, be a family of processes acting in a Banach space X with symbol space Σ ; then for any $\sigma \in \Sigma$

$$U_\sigma(t, s) \circ U_\sigma(s, \tau) = U_\sigma(t, \tau), \quad (23)$$

for any $t \geq s \geq \tau$, $\tau \in \mathbb{R}$,

$$U_\sigma(\tau, \tau) = \text{Id (identity)}, \quad \text{for any } \tau \in \mathbb{R}. \quad (24)$$

The operators $\{H(s)\}_{s \geq 0}$ are the translation semigroup on Σ ; a family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, is called to satisfy the translation identity if

$$U_\sigma(t + s, s + \tau) = U_{H(s)\sigma}(t, \tau), \quad (25)$$

for any $\sigma \in \Sigma$, $t \geq \tau$, $\tau \in \mathbb{R}$, $s \geq 0$,

$$H(s)\Sigma = \Sigma, \quad \text{for any } s \geq 0. \quad (26)$$

We recall (e.g., see [28]) also that the kernel $\mathcal{K}$ of the process $U(t, \tau)$ consists of all bounded complete trajectories of the process; that is,

$$\mathcal{K} = \{u(\cdot) : \|u(t)\|_X \leq C_u, \quad u(t) = U(t, \tau)u(\tau), \quad \forall t \geq \tau, \quad \tau \in \mathbb{R}\} \quad (27)$$

and $\mathcal{K}(s) = \{u(s) : u(\cdot) \in \mathcal{K}\} \subset X$ is denoted by the kernel section at a time moment $s \in \mathbb{R}$.

It is well-known that the key point is to obtain certain asymptotic compactness for the solution operator in the study of the long time behavior, especially for attractors. The nonlinearity having an arbitrary polynomial growth brings a difficulty here even for the autonomous and without memory case; see, for example, [22, 24, 27]. The main contribution of this paper is to extend the method in [15, 22] to overcome the difficulty caused by a lack of Sobolev compact embedding theorems. The conception, asymptotic contractive function, and new a priori estimates for verifying uniform asymptotic

compactness of the family of processes are devised. We also prove some *weak* continuity for the family of processes and then obtain the structure of the compact uniform attractors.

The main results of this paper are given expression to in the following two theorems, which will be proved in Sections 2 and 3, respectively.

Theorem 2. *Let X be a Banach space and let $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, be a family of processes on X . Assume further that $g_0 \in L^2_b(\mathbb{R}; L^2(\Omega))$ and Σ is the hull of g_0 in $L^2_{w,loc}(\mathbb{R}; L^2(\Omega))$. Then $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, has a uniform attractor in X provided that the following conditions hold true:*

- (i) $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, has a bounded absorbing set B_0 in X .
- (ii) For any $\varepsilon > 0$, there exist $T = T(B_0, \varepsilon) > 0$ and an asymptotic contractive function ψ_T on $B_0 \times B_0$ such that

$$\|U_{\sigma_1}(T, \tau)x - U_{\sigma_2}(T, \tau)y\|_X \leq \varepsilon + \psi_T(x, y, \sigma_1, \sigma_2), \quad (28)$$

$\forall x, y \in B_0, \quad \sigma_1, \sigma_2 \in \Sigma$,

where ψ_T depend on T .

Theorem 3 (uniform attractor). *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, and μ and f satisfy (4)–(6) and (13)–(14), respectively. Assume further that $g_0 \in L^2_b(\mathbb{R}; L^2(\Omega))$ and Σ is the hull of g_0 in $L^2_{w,loc}(\mathbb{R}; L^2(\Omega))$. Then the family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, corresponding to (10)–(11), has a compact uniform (with respect to $\sigma \in \Sigma$) attractor $\mathcal{A}$ in $\mathcal{M}_1$. Moreover, this attractor can be decomposed as follows:*

$$\mathcal{A} = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0), \quad (29)$$

where $\mathcal{K}_\sigma$ is the kernel of the process U_σ and $\mathcal{K}_\sigma(0)$ is the kernel section at time 0.

This theorem gives the existence of the uniform attractor and its structure as the union of the kernel sections (see [29]) of the nonautonomous process.

It is worth noting that Theorem 3 is also interesting in the autonomous case (see, e.g., [5]) or in the unbounded domain case (see, e.g., [22, 23]). This is the basis for further considering the asymptotic behavior, such as, to construct the nonautonomous (pullback) exponential attractor [2, 25]. On the other hand, from Theorem 16, the existence of the uniform attractor is obtained directly. However, since the external forcing term $g(x, t)$ is only assumed to be *translation-bounded*, consequently the symbol space is only weak compact with respect to the local *weak* convergence topology. So, in order to obtain the structure equality (29), we need to verify some *weak* continuity for the family of processes, which is different from the usual *strong* continuity. This may be the reason why some authors (e.g., see [1]) have to assume further that the external forcing term $g(x, t)$ is *translation-compact*.

2. Preliminaries

Definition 4 (see [15]). Let X be a Banach space, B be a bounded subset of X and Σ be a symbol (or parameter) space. We call a function $\phi(\cdot, \cdot, \cdot, \cdot)$, defined on $(X \times X) \times (\Sigma \times \Sigma)$, to be a contractive function on $B \times B$, if, for any sequence $\{x_n\}_{n=1}^\infty \subset B$, and any $\{\sigma_n\}_{n=1}^\infty \subset \Sigma$, there is a subsequence $\{x_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ and $\{\sigma_{n_k}\}_{k=1}^\infty \subset \{\sigma_n\}_{n=1}^\infty$ respectively, such that

$$\lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} \phi(x_{n_k}, x_{n_l}, \sigma_{n_k}, \sigma_{n_l}) = 0. \quad (30)$$

Definition 5. Let X be a Banach space and let B be a bounded subset of X and let Σ be a symbol (or parameter) space. We call a function $\psi(\cdot, \cdot, \cdot, \cdot)$, defined on $(X \times X) \times (\Sigma \times \Sigma)$, to be an asymptotic contractive function on $B \times B$, if for any $\varepsilon > 0$ and $x_i \in B, \sigma_i \in \Sigma$ ($i = 1, 2$), there is a contractive function ϕ on $B \times B$ such that

$$\psi(x_1, x_2, \sigma_1, \sigma_2) \leq \varepsilon + \phi(x_1, x_2, \sigma_1, \sigma_2). \quad (31)$$

We denote the set of all asymptotic contractive functions on $B \times B$ by $\mathfrak{C}(B, \Sigma)$.

In the following theorem, we present a new method (or technical) to verify the asymptotic compactness for the family of processes generated by evolutionary equations, which will be used in our later discussion.

Theorem 6. Let $(X, \|\cdot\|_X)$ be a Banach space; $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$, is a family of processes on X . Assume further that the following conditions hold true:

- (i) $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$, has a bounded absorbing set B_0 in X .
- (ii) For any $\varepsilon > 0$ and $x, y \in B_0, \sigma_1, \sigma_2 \in \Sigma$, there exist $T = T(B_0, \varepsilon) > 0$ and $\psi_T \in \mathfrak{C}(B_0, \Sigma)$ such that

$$\|U_{\sigma_1}(T, \tau)x - U_{\sigma_2}(T, \tau)y\|_X \leq \varepsilon + \psi_T(x, y, \sigma_1, \sigma_2). \quad (32)$$

Then $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$, is uniform (with respect to $\sigma \in \Sigma$) asymptotically compact in X , where ψ_T depend on T .

Proof. For any $\tau \in \mathbb{R}$, we assume that B is any bounded subset of X , and $\{x_n\}_{n=1}^\infty \subset B, \sigma_n \in \Sigma$ and $t_n \geq \tau$ satisfy $t_n \rightarrow +\infty$ as $n \rightarrow \infty$. It is enough to show that, from the assumptions, there is $N > 0$ such that $U_\sigma(t_n, \tau)x \in B_0$ and $t_n > t_N - \tau$ for each $n > N$ large enough and $x \in B$ and furthermore that

$$\begin{aligned} U_\sigma(t_n, \tau)x &= U_\sigma(t_n, t_N) \circ U_\sigma(t_N, \tau)x \\ &= U_\sigma(t_n, t_N)y = U_{\sigma'}(t'_n, \tau)y, \end{aligned} \quad (33)$$

where $y = U_\sigma(t_N, \tau)x \in B_0, t'_n = t_n - t_N + \tau > 0$ and $\sigma' = H(t_N - \tau)\sigma \in \Sigma$. So we only need to consider the case as $\{x_n\}_{n=1}^\infty \subset B_0$.

In the following, the existence of a Cauchy subsequence of $\{U_{\sigma_n}(t_n, \tau)x_n\}_{n=1}^\infty$ is proved by the diagonal method.

Take $\varepsilon_m > 0$ with $\varepsilon_m \rightarrow 0$ as $m \rightarrow \infty$.

At first, for ε_1 , by the assumptions, there exist $T_1 = T_1(\varepsilon_1)$ and $\psi_{T_1} \in \mathfrak{C}(B_0, \Sigma)$ such that

$$\begin{aligned} &\|U_{\sigma_1}(T_1, \tau)x - U_{\sigma_2}(T_1, \tau)y\|_X \\ &\leq \varepsilon_1 + \psi_{T_1}(x, y, \sigma_1, \sigma_2) \end{aligned} \quad (34)$$

for any $x, y \in B_0, \sigma_1, \sigma_2 \in \Sigma$.

Since $t_n \rightarrow +\infty$, for some T_1 fixed we assume that $t_n \gg T_1$ is so large that $U_{g_n}(t_n - T_1 + \tau, \tau)x_n \in B_0$ for each $n \geq N$ and $g_n \in \Sigma$.

Let $y_n = U_{g_n}(t_n - T_1 + \tau, \tau)x_n$; then from (23)–(26) we have

$$\begin{aligned} &\|U_{g_n}(t_n, \tau)x_n - U_{g_m}(t_m, \tau)x_m\|_X \\ &= \|U_{g_n}(t_n, t_n - T_1 + \tau) \circ U_{g_n}(t_n - T_1 + \tau, \tau)x_n \\ &\quad - U_{g_m}(t_m, t_m - T_1 + \tau) \\ &\quad \circ U_{g_m}(t_m - T_1 + \tau, \tau)x_m\|_X = \|U_{\sigma_n}(T_1, \tau)y_n \\ &\quad - U_{\sigma_m}(T_1, \tau)y_m\|_X \leq \varepsilon_1 + \psi_{T_1}(y_n, y_m, \sigma_n, \sigma_m), \end{aligned} \quad (35)$$

where $\sigma_n = H(t_n - T_1)g_n$.

Due to the definition of $\mathfrak{C}(B_0, \Sigma)$ and $\psi_{T_1} \in \mathfrak{C}(B_0, \Sigma)$, we know that, for ε_1 from (35), there is a contractive function ϕ on $B_0 \times B_0$ and $\{(y_n, \sigma_n)\}_{n=1}^\infty$ has a subsequence $\{(y_{n_k}^{(1)}, \sigma_{n_k}^{(1)})\}_{k=1}^\infty$ such that one gets the following estimation:

$$\begin{aligned} &\psi_{T_1}(y_{n_k}^{(1)}, y_{n_l}^{(1)}, \sigma_{n_k}^{(1)}, \sigma_{n_l}^{(1)}) \\ &\leq \varepsilon_1 + \phi_{T_1}(y_{n_k}^{(1)}, y_{n_l}^{(1)}, \sigma_{n_k}^{(1)}, \sigma_{n_l}^{(1)}), \end{aligned} \quad (36)$$

$$\lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} \phi_{T_1}(y_{n_k}^{(1)}, y_{n_l}^{(1)}, \sigma_{n_k}^{(1)}, \sigma_{n_l}^{(1)}) = 0. \quad (37)$$

And similar to [15, 22, 30], we have

$$\begin{aligned} &\lim_{k \rightarrow \infty} \sup_{p \in \mathbb{N}} \|U_{\sigma_{n_k+p}^{(1)}}(t_{n_k+p}^{(1)}, \tau)x_{n_k+p}^{(1)} - U_{\sigma_{n_k}^{(1)}}(t_{n_k}^{(1)}, \tau)x_{n_k}^{(1)}\|_X \\ &\leq \lim_{k \rightarrow \infty} \sup_{p \in \mathbb{N}} \lim_{l \rightarrow \infty} \|U_{\sigma_{n_k+p}^{(1)}}(t_{n_k+p}^{(1)})x_{n_k+p}^{(1)} \\ &\quad - U_{\sigma_{n_l}^{(1)}}(t_{n_l}^{(1)}, \tau)x_{n_l}^{(1)}\|_X \\ &\quad + \lim_{k \rightarrow \infty} \sup_{p \in \mathbb{N}} \lim_{l \rightarrow \infty} \|U_{\sigma_{n_k}^{(1)}}(t_{n_k}^{(1)}, \tau)x_{n_k}^{(1)} \\ &\quad - U_{\sigma_{n_l}^{(1)}}(t_{n_l}^{(1)}, \tau)x_{n_l}^{(1)}\|_X \leq \varepsilon_1 \\ &\quad + \lim_{k \rightarrow \infty} \sup_{p \in \mathbb{N}} \lim_{l \rightarrow \infty} \psi_{T_1}(y_{n_k+p}^{(1)}, y_{n_l}^{(1)}, \sigma_{n_k+p}^{(1)}, \sigma_{n_l}^{(1)}) + \varepsilon_1 \\ &\quad + \lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} \psi_{T_1}(y_{n_k}^{(1)}, y_{n_l}^{(1)}, \sigma_{n_k}^{(1)}, \sigma_{n_l}^{(1)}) \leq 4\varepsilon_1 \\ &\quad + \lim_{k \rightarrow \infty} \sup_{p \in \mathbb{N}} \lim_{l \rightarrow \infty} \phi_{T_1}(y_{n_k+p}^{(1)}, y_{n_l}^{(1)}, \sigma_{n_k+p}^{(1)}, \sigma_{n_l}^{(1)}) \\ &\quad + \lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} \phi_{T_1}(y_{n_k}^{(1)}, y_{n_l}^{(1)}, \sigma_{n_k}^{(1)}, \sigma_{n_l}^{(1)}) \end{aligned} \quad (38)$$

which, combined with (35) and (37), implies that

$$\lim_{k \rightarrow \infty} \sup_{p \in \mathbb{N}} \left\| U_{\sigma_{n_{k+p}}^{(1)}} \left(t_{n_{k+p}}^{(1)}, \tau \right) x_{n_{k+p}}^{(1)} - U_{\sigma_{n_k}^{(1)}} \left(t_{n_k}^{(1)}, \tau \right) x_{n_k}^{(1)} \right\|_X \leq 5\varepsilon_1. \quad (39)$$

Therefore, there is K_1 such that

$$\left\| U_{\sigma_{n_k}^{(1)}} \left(t_{n_k}^{(1)}, \tau \right) x_{n_k}^{(1)} - U_{\sigma_{n_l}^{(1)}} \left(t_{n_l}^{(1)}, \tau \right) x_{n_l}^{(1)} \right\|_X \leq 6\varepsilon_1 \quad \forall k, l \geq K_1. \quad (40)$$

By induction, we obtain that, for each $m \geq 1$, there is a subsequence $\{U_{\sigma_{n_k}^{(m+1)}}(t_{n_k}^{(m+1)}, \tau)x_{n_k}^{(m+1)}\}_{k=1}^\infty$ of $\{U_{\sigma_{n_k}^{(m)}}(t_{n_k}^{(m)}, \tau)x_{n_k}^{(m)}\}_{k=1}^\infty$ and certain K_{m+1} such that

$$\left\| U_{\sigma_{n_k}^{(m+1)}} \left(t_{n_k}^{(m+1)}, \tau \right) x_{n_k}^{(m+1)} - U_{\sigma_{n_l}^{(m+1)}} \left(t_{n_l}^{(m+1)}, \tau \right) x_{n_l}^{(m+1)} \right\|_X \leq 6\varepsilon_{m+1} \quad (41)$$

holds for all $k, l \geq K_{m+1}$. Now, we consider the diagonal subsequence $\{U_{\sigma_{n_k}^{(k)}}(t_{n_k}^{(k)}, \tau)x_{n_k}^{(k)}\}_{k=1}^\infty$. Since, for each $m \in \mathbb{N}$, $\{U_{\sigma_{n_k}^{(k)}}(t_{n_k}^{(k)}, \tau)x_{n_k}^{(k)}\}_{k=1}^\infty$ is a subsequence of $\{U_{\sigma_{n_k}^{(m)}}(t_{n_k}^{(m)}, \tau)x_{n_k}^{(m)}\}_{k=1}^\infty$, then

$$\left\| U_{\sigma_{n_k}^{(k)}} \left(t_{n_k}^{(k)}, \tau \right) x_{n_k}^{(k)} - U_{\sigma_{n_l}^{(l)}} \left(t_{n_l}^{(l)}, \tau \right) x_{n_l}^{(l)} \right\|_X \leq 6\varepsilon_m \quad \forall k, l \geq \max\{m, K_m\}, \quad (42)$$

which, combined with $\varepsilon_m \rightarrow 0$ as $m \rightarrow \infty$, implies that $\{U_{\sigma_{n_k}^{(k)}}(t_{n_k}^{(k)}, \tau)x_{n_k}^{(k)}\}_{k=1}^\infty$ is a Cauchy sequence in X . This shows that $\{U_{\sigma_n}(t_n, \tau)x_n\}_{n=1}^\infty$ is precompact in X . Then the proof is complete. $\square$

Proof of Theorem 2. From Theorem 6, the family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, on X satisfy the following conditions:

- (i) $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, has a bounded absorbing set B_0 in X ;
- (ii) $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, is asymp
- (iii) totically compact in X .

Then $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, has a uniform attractor in X . $\square$

Lemma 7 (see [31]). *Let $X \in H \subset Y$ be Banach spaces, with X reflexive. Suppose that u_n is a sequence that is uniformly bounded in $L^2(0, T; X)$ and du_n/dt is uniformly bounded in $L^p(0, T; Y)$, for some $p > 1$. Then there is a subsequence of u_n that converges strongly in $L^2(0, T; H)$.*

3. Uniform Attractors in $H_0^1(\Omega) \times L_\mu^2(\mathbb{R}^+; H_0^1(\Omega))$

3.1. A Priori Estimates. We start with the following general existence and uniqueness of solutions for the nonclassical diffusion equations with fading memory which can be obtained by the Galerkin approximation methods; here we formulate only the results.

Lemma 8. *Assume that $g_0 \in L_b^2(\mathbb{R}; L^2(\Omega))$ and $g \in \Sigma$ and f satisfies (13)-(14). Then, for any initial data $z_\tau \in \mathcal{M}_1$ and any $T > 0$, there exists a unique solution $z = (u, \eta^t)$ for problem (10)-(11). Moreover, we have the following Lipschitz continuity: for any (z_τ^i, g_i) ($z_\tau^i \in \mathcal{M}_1, g_i \in \Sigma$), denote by z_i ($i = 1, 2$) the corresponding solutions of (10); then for all $\tau \leq t \leq T + \tau$*

$$\|z_1(t) - z_2(t)\|_{\mathcal{M}_1}^2 \leq Q\left(\|z_\tau^1\|_{\mathcal{M}_1}, \|z_\tau^2\|_{\mathcal{M}_1}, T\right) \cdot \left(\|z_\tau^1 - z_\tau^2\|_{\mathcal{M}_1}^2 + \|g_1 - g_2\|_{L_b^2}^2\right), \quad (43)$$

where $Q(\cdot)$ is a monotonically increasing function.

By Lemma 8, we can define a process $U_g(t, \tau)$ in $\mathcal{M}_1$ as the following:

$$\begin{aligned} U_g(t, \tau) : \mathbb{R}_\tau \times \mathcal{M}_1 &\longrightarrow \mathcal{M}_1, \\ z_\tau = (u_\tau, \eta^\tau) &\longrightarrow z(t) = (u(t), \eta^t) \\ &= U_g(t, \tau) z_\tau, \end{aligned} \quad (44)$$

and $\{U_g(t, \tau)\}$, $g \in \Sigma$, is a family of processes on $\mathcal{M}_1$. See [5] for more details.

Lemma 9. *Let (13)-(14) hold, and $g \in \Sigma$. Then there exists a positive ρ_0 , which depends only on $\|g\|_{L_b^2}$, such that, for any bounded subset $B \subset \mathcal{M}_1$, there is $T_0 = T_0(\|B\|_{\mathcal{M}_1})$ such that*

$$\|U_g(t, \tau) z_\tau\|_{\mathcal{M}_1}^2 \leq \rho_0, \quad (45)$$

for all $t - \tau \geq T_0$ and all $z_\tau \in B$.

Proof. Multiplying (10) by u and then integrating over Ω , it follows that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(|u|_2^2 + \|u\|_0^2 + \|\eta^t\|_{\mu, \mathcal{V}}^2 \right) + \|u\|_0^2 + \langle \eta^t, \eta_s^t \rangle_{\mu, \mathcal{V}} \\ + \langle f(u), u \rangle = \langle g, u \rangle. \end{aligned} \quad (46)$$

Observe that

$$\langle f(u), u \rangle = \int_\Omega f(u) u \geq \gamma_1 |u|_p^p - \beta_1 \text{mes}(\Omega). \quad (47)$$

Using Lemma 1, we have

$$\langle \eta^t, \eta_s^t \rangle_{\mu, \mathcal{V}} \geq \frac{\delta}{2} \|\eta^t\|_{\mu, \mathcal{V}}^2. \quad (48)$$

Using the Hölder inequality, combining with (47) and (48), then (46) can be reformulated as follows:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(|u|_2^2 + \|u\|_0^2 + \|\eta^t\|_{\mu, \mathcal{V}}^2 \right) + \frac{1}{2} \|u\|_0^2 + \frac{\delta}{2} \|\eta^t\|_{\mu, \mathcal{V}}^2 \\ + \gamma_1 |u|_p^p \leq \beta_1 \text{mes}(\Omega) + \frac{1}{2\lambda_1^2} |g|_2^2. \end{aligned} \quad (49)$$

According to Poincaré inequality, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(|u|_2^2 + \|u\|_0^2 + \|\eta^t\|_{\mu, \mathcal{V}}^2 \right) + \frac{\lambda_1}{4} |u|_2^2 + \frac{1}{4} \|u\|_0^2 \\ + \frac{\delta}{2} \|\eta^t\|_{\mu, \mathcal{V}}^2 + \gamma_1 |u|_p^p \leq \frac{C}{2} (1 + |g|_2^2). \end{aligned} \quad (50)$$

Let $\alpha = \min(\lambda_1/2, 1/2, \delta)$; again, we have

$$\begin{aligned} \frac{d}{dt} \left(|u|_2^2 + \|u\|_0^2 + \|\eta^t\|_{\mu, \mathcal{V}}^2 \right) \\ + \alpha \left(|u|_2^2 + \|u\|_0^2 + \|\eta^t\|_{\mu, \mathcal{V}}^2 \right) \leq C (1 + |g|_2^2). \end{aligned} \quad (51)$$

Using the Gronwall Lemma, it follows from (51) that

$$\begin{aligned} |u|_2^2 + \|u\|_0^2 + \|\eta^t\|_{\mu, \mathcal{V}}^2 \\ \leq e^{-\alpha(t-\tau)} \left(|u_\tau|_2^2 + \|u_\tau\|_0^2 + \|\eta^\tau\|_{\mu, \mathcal{V}}^2 \right) \\ + C \left(1 + \|g\|_{L_b^2}^2 \right). \end{aligned} \quad (52)$$

We know that $\|g\|_{L_b^2}^2$ is bounded, and by (12) and (6)

$$\begin{aligned} \|\eta^\tau\|_{\mu, \mathcal{V}}^2 &= \int_0^{+\infty} \mu(s) \|\eta^\tau(s)\|_0^2 ds \\ &\leq \int_0^{+\infty} \int_0^s \mu(s) \|u_\tau(\tau-r)\|_0^2 dr ds \\ &\leq \int_0^{+\infty} dr \int_r^{+\infty} e^{-\delta s} \|u_\tau(\tau-r)\|_0^2 ds \\ &\leq \frac{1}{\delta} \int_0^{+\infty} e^{-\delta s} \|u_\tau(\tau-s)\|_0^2 ds \leq \delta^{-1} \mathfrak{R}. \end{aligned} \quad (53)$$

So $\|z_\tau\|_{\mathcal{M}_1}^2 = |u_\tau|_2^2 + \|u_\tau\|_0^2 + \|\eta^\tau\|_{\mu, \mathcal{V}}^2$ is bounded.

For any $t - \tau > T_0 = (1/\alpha) \ln(\|z_\tau\|_{\mathcal{M}_1}^2 / C(1 + \|g\|_{L_b^2}^2))$, we infer from (52) that

$$|u(t)|_2^2 + \|u(t)\|_0^2 + \|\eta^t\|_{\mu, \mathcal{V}}^2 \leq 2C \left(1 + \|g\|_{L_b^2}^2 \right). \quad (54)$$

For any $t - \tau > T_0$, we get

$$|u(t)|_2^2 + \|u(t)\|_0^2 + \|\eta^t\|_{\mu, \mathcal{V}}^2 \leq \rho_0, \quad (55)$$

where $\rho_0 = 2C(1 + \|g\|_{L_b^2}^2)$. Then we get

$$B_0 = \{z = (u, \eta^t) \in \mathcal{M}_1 : \|z\|_{\mathcal{M}_1}^2 \leq \rho_0\}. \quad (56)$$

The proof is complete. $\square$

Combining with (43), we know that, for any $\tau \in \mathbb{R}$, U_g maps the bounded set of $\mathcal{M}_1$ into a bounded set for all $t \geq \tau$ that is as follows.

Corollary 10. Let (13)-(14) hold, and $g \in \Sigma$. Then, for any bounded (in $\mathcal{M}_1$) subset B , there is $M_B = M(\|B\|_{\mathcal{M}_1}, \|g\|_{L_b^2})$ such that

$$\|U_g(t, \tau) z_\tau\|_{\mathcal{M}_1}^2 \leq M_B \quad (57)$$

for all $t \geq \tau$ and all $z_\tau \in B$.

Lemma 11. Let (13)-(14) hold, and $g \in \Sigma$. Then, for any bounded (in $\mathcal{M}_1$) subset B , there exists a positive constant $C = C(\alpha, \gamma_1, \rho_0)$, such that

$$\begin{aligned} \int_t^{t+1} |u(s)|_p^p ds &< C \left(1 + \|g\|_{L_b^2}^2 \right), \\ \int_t^{t+1} \|z(s)\|_{\mathcal{M}_1}^2 ds &< C \left(1 + \|g\|_{L_b^2}^2 \right) \end{aligned} \quad (58)$$

hold for any $t - \tau \geq T_0$ (from Lemma 9).

Proof. Combining with (49), let $\alpha = \min(\lambda_1/2, 1/2, \delta)$; it follows that

$$\begin{aligned} \frac{d}{dt} \left(|u|_2^2 + \|u\|_0^2 + \|\eta^t\|_{\mu, \mathcal{V}}^2 \right) \\ + \alpha \left(|u|_2^2 + \|u\|_0^2 + \|\eta^t\|_{\mu, \mathcal{V}}^2 \right) + \gamma_1 |u|_p^p \\ \leq 2C \left(1 + |g|_2^2 \right). \end{aligned} \quad (59)$$

For any $t - \tau \geq T_0$ (from Lemma 9) and integrating the inequality above from t to $t + 1$, we can get

$$\begin{aligned} \alpha \int_t^{t+1} \|z(s)\|_{\mathcal{M}_1}^2 ds + 2\gamma_1 \int_t^{t+1} |u(s)|_p^p ds \\ \leq \|z(t)\|_{\mathcal{M}_1}^2 + 2C \left(1 + \|g\|_{L_b^2}^2 \right). \end{aligned} \quad (60)$$

By Lemma 9, we have

$$\begin{aligned} \alpha \int_t^{t+1} \|z(s)\|_{\mathcal{M}_1}^2 ds + 2\gamma_1 \int_t^{t+1} |u(s)|_p^p ds \\ \leq \rho_0 + 2C \left(1 + \|g\|_{L_b^2}^2 \right). \end{aligned} \quad (61)$$

Setting $\alpha_0 = \min(\alpha, 2\gamma_1)$, then

$$\begin{aligned} \int_t^{t+1} \|z(s)\|_{\mathcal{M}_1}^2 ds &\leq \frac{1}{\alpha_0} \left(\rho_0 + 2C \left(1 + \|g\|_{L_b^2}^2 \right) \right), \\ \int_t^{t+1} |u(s)|_p^p ds &\leq \frac{1}{\alpha_0} \left(\rho_0 + 2C \left(1 + \|g\|_{L_b^2}^2 \right) \right). \end{aligned} \quad (62)$$

Let $C = C(\alpha, \gamma_1, \rho_0) = (1/\alpha_0)(\rho_0 + 2C)$; then we have that

$$\begin{aligned} \int_t^{t+1} \|z(s)\|_{\mathcal{M}_1}^2 ds &\leq C \left(1 + \|g\|_{L_b^2}^2 \right), \\ \int_t^{t+1} |u(s)|_p^p ds &\leq C \left(1 + \|g\|_{L_b^2}^2 \right) \end{aligned} \quad (63)$$

hold for any $t - \tau \geq T_0$. $\square$

Let $F(s) = \int_0^s f(v)dv$; from assumption (3), there are positive constants $\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\gamma}_1, \tilde{\gamma}_2$, such that

$$\tilde{\gamma}_1 |s|^p - \tilde{\beta}_1 \leq F(s) \leq \tilde{\gamma}_2 |s|^p + \tilde{\beta}_2. \quad (64)$$

Lemma 12. *There is a positive constant ρ_1 , for any $z_\tau = (u_\tau, \eta^\tau) \in B_0$ (from Lemma 9); the following estimate*

$$|u(t)|_p^p \leq \rho_1 \quad (65)$$

holds for any $t - \tau \geq 1$.

Proof. Multiplying (10) by u_t and then integrating in Ω , we get

$$\begin{aligned} & |u_t(t)|_2^2 + \|u_t(t)\|_0^2 + \frac{d}{dt} \left(\frac{1}{2} \|u(t)\|_0^2 + \int_\Omega F(u(t)) \right) \\ &= \int_0^{+\infty} \mu(s) \langle \Delta \eta^t(s), u_t(t) \rangle ds + (g(t), u_t(t)). \end{aligned} \quad (66)$$

By the Hölder inequality and the Young inequality, we have

$$\begin{aligned} & \int_0^{+\infty} \mu(s) \langle \Delta \eta^t(s), u_t(t) \rangle ds \\ & \leq \int_0^{+\infty} \mu(s) \|\eta^t(s)\|_0 \|u_t(t)\|_0 ds \\ & \leq \frac{m_0}{2} \int_0^{+\infty} \mu(s) \|\eta^t(s)\|_0^2 ds \\ & \quad + \frac{1}{2m_0} \|u_t(t)\|_0^2 \int_0^{+\infty} \mu(s) ds \\ & \leq \frac{m_0}{2} \|\eta^t\|_{\mu, \mathcal{V}}^2 + \frac{1}{2} \|u_t(t)\|_0^2 \\ & \leq \frac{m_0}{2} \|z(t)\|_{\mathcal{M}_1}^2 + \frac{1}{2} \|u_t(t)\|_0^2, \end{aligned} \quad (67)$$

here m_0 is from (7). And

$$(g(t), u_t(t)) \leq \frac{1}{2} |g(t)|_2^2 + \frac{1}{2} |u_t(t)|_2^2. \quad (68)$$

Let

$$E(t) = \frac{1}{2} \|u(t)\|_0^2 + \int_\Omega F(u(t)). \quad (69)$$

Then

$$\begin{aligned} & \frac{1}{2} |u_t(t)|_2^2 + \frac{1}{2} \|u_t(t)\|_0^2 + \frac{d}{dt} E(t) \\ & \leq \frac{m_0}{2} \|z(t)\|_{\mathcal{M}_1}^2 + \frac{1}{2} |g(t)|_2^2. \end{aligned} \quad (70)$$

So we get the following inequality:

$$\frac{d}{dt} E(t) \leq \frac{m_0}{2} \|z(t)\|_{\mathcal{M}_1}^2 + \frac{1}{2} |g(t)|_2^2. \quad (71)$$

Integrating the inequality above to t from s ($t \leq s \leq t+1$) to $t+1$, we have

$$E(t+1) \leq E(s) + \frac{m_0}{2} \int_t^{t+1} \|z(s)\|_{\mathcal{M}_1}^2 ds + \frac{1}{2} \|g\|_{L_b^2}^2, \quad (72)$$

and integrating (72) to s from t to $t+1$,

$$\begin{aligned} E(t+1) & \leq \int_t^{t+1} E(s) ds + \frac{m_0}{2} \int_t^{t+1} \|z(s)\|_{\mathcal{M}_1}^2 ds \\ & \quad + \frac{1}{2} \|g\|_{L_b^2}^2. \end{aligned} \quad (73)$$

Besides, by the Hölder inequality, Young inequality, and (13), we can infer from (69) that

$$\begin{aligned} E(s) & \leq \frac{1}{2} \|u(s)\|_0^2 + \tilde{\gamma}_2 |u(s)|_p^p + \tilde{\beta}_2 \text{mes}(\Omega) \\ & \leq \frac{1}{2} \|z(s)\|_{\mathcal{M}_1}^2 + \tilde{\gamma}_2 |u(s)|_p^p + \tilde{\beta}_2 \text{mes}(\Omega). \end{aligned} \quad (74)$$

On the contrary,

$$\begin{aligned} E(t+1) & \geq \frac{1}{2} \|u(t+1)\|_0^2 + \tilde{\gamma}_1 |u(t+1)|_p^p \\ & \quad - \tilde{\beta}_1 \text{mes}(\Omega) \\ & \geq \tilde{\gamma}_1 |u(t+1)|_p^p - \tilde{\beta}_1 \text{mes}(\Omega). \end{aligned} \quad (75)$$

Combining (73) and (74), we have

$$\begin{aligned} E(t+1) & \leq \frac{1+m_0}{2} \int_t^{t+1} \|z(s)\|_{\mathcal{M}_1}^2 ds \\ & \quad + \tilde{\gamma}_2 \int_t^{t+1} |u(s)|_p^p ds + \frac{1}{2} \|g\|_{L_b^2}^2 \\ & \quad + \tilde{\beta}_2 \text{mes}(\Omega). \end{aligned} \quad (76)$$

By Lemma 11, it follows from (76) that

$$\begin{aligned} E(t+1) & \leq C \left(\frac{1+m_0}{2} + \tilde{\gamma}_2 \right) \left(1 + \|g\|_{L_b^2}^2 \right) + \frac{1}{2} \|g\|_{L_b^2}^2 \\ & \quad + \tilde{\beta}_2 \text{mes}(\Omega), \end{aligned} \quad (77)$$

and by (75)–(77), we have

$$\begin{aligned} |u(t+1)|_p^p & \leq \frac{1}{\tilde{\gamma}_1} \left(\left(\frac{1+m_0}{2} + \tilde{\gamma}_2 \right) C \left(1 + \|g\|_{L_b^2}^2 \right) \right. \\ & \quad \left. + \frac{1}{2} \|g\|_{L_b^2}^2 + (\tilde{\beta}_1 + \tilde{\beta}_2) \text{mes}(\Omega) \right). \end{aligned} \quad (78)$$

Let

$$\begin{aligned} \rho_1 & = \frac{1}{\tilde{\gamma}_1} \left(\left(\frac{1+m_0}{2} + \tilde{\gamma}_2 \right) C \left(1 + \|g\|_{L_b^2}^2 \right) + \frac{1}{2} \|g\|_{L_b^2}^2 \right. \\ & \quad \left. + (\tilde{\beta}_1 + \tilde{\beta}_2) \text{mes}(\Omega) \right). \end{aligned} \quad (79)$$

Then the proof is complete. $\square$

Lemma 13 (bounded uniformly absorbing set). *Let (13)-(14) hold, and $g \in \Sigma$. Then there exist positive ρ_0, ρ_1 , which depend only on $\|g\|_{L_b^2}$, such that, for any bounded (in $\mathcal{M}_1$) subset B , there is $T_1 = T_1(\|B\|_{\mathcal{M}_1}) = T_0 + 1$ such that*

$$\begin{aligned} \|U_g(t, \tau) z_\tau\|_{\mathcal{M}_1}^2 &\leq \rho_0, \\ |u(t)|_p^p &\leq \rho_1 \end{aligned} \quad (80)$$

hold for all $t - \tau \geq T_1$ and all $z_\tau \in B$.

For brevity, in the sequel, let B_0 be the bounded absorbing set obtained in Lemmas 9 and 12; that is,

$$B_0 = \{z = (u, \eta^t) \in \mathcal{M}_1 : \|z\|_{\mathcal{M}_1}^2 \leq \rho_0, |u|_p^p \leq \rho_1\}. \quad (81)$$

Lemma 14. *There is a positive constant ρ_2 , for any $z_\tau = (u_\tau, \eta^\tau) \in B_0$ (from (81)); the following estimate*

$$\int_t^{t+1} (|u_t(s)|_2^2 + \|u_t(s)\|_0^2) ds \leq \rho_2, \quad (82)$$

holds for any $t - \tau \geq 0$.

Proof. By Lemma 12, for any $t - \tau \geq 0$, we integrate (70) to t from t to $t + 1$; then we have

$$E(t + 1) + \int_t^{t+1} (|u_t(s)|_2^2 + \|u_t(s)\|_0^2) ds \leq E(t). \quad (83)$$

According to Lemma 13, Corollary 10, and Lemma 12, we know that $E(t)$ is bounded for any $t - \tau \geq 0$, so there is a positive constant ρ_2 , and the conclusion is true. $\square$

3.2. Uniform Attractors. In the following, we will prove the existence of uniform attractors for system (10) with initial-boundary conditions (11) in $\mathcal{M}_1$ by using the method of asymptotic contractive function.

Lemma 15. *Let f satisfy (13)-(14); $z_n(t) = (u_n(t), \eta^{tn})$ is the sequence of solutions of (10) with initial data $z_{\tau n} = (u_{\tau n}, \eta^{\tau n}) \in B_0$ and $g_n \in \Sigma$ ($n = 1, 2, \dots$); then there is a subsequence u_{n_k} of $u_n(t)$ that converges strongly in $L^2(0, T; L^2(\Omega))$ and*

$$\begin{aligned} \lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \int_0^T \int_\Omega (g_{n_k}(x, s) - g_{n_j}(x, s)) \\ \cdot (u_{n_k}(x, s) - u_{n_j}(x, s)) ds = 0. \end{aligned} \quad (84)$$

Proof. By Lemmas 13 and 14, the sequence u_n is uniformly bounded in $L^2(0, T; H_0^1(\Omega))$ and du_n/dt is uniformly bounded in $L^2(0, T; H^{-1}(\Omega))$. Since $H_0^1(\Omega)$ is reflexive, is $L^2(0, T; H_0^1(\Omega))$. Let $X = H_0^1(\Omega)$, $H = L^2(\Omega)$, and $Y = H^{-1}(\Omega)$; then there is a subsequence $u_{n_k}(t)$ of $\{u_n(t)\}$ that converges strongly in $L^2(0, T; L^2(\Omega))$ by Lemma 7. One can write these as

$$\lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \int_0^T \int_\Omega |u_{n_k}(x, s) - u_{n_j}(x, s)|^2 ds = 0. \quad (85)$$

Then by using Hölder inequality, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \int_0^T \int_\Omega (g_{n_k}(x, s) - g_{n_j}(x, s)) \\ \cdot (u_{n_k}(x, s) - u_{n_j}(x, s)) ds \\ \leq \lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \left(\int_0^T \int_\Omega |g_{n_k}(x, s) - g_{n_j}(x, s)|^2 ds \right)^{1/2} \\ \cdot \left(\int_0^T \int_\Omega |u_{n_k}(x, s) - u_{n_j}(x, s)|^2 ds \right)^{1/2} \leq 2T \|g\|_{L_b^2} \\ \cdot \lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \left(\int_0^T \int_\Omega |u_{n_k}(x, s) - u_{n_j}(x, s)|^2 ds \right)^{1/2} \\ = 0. \end{aligned} \quad (86)$$

$\square$

Theorem 16. *Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary, and let μ and f satisfy (4)-(5) and (13)-(14), respectively. Assume further that $g_0 \in L_b^2(\mathbb{R}; L^2(\Omega))$ and Σ is the hull of g_0 in $L_{w,loc}^2(\mathbb{R}; L^2(\Omega))$. $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, is the family of processes generated by the solutions of (10) with initial data $z_\tau \in \mathcal{M}_1$. Then $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, is uniformly (with respect to $\sigma \in \Sigma$) asymptotic compact in $\mathcal{M}_1$.*

Proof. For any $z_\tau^i = (u_\tau^i, \eta_\tau^i) \in B_0$ and $g^i \in \Sigma$, let $z_i(t) = (u^i(t), \eta_i^t)$ be the corresponding solution with respect to initial data z_τ^i ($i = 1, 2$) and $g^i \in \Sigma$; that is, $z_i(t)$ is the solution of (10) with initial-boundary conditions (11).

For convenience, we denote $\omega(t) = u^1(t) - u^2(t)$ and $\xi^t = \eta_1^t - \eta_2^t$; then $\omega(t)$ satisfies the following equation:

$$\begin{aligned} \omega_t(t) - \Delta \omega_t(t) - \Delta \omega(t) - \int_0^\infty \mu(s) \Delta \xi^t(s) ds \\ + f(u_1(t)) - f(u_2(t)) = g_1(t) - g_2(t) \end{aligned} \quad (87)$$

$$\xi_t^t = -\xi_s^t + \omega,$$

with initial-boundary conditions

$$\begin{aligned} \omega|_{\partial\Omega} &= 0, \\ \xi^t(x, s)|_{\partial\Omega \times \mathbb{R}^+} &= 0, \quad t \geq \tau, \\ \omega(x, \tau) &= \omega_\tau(x) = u_\tau^1(x) - u_\tau^2(x), \quad x \in \Omega, \\ \xi^\tau(x, s) &= \eta_1^\tau(x, s) - \eta_2^\tau(x, s), \\ (x, s) &\in \Omega \times \mathbb{R}^+. \end{aligned} \quad (88)$$

Multiplying (87) by $\omega(t)$ and integrating in Ω , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (|\omega|_2^2 + \|\omega\|_0^2 + \|\xi^t\|_{\mu, \mathcal{V}}^2) + \|\omega\|_0^2 \\ + \langle \xi^t(s), \xi_s^t(s) \rangle_{\mu, \mathcal{V}} \end{aligned}$$

$$\begin{aligned}
& + \int_{\Omega} (f(u_1) - f(u_2))(u_1 - u_2) \\
& = \int_{\Omega} (g_1(t) - g_2(t))(u_1 - u_2).
\end{aligned} \tag{89}$$

By Lemma 1 and assumption (14), we have

$$\begin{aligned}
\langle \xi^t(s), \xi_s^t(s) \rangle_{\mu, \mathcal{V}} & \geq \frac{\delta}{2} \|\xi^t\|_{\mu, \mathcal{V}}^2, \\
\int_{\Omega} (f(u_1) - f(u_2))(u_1 - u_2) & \geq -l|\omega|_2^2.
\end{aligned} \tag{90}$$

Using the Poincaré inequality, it follows that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (|\omega|_2^2 + \|\omega\|_0^2 + \|\xi^t\|_{\mu, \mathcal{V}}^2) + \frac{\lambda_1}{2} |\omega|_2^2 + \frac{1}{2} \|\omega\|_0^2 \\
& + \frac{\delta}{2} \|\xi^t\|_{\mu, \mathcal{V}}^2 \\
& \leq l|\omega|_2^2 + \int_{\Omega} (g_1(t) - g_2(t))(u_1 - u_2).
\end{aligned} \tag{91}$$

Let $\varrho = \min(\lambda_1, 1, \delta)$; then we get

$$\begin{aligned}
& \frac{d}{dt} (|\omega|_2^2 + \|\omega\|_0^2 + \|\xi^t\|_{\mu, \mathcal{V}}^2) \\
& + \varrho (|\omega|_2^2 + \|\omega\|_0^2 + \|\xi^t\|_{\mu, \mathcal{V}}^2) \\
& \leq 2l|u_1 - u_2|_2^2 + 2 \int_{\Omega} (g_1(t) - g_2(t))(u_1 - u_2).
\end{aligned} \tag{92}$$

By the Gronwall Lemma, we have

$$\begin{aligned}
|\omega(t)|_2^2 + \|\omega(t)\|_0^2 + \|\xi^t\|_{\mu, \mathcal{V}}^2 & \leq e^{-\varrho(t-\tau)} (|\omega_{\tau}|_2^2 + \|\omega_{\tau}\|_0^2 \\
& + \|\xi^{\tau}\|_{\mu, \mathcal{V}}^2) + 2l \int_{\tau}^t e^{-\varrho(t-s)} |u_1(s) - u_2(s)|_2^2 ds \\
& + 2 \int_{\tau}^t e^{-\varrho(t-s)} \int_{\Omega} (g_1(s) - g_2(s)) \\
& \cdot (u_1(s) - u_2(s)) ds.
\end{aligned} \tag{93}$$

For any $\varepsilon > 0$, there exists $T_2 = T_2(\varepsilon) = \max(T_1, (1/\varrho) \ln(\varepsilon/\|z_{\tau}\|_{\mathcal{M}_1}^2)) \geq T_1$, such that

$$|\omega(t)|_2^2 + \|\omega(t)\|_0^2 + \|\xi^t\|_{\mu, \mathcal{V}}^2 \leq \varepsilon + \psi_t \tag{94}$$

holds for any $t \geq T_2$; here

$$\begin{aligned}
\psi_t & = \psi_t(u_1, u_2, g_1, g_2) = 2l \int_{\tau}^t e^{-\varrho(t-s)} |u_1(s) \\
& - u_2(s)|_2^2 ds + 2 \int_{\tau}^t e^{-\varrho(t-s)} \int_{\Omega} (g_1(s) - g_2(s)) \\
& \cdot (u_1(s) - u_2(s)) ds.
\end{aligned} \tag{95}$$

Hereafter, we verify that there is $T \geq T_2$ such that ψ_T is an asymptotic contractive function on B_0 . In fact

$$\begin{aligned}
\psi_t & = 2l \int_{\tau}^t e^{-\varrho(t-s)} |u_1(s) - u_2(s)|_2^2 ds \\
& + 2 \int_{\tau}^t e^{-\varrho(t-s)} \int_{\Omega} (g_1(s) - g_2(s)) \\
& \cdot (u_1(s) - u_2(s)) ds \leq 2l \int_{\tau}^{T_1} e^{-\varrho(t-s)} |u_1(s) \\
& - u_2(s)|_2^2 ds \\
& + 2 \int_{\tau}^{T_1} e^{-\varrho(t-s)} \int_{\Omega} (g_1(s) - g_2(s)) \\
& \cdot (u_1(s) - u_2(s)) ds + 2l \int_{T_1}^t e^{-\varrho(t-s)} |u_1(s) \\
& - u_2(s)|_2^2 ds \\
& + 2 \int_{T_1}^t e^{-\varrho(t-s)} \int_{\Omega} (g_1(s) - g_2(s)) \\
& \cdot (u_1(s) - u_2(s)) ds
\end{aligned} \tag{96}$$

$$\begin{aligned}
& \leq e^{-\varrho(t-T_1)} \left(2l \int_{\tau}^{T_1} |u_1(s) - u_2(s)|_2^2 ds \right. \\
& + 2 \int_{\tau}^{T_1} \int_{\Omega} (g_1(s) - g_2(s))(u_1(s) - u_2(s)) ds \\
& + 2l \int_{T_1}^t |u_1(s) - u_2(s)|_2^2 ds \\
& + 2 \int_{T_1}^t \int_{\Omega} (g_1(s) - g_2(s))(u_1(s) - u_2(s)) ds \\
& \leq 2T_1 e^{-\varrho(t-T_1)} \left((2l+1)M_B + \|g\|_{L_b^2}^2 \right) \\
& + 2l \int_0^{t-T_1} |u_1(s+T_1) - u_2(s+T_1)|_2^2 ds \\
& + 2 \int_0^{t-T_1} \int_{\Omega} (g_1(s+T_1) - g_2(s+T_1)) \\
& \cdot (u_1(s+T_1) - u_2(s+T_1)) ds,
\end{aligned}$$

where M_B is from Corollary 10. Let

$$\begin{aligned}
\varphi_t & = 2l \int_0^{t-T_1} |v_1(s) - v_2(s)|_2^2 ds \\
& + 2 \int_0^{t-T_1} \int_{\Omega} (\sigma_1(s) - \sigma_2(s))(v_1(s) - v_2(s)) ds,
\end{aligned} \tag{97}$$

where $v_i(s) = u_i(s+T_1) \in B_0$ and $\sigma_i(s) = g_i(s+T_1) \in \Sigma$ ($i = 1, 2$). For any $\varepsilon > 0$, let $t = T \geq T_1 + (1/\varrho) \ln(2T_1((2l+1)M_B + \|g\|_{L_b^2}^2)/\varepsilon)$ be fixed; then

$$\psi_T(u_1, u_2, g_1, g_2) \leq \varepsilon + \varphi_T(u_1, u_2, g_1, g_2). \tag{98}$$

Combining Lemma 15 and Definition 5, we know that φ_T is the contractive function on $B_0 \times B_0$ and ψ_T is the asymptotic contractive function on $B_0 \times B_0$.

Then $\{U_g(t, \tau)\}_{t \geq 0}$, $g \in \Sigma$, is uniform (with respect to $g \in \Sigma$) asymptotic compact in $\mathcal{M}_1$ by Theorem 6. $\square$

We also recall the following useful result, whose proof is simple and we omit it.

Lemma 17 (see [15]). *Let X be a reflexive Banach space and $x_n \rightarrow 0$ in X . Then, for each compact (in X^*) subset $B \subset X^*$, the uniform convergence holds: for any $\varepsilon > 0$, there is N_ε , depending only on ε , such that*

$$|\langle f, x_n \rangle_{X^*}| \leq \varepsilon \quad \text{for all } n \geq N_\varepsilon \text{ and all } f \in B. \quad (99)$$

Proof of Theorem 3. Theorem 2 implies that the family of processes $\{U_g(t, \tau)\}$, $g \in \Sigma$, corresponding to (1), has a compact (in $\mathcal{M}_1$) uniform (with respect to $g \in \Sigma$) attracting set which is bounded in $\mathcal{M}_1$; consequently, as a direct application of the abstract theorem [29, Chapter IV, Theorem 3.1], we obtain the existence of a compact uniform (with respect to $g \in \Sigma$) attractor $\mathcal{A}$, and $\mathcal{A} \subset \mathcal{M}_1$.

We remark that the above existence does not require any continuity of the family of processes (e.g., see [29]). However, in order to obtain the explicit form of $\mathcal{A}$, that is, equality (29), we need some continuity. Moreover, since the symbol space Σ now has only *weak compactness*, we need to verify the corresponding *weak continuity*.

Hence, in order to obtain (29), we need to verify the following: for any fixed $\tau \in \mathbb{R}$ and $t \geq \tau$, if $z_{n\tau} \rightarrow z_\tau$ in $\mathcal{M}_1$ and $g_n \rightarrow g$ with respect to the local weak convergence topology of $L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$, then

$$U_{g_n}(t, \tau) z_{n\tau} \rightarrow U_g(t, \tau) z_\tau \quad \text{in } \mathcal{M}_1. \quad (100)$$

In the following, we will verify that the continuity above holds on $\mathcal{A} \times \Sigma$, which obviously is a compact uniformly attracting set for the corresponding skew-product flow semigroup (generated by the family processes $U_g(t, \tau) z_\tau$, $g \in \Sigma$, and the shift semigroup $\{T(t)\}_{t \geq 0}$). We will complete the proof of the weak continuity.

Let $z_n(t) = (u_n, \eta_n^t)$, let $z_n(t)$ be the solutions of (10) with initial data $z_{n\tau} \in \mathcal{A}$ ($n = 1, 2, \dots$), and let the time-dependent forcing $g_n \in \Sigma$, $z(t)$ be the solutions of (10) with initial data $z_\tau \in \mathcal{A}$ and the time-dependent forcing $g \in \Sigma$; we assume the following hypotheses:

$$z_{n\tau} \rightarrow z_\tau, \quad \text{in } \mathcal{M}_1, \quad (101)$$

$$g_n \rightarrow g, \quad \text{in } L^2_{w, \text{loc}}(\mathbb{R}, L^2(\Omega)). \quad (102)$$

Since (10) holds as an equality in $L^q(\tau, T; H^{-1}(\Omega))$ (q is conjugate to p from (13)), this means that for any $T(> \tau)$ fixed

$$u_n \rightharpoonup u, \quad \text{in } L^2(\tau, T; H_0^1(\Omega)), \quad (103)$$

where $u = \Pi_1 U_g(t, \tau) z_\tau \in L^2(\tau, T; H_0^1(\Omega))$, and Π_1 is the projector from $X \times Y$ to X . Since $\mathcal{A}$ is bounded in $\mathcal{M}_1$, following Corollary 10, we know that there is M such that

$$\sup_{g \in \Sigma} \sup_{\tau \in \mathbb{R}} \sup_{t \geq \tau} \|U_g(t, \tau) \mathcal{A}\|_{\mathcal{M}_1}^2 \leq M < \infty. \quad (104)$$

Hence,

$$\begin{aligned} & \bigcup_{g \in \Sigma} \{\Pi_1 U_g(t, \tau) z_\tau : t \in [\tau, T], z_\tau \in \mathcal{A}\} \\ & \text{is bounded in } L^2(\tau, T; H_0^1(\Omega)), \end{aligned} \quad (105)$$

$$\begin{aligned} & \bigcup_{g \in \Sigma} \{\partial_t \Pi_1 U_g(t, \tau) z_\tau : t \in [\tau, T], z_\tau \in \mathcal{A}\} \\ & \text{is bounded in } L^2(\tau, T; H^{-1}(\Omega)). \end{aligned}$$

Then by the compactness lemma (Lemma 7) we know that

$$\begin{aligned} & \bigcup_{g \in \Sigma} \{\Pi_1 U_g(t, \tau) z_\tau : t \in [\tau, T], z_\tau \in \mathcal{A}\} \\ & \text{is compact in } L^2(\tau, T; L^2(\Omega)). \end{aligned} \quad (106)$$

It is

$$\lim_{n \rightarrow \infty} \int_\tau^T \int_\Omega |u_n(s) - u(s)|^2 ds = 0. \quad (107)$$

Denote $\omega(t) = u_n(t) - u(t)$, $\xi^t = \eta_n^t - \eta^t$, $\omega_\tau = u_{n\tau} - u_\tau$, and $z_n(t) = (u_n(t), \eta_n^t) = U_{g_n}(t, \tau) z_{n\tau} - U_g(t, \tau) z_\tau$, $z_{n\tau}, z_\tau \in \mathcal{A}$, $n = 1, 2, \dots$. The proof of the following inequality is similar to (93):

$$\begin{aligned} & |\omega(t)|_2^2 + \|\omega(t)\|_0^2 + \|\xi^t\|_{\mu, \gamma}^2 \\ & \leq e^{-\varrho(t-\tau)} \|\omega_\tau\|_{\mathcal{M}_1}^2 + 2l \int_\tau^t |u_n(s) - u(s)|_2^2 ds \\ & \quad + 2 \int_\tau^t \int_\Omega (g_n(s) - g(s)) (u_n(s) - u(s)) ds. \end{aligned} \quad (108)$$

Due to (107) and (102), we only need to show that if $g_n \rightarrow g$ in $L^2_{w, \text{loc}}(\mathbb{R}; L^2(\Omega))$, then

$$\left| \int_\tau^t \langle g_n(x, s) - g(x, s), w(x, s) \rangle ds \right| \rightarrow 0 \quad (109)$$

uniformly on $L^2(\tau, t; L^2(\Omega))$ -compact set. But this is a direct application of Lemma 17. Therefore we complete the proof of the *continuity* on $\mathcal{A} \times \Sigma$.

Based on the *continuity claim* above, and by constructing a skew-product flow on $\mathcal{A} \times \Sigma$ and applying [29, Chapter IV, Theorem 5.1], we obtain the structure equality (29). This completes the proof of Theorem 3. $\square$

Competing Interests

The authors declare that they have no competing interests.

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