Research Article

On the Stability of Quadratic Functional Equations in F -Spaces

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The Hyers-Ulam-Rassias stability of quadratic functional equation $f(2x + y) + f(2x - y) = f(x + y) + f(x - y) + 6f(x)$ and orthogonal stability of the Pexiderized quadratic functional equation $f(x + y) + f(x - y) = 2g(x) + 2h(y)$ in F -spaces are proved.

1. Introduction

In 1940, Ulam [1] proposed the following stability problem: given metric group $G(\cdot, \rho)$, number $\varepsilon > 0$ and mapping $f : G \rightarrow G$ which satisfies inequality $\rho(f(x \cdot y), f(x) \cdot f(y)) < \varepsilon$ for all x, y in G , do automorphism a of G and constant $k > 0$, depending only on G , such that $\rho(a(x), f(x)) \leq k\varepsilon$ for all x in G , exist? If the answer is affirmative, we call equation $a(x \cdot y) = a(x) \cdot a(y)$ of automorphism stable. One year later, Hyers [2] provided a positive partial answer to Ulam's problem. In 1978, a generalized version of Hyers' result was proved by Rassias in [3]. Since then, the stability problems of several functional equations have been extensively investigated by a number of authors [4–17]. In fact, we also refer the readers to the paper [18] for recent developments in Ulam's type stability, [19] for recent developments of the conditional stability of the homomorphism equation and books, and [8, 20] for the general understanding of the stability theory.

Another important stability problem is orthogonal stability, which is closely related to the notion of orthogonality spaces; we know that a number of definitions of orthogonality in vector spaces, in addition to the usual one for inner product spaces, have appeared in the literature during the past half century. Many of these are mentioned in an article by Drljević [21]. Perhaps the best known of these is the “Birkhoff-James” orthogonality (see James [22]) for real normed vector spaces, where x is orthogonal to y meaning that $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$. In giving his axiomatic definition of orthogonality, Rätz in a 1985 paper [23] modified the definition given on

pp. 427–428 of Gudder and Strawther [24] and arrived at the following.

Definition 1. Suppose that X is a real vector space with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

- (O1) Totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$.
- (O2) Independence: if $x, y \in X \setminus 0$, $x \perp y$, then x, y are linearly independent.
- (O3) Homogeneity: if $x, y \in X$, $x \perp y$, then $ax \perp by$ for all $a, b \in \mathbb{R}$.
- (O4) The Thalesian property: Let P be a 2-dimensional subspace of X . If $x \in P$ and $\lambda \in \mathbb{R}^+$, then there exists $y \in P$ such that $x \perp y$ and $x + y \perp \lambda x - y$.

The pair $(X, \perp)$ is called an orthogonality space.

Rätz points out that this definition is more restrictive than that given by Gudder and Strawther [24], but he showed that his definition includes the following basic examples.

Example 2. The trivial orthogonality on vector space X is defined by (O1), and for nonzero elements $x, y \in X$, $x \perp y$, if and only if x, y are linearly independent.

Example 3. The ordinary orthogonality on inner product space $(X, \langle \cdot, \cdot \rangle)$ is given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.

Example 4. The Birkhoff-James orthogonality on a normed space $(X, \|\cdot\|)$ is defined by $x \perp y$ if and only if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$.

It is well-known that two orthogonal vectors can not be commutated, and it is necessary to introduce the following definition.

Definition 5. Relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$.

Clearly, Examples 2 and 3 are symmetric, but Example 4 is not. It is remarkable to note, however, that a real normed space of dimension greater than or equal to 3 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric. Some notions of orthogonally additive or orthogonally quadratic function equation are given as follows.

Definition 6. Let X be a vector space (an orthogonality space) and $(Y, +)$ be an abelian group. Mapping $f : X \rightarrow Y$ is called (orthogonally) additive if it satisfies the so-called (orthogonal) additive functional equation:

$$f(x + y) = f(x) + f(y), \quad (1)$$

for all $x, y \in X$ with $x \perp y$.

Definition 7. Mapping $f : X \rightarrow Y$ is said to be (orthogonally) quadratic if it satisfies the so-called (orthogonally) Jordan-von Neumann quadratic function equation:

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad (2)$$

for all $x, y \in X$ with $x \perp y$.

The orthogonal quadratic equation (2) was first investigated by Vajzović [25] when X is a Hilbert space, Y is the scalar field, f is continuous, and $\perp$ means the Hilbert space orthogonality. Later Drljević [21], Fochi [26], and Szabó [27] generalized this result. One of the significant conditional equations is the so-called orthogonally quadratic functional equation of Pexider type:

$$f(x + y) + f(x - y) = 2g(x) + 2h(y), \quad x \perp y. \quad (3)$$

Moslehian [28] considered the stability of this equation in the spirit of Hyers-Ulam under certain conditions. We will examine the stability of this equation in more general setting such that the target spaces are F -spaces or complete β -normed spaces. We have obtained some results which generalize work of Moslehian [28] and will be presented in Section 3. First, we recall some notions of F -spaces and β -normed spaces; for detailed understanding of the properties of the above spaces, the readers are required to read the book [29].

Definition 8. Let X be a linear space over $\mathbb{K}$ that denotes either complex or real numbers. A nonnegative valued

function $\|\cdot\|$ defined on X is called F -norm (or briefly a norm) if it satisfies the following conditions:

- (n1) $\|x\| = 0$ if and only if $x = 0$.
- (n2) $\|ax\| = \|x\|$ for all $a \in \mathbb{K}$, $|a| = 1$.
- (n3) $\|x + y\| \leq \|x\| + \|y\|$.
- (n4) $\|a_n x\| \rightarrow 0$ provided $a_n \rightarrow 0$.
- (n5) $\|ax_n\| \rightarrow 0$ provided $x_n \rightarrow 0$.
- (n6) $\|a_n x_n\| \rightarrow 0$ provided $a_n \rightarrow 0$, $x_n \rightarrow 0$.

A linear space equipped with F -norm is called F^* -space which will be denoted by $(X, \|\cdot\|)$ or X . A complete F^* -space is called F -space.

Definition 9. Let X be a linear space over $\mathbb{K}$ that denotes either complex or real numbers and $0 < \beta \leq 1$. A nonnegative valued function $\|\cdot\|$ defined on X is called β -norm if it satisfies the following conditions:

- (n1) $\|x\| = 0$ if and only if $x = 0$.
- (n2) $\|ax\| = |a|^\beta \|x\|$ for all $a \in \mathbb{K}$.
- (n3) $\|x + y\| \leq \|x\| + \|y\|$.

A linear space equipped with β -norm is called β -normed space, which will be denoted by $(X, \|\cdot\|)$ or briefly X .

Remark 10. It is clear that β -normed space is a special F^* -space, and when $\beta = 1$, β -normed space become normed space. Comparing with the normed space, F^* -space does not possess good metric properties, and the study of the stability of functional equations becomes more difficult.

There are many forms of the quadratic functional equation among them of great interest to us is the following:

$$\begin{aligned} & f(2x + y) + f(2x - y) \\ &= f(x + y) + f(x - y) + 6f(x). \end{aligned} \quad (4)$$

The purpose of this paper is to study the stability or orthogonal stability of equations in F^* -space or β -normed space. Section 2 is devoted to the study of stability of (4) in more general setting such that the target spaces are F -spaces or complete β -normed spaces. In Section 3, we focus on the study of orthogonal stability of (3) under the condition that the target spaces are F -spaces or complete β -normed space; some open problems are also proposed.

Throughout this paper, let X be a real β -normed space or orthogonality space, and Y be complete F -spaces or β -normed space. Also $\mathbb{R}$, $\mathbb{K}$, and $\mathbb{N}$ stand for the set of all real numbers, real numbers, or complex numbers and natural numbers, respectively.

2. On the Hyers-Ulam-Rassias Stability of (4)

From now on, let X be a real vector space and let Y be F -space in which there exists $1/2 \leq c < 1$ such that $\|y/2\| \leq c\|y\|$ for all $x \in Y$, unless we give any specific reference.

We will investigate the Hyers-Ulam-Rassias stability problem for functional equation (4). Thus, we find the condition that there exists a true quadratic function near an approximately quadratic function.

Theorem 11. *Let X be a real vector space and Y be F -space in which there exists $1/2 \leq c < 1$ such that $\|y/2\| \leq c\|y\|$ for all $x \in Y$, and let $\varphi : X \times X \rightarrow \mathbb{R}^+$ be a function such that*

$$\sum_{i=0}^{\infty} c^{2i} \varphi(2^i x, 0) \quad (5)$$

converges and

$$\lim_{n \rightarrow \infty} c^{2n} \varphi(2^n x, 2^n y) = 0, \quad (6)$$

for all $x, y \in X$. Suppose that f satisfies

$$\begin{aligned} & \|f(2x + y) + f(2x - y) - f(x + y) - f(x - y) \\ & - 6f(x)\| \leq \varphi(x, y), \end{aligned} \quad (7)$$

for all $x, y \in X$. Then, there exists unique quadratic function $g : X \rightarrow Y$ which satisfies (4) and the inequality

$$\|f(x) - g(x)\| \leq c^3 \sum_{i=0}^{\infty} c^{2i} \varphi(2^i x, 0), \quad (8)$$

for all $x \in X$. Function g is given by

$$g(x) = \lim_{n \rightarrow \infty} 2^{-2n} f(2^n x), \quad (9)$$

for all $x \in X$.

Proof. Putting $y = 0$ in (7), we have

$$\|2f(2x) - 8f(x)\| \leq \varphi(x, 0). \quad (10)$$

It follows that

$$\|f(x) - 4^{-1} f(2x)\| \leq c^3 \varphi(x, 0), \quad (11)$$

for all $x \in X$. Replacing x by $2x$ in (11) and by the assumption on norm, we get

$$\|4^{-1} f(2x) - 4^{-2} f(2^2 x)\| \leq c^3 \cdot c^2 \varphi(2x, 0). \quad (12)$$

Hence,

$$\|f(x) - 4^{-2} f(2^2 x)\| \leq c^3 [\varphi(x, 0) + c^2 \varphi(2x, 0)] \quad (13)$$

for all $x \in X$. Using the induction on positive integer n , we obtain that

$$\|f(x) - 4^{-n} f(2^n x)\| \leq c^3 \sum_{i=0}^{n-1} c^{2i} \varphi(2^i x, 0), \quad (14)$$

for all $x \in X$. In order to prove convergence of sequence $\{4^{-n} f(2^n x)\}$, replace x by $2^m x$ to find that, for $n, m > 0$,

$$\begin{aligned} & \|4^{-(n+m)} f(2^{n+m} x) - 4^{-m} f(2^m x)\| \\ & = \|4^{-m} [4^{-n} f(2^{n+m} x) - f(2^m x)]\| \\ & \leq c^3 c^3 \sum_{i=0}^n c^{2(m+i)} \varphi(2^{m+i} x, 0) \\ & = c^3 c^3 \sum_{i=m}^n c^{2i} \varphi(2^i x, 0). \end{aligned} \quad (15)$$

Since the right hand side of the inequality tends to 0 as m tends to infinity, sequence $\{4^{-n} f(2^n x)\}$ is a Cauchy sequence. Therefore, we may define $g(x) = \lim_{n \rightarrow \infty} 2^{-2n} f(2^n x)$ for all $x \in X$. By letting $n \rightarrow \infty$ in (14), we arrive at formula (8). To show that T satisfies (4), replace x, y by $2^n x, 2^n y$, respectively; then, it follows that

$$\begin{aligned} & \|4^{-n} [f(2^n(2x + y)) + f(2^n(2x - y)) \\ & - f(2^n(x + y)) - f(2^n(x - y)) - 6f(2^n x)]\| \\ & \leq c^{2n} \varphi(2^n x, 2^n y). \end{aligned} \quad (16)$$

Taking the limit as $n \rightarrow \infty$, we find that g satisfies (4) for all $x, y \in X$. To prove the uniqueness of quadratic function T subject to (8), let us assume that there exists quadratic function $S : X \rightarrow Y$ which satisfies (4) and inequality (8). Obviously, we have $S(2^n x) = 4^n S(x)$ and $T(2^n x) = 4^n T(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Hence, it follows from (8) that

$$\begin{aligned} & \|S(x) - T(x)\| = \|4^{-n} [S(2^n x) - T(2^n x)]\| \\ & \leq c^{2n} \|S(2^n x) - f(2^n x)\| \\ & \quad + \|f(2^n x) - T(2^n x)\| \\ & \leq c^3 \sum_{i=0}^{\infty} c^{2(n+i)} \varphi(2^{n+i} x, 0), \end{aligned} \quad (17)$$

for all $x \in X$. By letting $n \rightarrow \infty$ in the preceding inequality, we immediately find the uniqueness of g . This completes the proof of the theorem. $\square$

Corollary 12. *Let X be a real vector space and Y be a complete β -normed space ($0 < \beta \leq 1$), and let $\varphi : X \times X \rightarrow \mathbb{R}^+$ be a function such that*

$$\sum_{i=0}^{\infty} 4^{-\beta i} \varphi(2^i x, 0) \quad (18)$$

converges and

$$\lim_{n \rightarrow \infty} 4^{-\beta n} \varphi(2^n x, 2^n y) = 0, \quad (19)$$

for all $x, y \in X$. Suppose that f satisfies

$$\begin{aligned} & \|f(2x + y) + f(2x - y) - f(x + y) - f(x - y) \\ & - 6f(x)\| \leq \varphi(x, y), \end{aligned} \quad (20)$$

for all $x, y \in X$. Then, there exists unique quadratic function $g : X \rightarrow Y$ which satisfies (4) and inequality

$$\|f(x) - g(x)\| \leq 8^{-\beta} \sum_{i=0}^{\infty} 4^{-\beta i} \varphi(2^i x, 0), \quad (21)$$

for all $x \in X$. Function g is given by

$$g(x) = \lim_{n \rightarrow \infty} 2^{-2n} f(2^n x), \quad (22)$$

for all $x \in X$.

3. On the Orthogonal Stability of (3)

Applying some ideas from [30], we deal with the conditional stability problem of the following equation:

$$f(x+y) + f(x-y) = 2g(x) + 2h(y) \quad \text{for } x \perp y, \quad (23)$$

where f is odd and $\perp$ is symmetric. Throughout this section, $(X, \perp)$ denotes an orthogonality space in the sense of Rätz and $(Y, \|\cdot\|)$ is a real F -space or β -Banach space ($0 < \beta \leq 1$). First, we give a technical lemma.

Lemma 13. *If $A : X \rightarrow Y$ fulfills $A(x+y) + A(x-y) = 2A(x)$ for all $x, y \in X$ with $x \perp y$ and $\perp$ is symmetric, then $A(x) - A(0)$ is orthogonally additive.*

Proof. Assume that $A(x+y) + A(x-y) = 2A(x)$ for all $x, y \in X$ with $x \perp y$. Putting $x = 0$, we get $-A(y) = A(-y) - 2A(0)$, $y \in X$. Let $x \perp y$. Then, $y \perp x$ and so $A(y-x) = -A(y+x) + 2A(y)$. Hence,

$$\begin{aligned} A(x+y) &= -A(x-y) + 2A(x) \\ &= (A(y-x) - 2A(0)) + 2A(x) \\ &= (-A(y+x) + 2A(y)) - 2A(0) \\ &\quad + 2A(x). \end{aligned} \quad (24)$$

Thus,

$$\begin{aligned} A(x+y) - A(0) &= (A(x) - A(0)) \\ &\quad + (A(y) - A(0)). \end{aligned} \quad (25)$$

Therefore, $A(x) - A(0)$ is orthogonally additive. $\square$

Remark 14. Rätz gave example to demonstrate that there exists odd mapping A from an orthogonality space into a uniquely 2-divisible group $(Y, +)$ (i.e., an abelian group in which map $\varphi : Y \rightarrow Y$, $\varphi(x) = 2x$ is bijective) satisfying $A(x+y) + A(x-y) = 2A(x)$, $x \perp y$, such that $A(0) \neq 0$. He considered $Y = \mathbf{Z}_2 = \{0, 1\}$ and $A(x) = 1$, $x \in X$.

Theorem 15. *Let X be an orthogonality space, and Y be F -space in which there exists $1/2 \leq c < 1$ such that $\|y/2\| \leq c\|y\|$ for all $x \in Y$. Suppose that $\perp$ is symmetric on X and $f, g, h : X \rightarrow Y$ are mappings fulfilling*

$$\|f(x+y) + f(x-y) - 2g(x) - 2h(y)\| \leq \varepsilon, \quad (26)$$

for some $\varepsilon > 0$ and for all $x, y \in X$ with $x \perp y$. Assume that f is odd, and $g(0) = h(0) = 0$. Then, there exist one additive mapping $T : X \rightarrow Y$ and one quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - T(x) - Q(x)\| \leq c\varepsilon + (3\varepsilon + 6c\varepsilon) \frac{c}{1-c}, \quad x \in X, \quad (27)$$

$$\|g(x) - T(x) - Q(x)\| \leq c\varepsilon + (3\varepsilon + 6c\varepsilon) \frac{c}{1-c},$$

for all $x \in X$.

Proof. Put $x = 0$ in (26). We can do this because of (O1). Then,

$$\|f(y) + f(-y) - 2g(0) - 2h(y)\| \leq \varepsilon. \quad (28)$$

Therefore,

$$\|2h(y)\| \leq \varepsilon. \quad (29)$$

Similarly, by putting $y = 0$ in (26), we get

$$\|f(x) - g(x)\| \leq c\varepsilon, \quad (30)$$

for all $x \in X$. Hence,

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x)\| \\ &\leq \|f(x+y) + f(x-y) - 2g(x) - 2h(y)\| \\ &\quad + \|2[f(x) - g(x)]\| + \|2h(y)\| \leq 3\varepsilon, \end{aligned} \quad (31)$$

for all $x, y \in X$ with $x \perp y$. Fix $x \in X$. By (O4), there exists $y \in X$ such that $x \perp y$ and $x+y \perp x-y$. Since $\perp$ is symmetric, $x-y \perp x+y$ too. Using inequality (31) and the oddness of f , we get

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x)\| \leq 3\varepsilon, \\ &\|f(2x) + f(2y) - 2f(x+y)\| \leq 3\varepsilon, \\ &\|f(2x) - f(2y) - 2f(x-y)\| \leq 3\varepsilon. \end{aligned} \quad (32)$$

So that

$$\begin{aligned} &\|f(2x) - 2f(x)\| \\ &\leq \|f(x+y) + f(x-y) - 2f(x)\| \\ &\quad + \left\| \frac{1}{2} [f(2x) + f(2y) - 2f(x+y)] \right\| \\ &\quad + \left\| \frac{1}{2} [f(2x) - f(2y) - 2f(x-y)] \right\| \\ &\leq 3\varepsilon + 6c\varepsilon. \end{aligned} \quad (33)$$

It is not hard to see that

$$\|2^{-n} f(2^n x) - f(x)\| \leq (3\varepsilon + 6c\varepsilon) \sum_{k=1}^n c^k, \quad (34)$$

for all n . In fact, It follows from

$$\|2^{-1}f(2x) - f(x)\| \leq c(3\varepsilon + 6c\varepsilon) \quad (35)$$

that (34) holds for $n = 1$. Assume that (34) holds for $k = n$, when $k = n + 1$. Replacing x by $2x$ in (34), we get

$$\begin{aligned} \|2^{-n}f(2^{n+1}x) - f(2x)\| &\leq (3\varepsilon + 6c\varepsilon) \sum_{k=1}^n c^k, \\ \|2^{-n-1}f(2^{n+1}x) - 2^{-1}f(x)\| &\leq (3\varepsilon + 6c\varepsilon) \sum_{k=1}^n c^{k+1}. \end{aligned} \quad (36)$$

Hence,

$$\|2^{-n-1}f(2^{n+1}x) - f(x)\| \leq (3\varepsilon + 6c\varepsilon) \sum_{k=1}^{n+1} c^k. \quad (37)$$

So, formula (34) is proved. Replacing x by $2^m x$ in inequality (34), we have

$$\begin{aligned} \|2^{-n}f(2^{n+m}x) - f(2^m x)\| &\leq (3\varepsilon + 6c\varepsilon) \sum_{k=1}^n c^k, \\ \|2^{-n-m}f(2^{n+m}x) - 2^{-m}f(2^m x)\| &\leq (3\varepsilon + 6c\varepsilon) \frac{c^{m+1}}{1-c}, \end{aligned} \quad (38)$$

which implies that $\{2^{-n}f(2^n x)\}$ is a Cauchy sequence in F -space Y , and $\lim_{n \rightarrow \infty} 2^{-n}f(2^n x)$ exists and map $A(x) := \lim_{n \rightarrow \infty} 2^{-n}f(2^n x)$ is well defined odd map from X into Y satisfying

$$\|f(x) - A(x)\| \leq (3\varepsilon + 6c\varepsilon) \frac{c}{1-c}, \quad x \in X. \quad (39)$$

For all $x, y \in X$ with $x \perp y$, by applying inequality (31) and (O3), we obtain

$$\begin{aligned} &\|2^{-n}f(2^n(x+y)) + 2^{-n}f(2^n(x-y)) \\ &\quad - 2^{-n+1}f(2^n x)\| \leq c^n \cdot 3\varepsilon. \end{aligned} \quad (40)$$

If $n \rightarrow \infty$, then we deduce that

$$A(x+y) + A(x-y) - 2A(x) = 0, \quad (41)$$

for all $x, y \in X$ with $x \perp y$. Moreover, $A(0) = \lim_{n \rightarrow \infty} 2^{-n}f(2^n 0) = 0$. Using Lemma 13, we conclude that A is an orthogonally additive mapping. By Corollary 7 of [23], A is of form $T + Q$ with T additive and Q quadratic. If there are another quadratic mapping Q' and another additive mapping T' satisfying the required inequalities in our theorem and $A' = T' + Q'$, then

$$\begin{aligned} \|A(x) - A'(x)\| &\leq \|f(x) - A(x)\| \\ &\quad + \|f(x) - A'(x)\| \\ &\leq (3\varepsilon + 6c\varepsilon) \frac{2c}{1-c}, \end{aligned} \quad (42)$$

for all $x \in X$. Using the fact that additive mappings are odd and quadratic mappings are even, we obtain

$$\begin{aligned} \|T(x) - T'(x)\| &= \left\| \frac{1}{2} \right. \\ &\quad \cdot \left[(T(x) + Q(x) - T'(x) - Q'(x)) \right. \\ &\quad \left. + (T(x) - Q(x) - T'(x) + Q'(x)) \right] \leq c\|T(x) \\ &\quad + Q(x) - T'(x) - Q'(x)\| + c\|T(x) - Q(x) \\ &\quad - T'(x) + Q'(x)\| \leq c\|A(x) - A'(x)\| \\ &\quad + c\|A(-x) - A'(-x)\| \leq (3\varepsilon + 6c\varepsilon) \frac{4c^2}{1-c}. \end{aligned} \quad (43)$$

Hence,

$$\begin{aligned} \|T(x) - T'(x)\| &\leq c^n \|T(2^n x) - T'(2^n x)\| \\ &\leq (3\varepsilon + 6c\varepsilon) \frac{4c^{n+2}}{1-c}. \end{aligned} \quad (44)$$

Letting n tends to ∞ we infer that $T = T'$. Similarly,

$$\begin{aligned} \|Q(x) - Q'(x)\| &= \left\| \frac{1}{2} \left[(T(x) + Q(x) - T'(x) - Q'(x)) \right. \right. \\ &\quad \left. \left. - (T(x) - Q(x) - T'(x) + Q'(x)) \right] \right\| \\ &\leq 2^{-\beta} \|T(x) + Q(x) - T'(x) - Q'(x)\| + c\|T(x) \\ &\quad - Q(x) - T'(x) + Q'(x)\| \leq c\|A(x) - A'(x)\| \\ &\quad + c\|A(-x) - A'(-x)\| \leq (3\varepsilon + 6c\varepsilon) \frac{4c^2}{1-c}, \end{aligned} \quad (45)$$

for all $x \in X$. Hence,

$$\begin{aligned} \|Q(x) - Q'(x)\| &\leq c^{2n} \|Q(2^{2n}x) - Q'(2^{2n}x)\| \\ &\leq (3\varepsilon + 6c\varepsilon) \frac{4c^{2n+2}}{1-c}, \end{aligned} \quad (46)$$

for all $x \in X$. Taking the limit as $n \rightarrow \infty$, we conclude that $Q = Q'$. Using (30) and (39), we infer that, for all $x \in X$,

$$\begin{aligned} \|g(x) - A(x)\| &\leq \|g(x) - f(x)\| + \|f(x) - A(x)\| \\ &\leq c\varepsilon + (3\varepsilon + 6c\varepsilon) \frac{c}{1-c}. \end{aligned} \quad (47)$$

□

Corollary 16. Let X be an orthogonality space, and Y be β -Banach space. Suppose that $\perp$ is symmetric on X and $f, g, h : X \rightarrow Y$ are mappings fulfilling

$$\|f(x+y) + f(x-y) - 2g(x) - 2h(y)\| \leq \varepsilon, \quad (48)$$

for some $\varepsilon > 0$ and for all $x, y \in X$ with $x \perp y$. Assume that f is odd, and $g(0) = h(0) = 0$. Then, there exist one additive mapping $T : X \rightarrow Y$ and one quadratic mapping $Q : X \rightarrow Y$, such that

$$\begin{aligned} \|f(x) - T(x) - Q(x)\| &\leq \left(3\varepsilon + \frac{6\varepsilon}{2^\beta}\right) \frac{1}{2^{\beta-1}}, \quad x \in X, \\ \|g(x) - T(x) - Q(x)\| &\leq 2^{-\beta}\varepsilon + \frac{1}{2^\beta - 1} 3 \left(\varepsilon + \frac{6\varepsilon}{2^\beta}\right), \end{aligned} \quad (49)$$

for all $x \in X$.

Remark 17. (i) If $g = \lambda f$ for some number $\lambda \neq 1$, then inequality (30) implies that $\|(1 - \lambda)f(x)\| \leq c\varepsilon$, $x \in X$. Hence, $\|(1 - \lambda)2^{-n}f(2^n x)\| \leq c^{n+1}\varepsilon$, $x \in X$. So $A(x) = \lim_{n \rightarrow \infty} 2^{-n}f(2^n x) = 0$, $x \in X$.

(ii) Similarly, if $h = \lambda f$ for some number $\lambda \neq 0$, then it follows from (29) that $A(x) = 0$ for all $x \in X$.

As far as the author knows, unlike orthogonally additive maps (see Corollary 7 of [23]), there is no characterization for orthogonally quadratic maps. Every orthogonally quadratic mapping q into a uniquely 2-divisible abelian group $(Y, +)$ is even. In fact, $0 \perp 0$, so $q(0) + q(0) = 4q(0)$. Therefore, $q(0) = 0$. For all $y \in X$, we have $0 \perp y$ and hence $q(y) + q(-y) = 2q(0) + 2q(y)$.

Thus, $q(-y) = q(y)$. There are some characterizations of orthogonally quadratic maps in various notions of orthogonality. For example, if A -orthogonality on Hilbert space H is defined by $\perp_A = \{(x, y) : \langle Ax, y \rangle = 0\}$, where A is a bounded self-adjoint operator on H , then, as shown by Fochi, every A -orthogonally quadratic functional is quadratic if $\dim A(H) \geq 3$ (see [26, 27]).

To conclude this paper, we propose the following problem.

Problem. Let X be an orthogonality space, and Y be F -space in which there exists $1/2 \leq c < 1$ such that $\|y/2\| \leq c\|y\|$ for all $x \in Y$. Suppose that $\perp$ is symmetric on X and $f, g, h : X \rightarrow Y$ are mappings fulfilling

$$\|f(x + y) + f(x - y) - 2g(x) - 2h(y)\| \leq \varepsilon, \quad (50)$$

for some ε and for all $x, y \in X$ with $x \perp y$. Assume that f is even, and $g(0) = h(0) = 0$. Does there exist an orthogonally quadratic mapping $Q : X \rightarrow Y$, under certain conditions such that

$$\begin{aligned} \|f(x) - Q(x)\| &\leq \alpha\varepsilon, \\ \|g(x) - Q(x)\| &\leq \beta\varepsilon, \\ \|h(x) - Q(x)\| &\leq \gamma\varepsilon, \end{aligned} \quad (51)$$

for some scalars α, β, γ and for all x ?

Competing Interests

The author declares that there are no competing interests regarding the publication of this paper.

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