Research Article

Nonclassical Problem for Ultraparabolic Equation in Abstract Spaces

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Nonclassical problem for ultraparabolic equation with nonlocal initial condition with respect to one time variable is studied in abstract Hilbert spaces. We define the space of square integrable vector-functions with values in Hilbert spaces corresponding to the variational formulation of the nonlocal problem for ultraparabolic equation and prove trace theorem, which allows one to interpret initial conditions of the nonlocal problem. We obtain suitable a priori estimates and prove the existence and uniqueness of solution of the nonclassical problem and continuous dependence upon the data of the solution to the nonlocal problem. We consider an application of the obtained abstract results to nonlocal problem for ultraparabolic partial differential equation with second-order elliptic operator and obtain well-posedness result in Sobolev spaces.

1. Introduction

Evolution equations with several time-like variables are encountered in various models of science and technology, particularly, in mathematical models of multiparameter Brownian motion [1], theory of boundary layers [2], mathematical models of diffusion of pollutants in water flows [3], transport theory [4], mathematical models of age structured biological population dynamics [5, 6], mathematical finance [7], mechanics, physics, and cosmology [8, 9]. In these models one of time variables is usual time and others might denote various quantities, for example, coordinates, temperature, and age or size of individuals of biological population.

Various mathematical problems are investigated for multitime evolution equations. Ultraparabolic equations with classical Cauchy conditions with respect to one time variable, problems with classical initial conditions with respect to all time variables, and classical or nonlocal boundary conditions with respect to space variables, inverse problems are investigated in spaces of classical smooth functions and in suitable spaces of generalized functions, and some papers are also devoted to numerical methods for ultraparabolic equations

with classical initial conditions (see [10–26] and references given therein). Abstract rather general ultraparabolic equation was studied by Lions [27] in spaces of abstract vector-valued distributions. Applying methods of the theory of semigroups ultraparabolic integrodifferential equation with classical initial conditions with respect to time variables in the spaces of Hölder continuous vector-functions with values in a Banach space was investigated by Lorenzi [28] and ultraparabolic equations arising in age structured population dynamics with integral condition with respect to one of time variables were studied by many researchers; see [6, 29, 30] and their references.

Mathematical models of populations incorporating age structure [6] are given by ultraparabolic equations and include nonlocal integral condition with respect to age variable, that is, nonlocal initial condition with respect to the second time variable. Nonclassical problem with nonlocal initial condition for abstract first-order evolution equation was investigated by Krejn and Laptev [31] and in the case of parabolic equation was studied by Kerefov [32]. Various initial-boundary value problems with nonlocal initial conditions are generalizations of initial-boundary value problems

with classical and periodical initial conditions and they can be also considered as problems of controllability by initial conditions. Later on, initial-boundary value problems with nonlocal initial conditions were investigated for various time-dependent partial differential equations and systems (see [33–45] and references given therein). Nonlocal in time problems with discrete nonlocal initial conditions are used for investigation of radionuclides propagation in Stokes fluid and problems of predicting the state of a medium [44, 45].

In this paper, we consider nonclassical problem for ultraparabolic equation:

$$\frac{\partial u}{\partial t_1} + \frac{\partial u}{\partial t_2} + Au = f, \quad (t_1, t_2) \in (0, T_1) \times (0, T_2), \quad (1)$$

with the following classical and nonlocal initial conditions:

$$\begin{aligned} u(0, t_2) &= \varphi(t_2), \quad 0 \leq t_2 \leq T_2, \\ u(t_1, 0) &= \alpha u(t_1, \xi) + \psi(t_1), \quad 0 \leq t_1 \leq T_1, \end{aligned} \quad (2)$$

where $\alpha \in \mathbf{R}$, $0 < \xi < T_2$, φ , ψ , and f are given vector-functions from suitable spaces, and A is a given linear continuous operator in corresponding abstract Hilbert spaces.

As far as the authors know ultraparabolic equations with nonlocal initial condition (1) in abstract spaces have not been investigated yet. Moreover, in the case of classical initial conditions, that is, for $\alpha = 0$, the well-posedness results are obtained either in spaces of continuous functions or in spaces of distributions with respect to time variables, for which the trace operator on boundary of domain of time variables cannot be defined. So, there are no results on the existence and uniqueness of solution in the spaces with minimal regularity properties necessary to define the initial conditions (2). In the present paper we obtain well-posedness theorem for nonlocal problem (1), (2), which in the case of $\alpha = 0$ also gives new existence, uniqueness, and continuous dependence result for classical problem for the abstract ultraparabolic equation.

The paper is organized as follows. In Section 2, we introduce some notations and give properties of spaces of square integrable vector-functions defined on two-dimensional domain with values in Hilbert spaces. We prove new trace theorem and formulas for integration by parts. In addition, we show characteristic properties of some spaces of vector-valued functions, which are used to investigate the non-classical problem for ultraparabolic equation. In Section 3, we give variational formulation of the problem (1), (2) and prove our main result on the existence and uniqueness of its solution. Finally, in Section 4, we consider an application of the obtained abstract result to nonlocal problem for ultraparabolic partial differential equation in Sobolev spaces.

2. Preliminaries

We denote by $L(X; Y)$ the space of linear continuous operators from X to Y , where X and Y are Banach spaces. Let $C([0, T]; X)$ denote the space of continuous vector-functions on $[0, T]$ with values in X . We denote by $L^2(\Delta; X)$ the space

of measurable vector-functions $g : \Delta \rightarrow X$ such that $\|g\|_X \in L^2(\Delta)$ and the generalized derivatives of g are denoted by $\partial g / \partial \tau_i \in D'(\Delta; X) = L(D(\Delta); X)$, $i = 1, \dots, s$, $dg/d\tau_1 = g'$, for $s = 1$ (cf. [46]), where $D(\Delta)$ is the space of infinitely differentiable functions with compact support in Δ , where $\Delta \subset \mathbf{R}^s$, $s \in \mathbf{N}$ is a bounded domain.

Throughout this paper, we use c to denote generic constants that are independent of the main parameters involved, but whose values may differ from line to line and may change even within a single chain of estimates.

Let V and H be separable Hilbert spaces, such that V is dense in H and continuously embedded in it. We identify space H with its dual by using inner product $(\cdot, \cdot)$ in H and hence $V \subset H \subset V'$ with continuous and dense embeddings, where V' is the dual space of V [46]. We denote by $\langle \cdot, \cdot \rangle$ the duality relation between V' and V . To investigate problem (1), (2) let us consider the following space of vector-valued functions on $\omega = (0, T_1) \times (0, T_2)$:

$$W = \left\{ w \in L^2(\omega; V); \frac{\partial w}{\partial t_1}, \frac{\partial w}{\partial t_2} \in L^2(\omega; V') \right\}, \quad (3)$$

which is a Hilbert space equipped with the norm

$$\begin{aligned} \|v\|_W &= \left(\|v\|_{L^2(\omega; V)}^2 + \left\| \frac{\partial w}{\partial t_1} \right\|_{L^2(\omega; V')}^2 + \left\| \frac{\partial w}{\partial t_2} \right\|_{L^2(\omega; V')}^2 \right)^{1/2}. \end{aligned} \quad (4)$$

In the following three theorems we give some properties of the space W .

Theorem 1. *For each vector-function $w \in W$ and $0 \leq \tau_1 \leq T_1$, $0 \leq \tau_2 \leq T_2$, there exist traces $w(\tau_1, t_2) = (\text{tr}_{t_1=\tau_1} w)(t_2)$ and $w(t_1, \tau_2) = (\text{tr}_{t_2=\tau_2} w)(t_1)$, such that the trace operators $\text{tr}_{t_1=\tau_1} : W \rightarrow L^2(0, T_2; H)$ and $\text{tr}_{t_2=\tau_2} : W \rightarrow L^2(0, T_1; H)$ are continuous.*

Proof. We can identify each vector-function v from space $L^2(\omega; V)$ with vector-function $\tilde{v}$ from $L^2(0, T_1; L^2(0, T_2; V))$. If $w \in W$, then $\partial w / \partial t_1 \in L^2(\omega; V')$, and since the linear combinations of products $\theta \eta$, $\theta \in D(0, T_1)$, and $\eta \in D(0, T_2)$ are dense in $D(\omega)$, we have $\tilde{w} \in L^2(0, T_1; L^2(0, T_2; V))$, $d\tilde{w}/dt_1 \in L^2(0, T_1; L^2(0, T_2; V'))$. Note that $L^2(0, T_2; V) \subset L^2(0, T_2; H) \subset L^2(0, T_2; V')$ with continuous and dense embeddings and $L^2(0, T_2; V') = (L^2(0, T_2; V))'$. Consequently, from embedding theorem [46] we have that $\tilde{w} \in C([0, T_1]; L^2(0, T_2; H))$ and $C([0, T_1]; L^2(0, T_2; H))$ is continuously embedded in

$$\begin{aligned} W_1 &= \left\{ w_1 \in L^2(0, T_1; L^2(0, T_2; V)); \frac{dw}{dt_1} \right. \\ &\quad \left. \in L^2(0, T_1; L^2(0, T_2; V')) \right\}; \end{aligned} \quad (5)$$

that is,

$$\begin{aligned} & \|\tilde{w}(\tau_1)\|_{L^2(0,T_2;H)} \\ & \leq c \left(\|\tilde{w}\|_{L^2(0,T_1;L^2(0,T_2;V))}^2 + \left\| \frac{d\tilde{w}}{dt_1} \right\|_{L^2(0,T_1;L^2(0,T_2;V'))}^2 \right)^{1/2} \quad (6) \\ & \leq c \|w\|_W, \end{aligned}$$

for any $\tau_1 \in [0, T_1]$. Hence, we can define the trace operator $w(\tau_1, t_2) = (\text{tr}_{t_1=\tau_1} w)(t_2) = \tilde{w}(\tau_1)$, which is a linear continuous operator from W to $L^2(0, T_2; H)$. Likewise, for each $0 \leq \tau_2 \leq T_2$ we obtain the existence of the linear continuous trace operator $w(t_1, \tau_2) = (\text{tr}_{t_2=\tau_2} w)(t_1)$ from W to $L^2(0, T_1; H)$. $\square$

Theorem 2. For vector-functions $w, w_* \in W$ the following formulas for integration by parts are valid:

$$\begin{aligned} & \int_0^{T_2} \int_0^{T_1} \left\langle \frac{\partial w}{\partial t_1}, w_* \right\rangle dt_1 dt_2 \\ & + \int_0^{T_2} \int_0^{T_1} \left\langle \frac{\partial w_*}{\partial t_1}, w \right\rangle dt_1 dt_2 \\ & = \int_0^{T_2} (w(T_1, t_2), w_*(T_1, t_2)) dt_2 \\ & - \int_0^{T_2} (w(0, t_2), w_*(0, t_2)) dt_2, \\ & \int_0^{T_1} \int_0^{T_2} \left\langle \frac{\partial w}{\partial t_2}, w_* \right\rangle dt_2 dt_1 \\ & + \int_0^{T_1} \int_0^{T_2} \left\langle \frac{\partial w_*}{\partial t_2}, w \right\rangle dt_2 dt_1 \\ & = \int_0^{T_1} (w(t_1, T_2), w_*(t_1, T_2)) dt_1 \\ & - \int_0^{T_1} (w(t_1, 0), w_*(t_1, 0)) dt_1. \end{aligned} \quad (7)$$

Proof. For $w, w_* \in W$ we consider the corresponding vector-functions $\tilde{w}, \tilde{w}_* \in W_1$ defined in the proof of Theorem 1. For vector-functions $\tilde{w}, \tilde{w}_* \in W_1$ the following formula for integration by parts is valid [46]:

$$\begin{aligned} & \int_0^{T_1} \left\langle \frac{d\tilde{w}}{dt_1}, \tilde{w}_* \right\rangle_{T_2} dt_1 + \int_0^{T_1} \left\langle \frac{d\tilde{w}_*}{dt_1}, \tilde{w} \right\rangle_{T_2} dt_1 \\ & = (\tilde{w}(T_1), \tilde{w}_*(T_1))_{T_2} - (\tilde{w}(0), \tilde{w}_*(0))_{T_2}, \end{aligned} \quad (8)$$

where $\langle \cdot, \cdot \rangle_{T_2}$ denotes the duality relation between $L^2(0, T_2; V')$ and $L^2(0, T_2; V)$ and $(\cdot, \cdot)_{T_2}$ denotes the inner product in $L^2(0, T_2; H)$. The latter formula is equivalent to the first formula of the theorem. Likewise we obtain the second formula. $\square$

Theorem 3. For vector-functions $w \in W$ and $v \in V$ the following equalities are valid:

$$\begin{aligned} \left\langle \frac{\partial w}{\partial t_1}, v \right\rangle &= \frac{\partial}{\partial t_1} (w, v), \\ \left\langle \frac{\partial w}{\partial t_2}, v \right\rangle &= \frac{\partial}{\partial t_2} (w, v) \end{aligned} \quad (9)$$

in $D'(\omega)$.

Proof. Applying Theorem 2 for $w \in W$ and $w_* = \phi v$, $\phi \in D(\omega)$, $v \in V$, we obtain

$$\begin{aligned} & \int_0^{T_2} \int_0^{T_1} \left\langle \frac{\partial w}{\partial t_i}, v \right\rangle \phi(t_1, t_2) dt_1 dt_2 \\ & = - \int_0^{T_2} \int_0^{T_1} \langle w, v \rangle \frac{\partial \phi}{\partial t_i} dt_1 dt_2 = 0, \end{aligned} \quad (10)$$

for $i = 1, 2$. Since $w(t_1, t_2) \in V$, $(t_1, t_2) \in \omega$, we have $\langle w, v \rangle = (w, v)$, which proves the required equalities. $\square$

Let $A \in L(V; V')$ be a self-adjoint coercive operator, which means that the bilinear form $a(v, w) = \langle Av, w \rangle$ satisfies the conditions

$$\begin{aligned} |a(v, w)| &\leq c_a \|v\|_V \|w\|_V, \\ a(v, w) &= a(w, v), \\ a(v, v) &\geq \hat{c}_a \|v\|_V^2, \end{aligned} \quad (11)$$

$\forall v, w \in V$,

where $c_a, \hat{c}_a = \text{const} > 0$. We also suppose that there exists a system of eigenvectors $\{v_k\}_{k \in \mathbb{N}}$ of the operator A corresponding to the eigenvalues $\{\lambda_k^2\}_{k \in \mathbb{N}}$, that is, $a(v_k, v) = \lambda_k^2(v_k, v)$, for all $v \in V$, which is complete in V and orthonormal in H . From (11) it follows that the operator $A \in L(V; V')$ is invertible and $A^{-1} \in L(V'; V)$ [47]. Note that $a(v_k, v) = \langle Av_k, v \rangle = \lambda_k^2(v_k, v) = \langle \lambda_k^2 v_k, v \rangle$ and hence $Av_k = \lambda_k^2 v_k \in V$, $v_k / \lambda_k^2 = A^{-1} v_k$ for all $k \in \mathbb{N}$.

Let us define space $V_1 = \{v \in V; Av \in H\}$, which is a Hilbert space equipped with the scalar product $(v, w)_1 = (Av, Aw)$ for all $v, w \in V_1$. Space V_1 is continuously embedded in V , since

$$\begin{aligned} \|v\|_V &= \|A^{-1} Av\|_V \leq \|A^{-1}\| \|Av\|_{V'} \leq c \|A^{-1}\| \|Av\|_H \\ &= c \|A^{-1}\| \|v\|_1, \end{aligned} \quad (12)$$

where $\|v\|_1 = \sqrt{(v, v)_1}$. Space V_1 is dense in V . Indeed, if $v \in V$, then $Av \in V'$. Since H is dense in V' , there exists sequence $(h_n)_{n \geq 1}$, $h_n \in H$, $n \in \mathbb{N}$, such that $h_n \rightarrow Av$ in V' , as $n \rightarrow \infty$. Consequently, $A^{-1} h_n \rightarrow A^{-1} Av = v$ in V as $n \rightarrow \infty$ and $A^{-1} h_n \in V_1$, $n \in \mathbb{N}$.

So, $V_1 \subset V \subset H \subset V' \subset V'_1$ with continuous and dense embeddings, where V'_1 is the dual space of V_1 . The duality

relation between V_1' and V_1 is denoted by $\langle \cdot, \cdot \rangle_1$. Note that for all $v, w \in V_1$ we have

$$\begin{aligned} |(Av, w)| &\leq \|Av\|_H \|w\|_H = \|v\|_1 \|w\|_H, \\ |\langle Av, w \rangle_1| &= |(Av, w)| = |a(v, w)| = |a(w, v)| \\ &= |(Aw, v)| \leq \|Aw\|_H \|v\|_H = \|v\|_H \|w\|_1. \end{aligned} \quad (13)$$

Hence, $\|Av\|_H = \|v\|_1$ and $\|Av\|_{V_1'} \leq c\|v\|_H$. Therefore, the restriction of operator A on V_1 is a linear continuous operator from V_1 to H , and since V_1 is dense in H , there exists unique continuous continuation of A on H , which we denote by $A_1 : H \rightarrow V_1'$, $A_1 v = Av$ for all $v \in V_1$. From the definition of the operator A_1 it follows that

$$\langle A_1 h, z \rangle_1 = (h, Az) \quad \forall h \in H, z \in V_1. \quad (14)$$

Indeed, since V_1 is dense in H , there exists sequence $(z_n)_{n \geq 1}$, $z_n \in V_1$, $n \in \mathbf{N}$, such that $z_n \rightarrow h$ in H as $n \rightarrow \infty$. Consequently, $Az_n \rightarrow A_1 h$ in V_1' as $n \rightarrow \infty$, and

$$\begin{aligned} \langle A_1 h, z \rangle_1 &= \lim_{n \rightarrow \infty} \langle Az_n, z \rangle_1 = \lim_{n \rightarrow \infty} a(z_n, z) \\ &= \lim_{n \rightarrow \infty} a(z, z_n) = \lim_{n \rightarrow \infty} \langle z_n, Az \rangle_1 = (h, Az). \end{aligned} \quad (15)$$

In the following lemmas we give some properties of spaces $L^2(\Delta; H)$, $L^2(\Delta; V)$, $L^2(\Delta; V')$, $L^2(\Delta; V_1)$, and $L^2(\Delta; V_1')$, which will be used to prove the main theorem.

Lemma 4. (a) If $g \in L^2(\Delta; H)$, then

$$\sum_{n=1}^{\infty} \|g_n\|_{L^2(\Delta)}^2 \leq \|g\|_{L^2(\Delta; H)}^2, \quad g_n = (g, v_n), \quad n \in \mathbf{N}. \quad (16)$$

(b) Suppose that $g_n \in L^2(\Delta)$, $n \in \mathbf{N}$. The series $\sum_{n=1}^{\infty} \|g_n\|_{L^2(\Delta)}^2$ converges if and only if the series $\sum_{n=1}^{\infty} g_n v_n$ converges in the space $L^2(\Delta; H)$.

Proof. (a) Since the system of vector $\{v_k\}_{k \in \mathbf{N}}$ is orthonormal in H we have

$$\int_{\Delta} \left\| g - \sum_{n=1}^N g_n v_n \right\|_H^2 d\Delta = \|g\|_{L^2(\Delta; H)}^2 - \sum_{n=1}^N \|g_n\|_{L^2(\Delta)}^2 \geq 0 \quad (17)$$

for all $N \in \mathbf{N}$, and tending $N \rightarrow \infty$ we obtain the inequality.

(b) The equivalence of the convergences of the series directly follows from the following equalities:

$$\begin{aligned} \left\| \sum_{n=N}^M g_n v_n \right\|_{L^2(\Delta; H)}^2 &= \int_{\Delta} \left\| \sum_{n=N}^M g_n v_n \right\|_H^2 d\Delta \\ &= \int_{\Delta} \sum_{n=N}^M g_n^2 d\Delta = \sum_{n=N}^M \|g_n\|_{L^2(\Delta)}^2, \end{aligned} \quad (18)$$

$\forall N, M \in \mathbf{N}, N \leq M.$ □

Lemma 5. (a) If $g \in L^2(\Delta; V)$, then

$$\sum_{n=1}^{\infty} \lambda_n^2 \|g_n\|_{L^2(\Delta)}^2 \leq c_a \|g\|_{L^2(\Delta; V)}^2, \quad (19)$$

$$g_n = (g, v_n), \quad n \in \mathbf{N}.$$

(b) Suppose that $g_n \in L^2(\Delta)$, $n \in \mathbf{N}$. The series $\sum_{n=1}^{\infty} \lambda_n^2 \|g_n\|_{L^2(\Delta)}^2$ converges if and only if the series $\sum_{n=1}^{\infty} g_n v_n$ converges in the space $L^2(\Delta; V)$.

Proof. (a) Since $\{v_k\}_{k \in \mathbf{N}}$ are eigenvectors of the operator A and $\{v_k\}_{k \in \mathbf{N}}$ is orthonormal in H we have

$$\begin{aligned} &\int_{\Delta} a \left(g - \sum_{n=1}^N g_n v_n, g - \sum_{k=1}^N g_k v_k \right) d\Delta \\ &= \int_{\Delta} a(g, g) d\Delta - \int_{\Delta} \sum_{n=1}^N \lambda_n^2 g_n^2 d\Delta \geq 0 \end{aligned} \quad (20)$$

for all $N \in \mathbf{N}$, and tending $N \rightarrow \infty$ from (11) we obtain the required inequality.

(b) The equivalence of the convergences of the series directly follows from the following inequalities for all $N, M \in \mathbf{N}$, $N \leq M$,

$$\begin{aligned} \left\| \sum_{n=N}^M g_n v_n \right\|_{L^2(\Delta; V)}^2 &\leq \frac{1}{\hat{c}_a} \int_{\Delta} a \left(\sum_{n=N}^M g_n v_n, \sum_{k=N}^M g_k v_k \right) d\Delta \\ &= \frac{1}{\hat{c}_a} \sum_{n=N}^M \lambda_n^2 \|g_n\|_{L^2(\Delta)}^2 \\ &\leq \frac{c_a}{\hat{c}_a} \left\| \sum_{n=N}^M g_n v_n \right\|_{L^2(\Delta; V)}^2. \end{aligned} \quad (21)$$

□

Lemma 6. (a) If $g \in L^2(\Delta; V')$, then

$$\sum_{n=1}^{\infty} \frac{\|g_n\|_{L^2(\Delta)}^2}{\lambda_n^2} \leq c_a \|A^{-1}\|^2 \|g\|_{L^2(\Delta; V')}^2, \quad (22)$$

$$g_n = \langle g, v_n \rangle, \quad n \in \mathbf{N}.$$

(b) Suppose that $g_n \in L^2(\Delta)$, $n \in \mathbf{N}$. The series $\sum_{n=1}^{\infty} \|g_n\|_{L^2(\Delta)}^2 / \lambda_n^2$ converges if and only if the series $\sum_{n=1}^{\infty} g_n v_n$ converges in the space $L^2(\Delta; V')$.

Proof. (a) Let us define $\hat{g} = A^{-1}g \in L^2(\Delta; V)$. Applying part (a) of Lemma 5 we have $\sum_{n=1}^{\infty} \lambda_n^2 \|\hat{g}_n\|_{L^2(\Delta)}^2 \leq c_a \|\hat{g}\|_{L^2(\Delta; V)}^2$, where

$$\begin{aligned} \hat{g}_n &= (\hat{g}, v_n) = (v_n, \hat{g}) = \frac{a(v_n, \hat{g})}{\lambda_n^2} = \frac{a(\hat{g}, v_n)}{\lambda_n^2} \\ &= \frac{\langle A\hat{g}, v_n \rangle}{\lambda_n^2} = \frac{\langle g, v_n \rangle}{\lambda_n^2} = \frac{g_n}{\lambda_n^2}, \quad \forall n \in \mathbf{N}. \end{aligned} \quad (23)$$

Consequently,

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_n^2 \|\hat{g}_n\|_{L^2(\Delta)}^2 &= \sum_{n=1}^{\infty} \frac{\|g_n\|_{L^2(\Delta)}^2}{\lambda_n^2} \leq c_a \|A^{-1}g\|_{L^2(\Delta;V)}^2 \\ &\leq c_a \|A^{-1}\|^2 \|g\|_{L^2(\Delta;V')}^2. \end{aligned} \quad (24)$$

(b) The equivalence of the convergences of the series directly follows from the following inequalities

$$\begin{aligned} \left\| \sum_{n=N}^M g_n v_n \right\|_{L^2(\Delta;V')}^2 &= \left\| A A^{-1} \sum_{n=N}^M g_n v_n \right\|_{L^2(\Delta;V')}^2 \\ &\leq \|A\|^2 \left\| \sum_{n=N}^M g_n A^{-1} v_n \right\|_{L^2(\Delta;V)}^2 \\ &\leq \frac{\|A\|^2}{\hat{c}_a} \int_{\Delta} a \left(\sum_{n=N}^M g_n \frac{v_n}{\lambda_n^2}, \sum_{k=N}^M g_k \frac{v_k}{\lambda_k^2} \right) d\Delta \\ &= \frac{\|A\|^2}{\hat{c}_a} \int_{\Delta} \sum_{n=N}^M \frac{g_n^2}{\lambda_n^2} d\Delta \\ &\leq \frac{\|A\|^2}{\hat{c}_a} c_a \int_{\Delta} \left\| \sum_{n=N}^M g_n \frac{v_n}{\lambda_n^2} \right\|_V^2 d\Delta \\ &= \frac{\|A\|^2}{\hat{c}_a} c_a \int_{\Delta} \left\| \sum_{n=N}^M g_n A^{-1} v_n \right\|_V^2 d\Delta \\ &= \frac{\|A\|^2}{\hat{c}_a} c_a \left\| A^{-1} \sum_{n=N}^M g_n v_n \right\|_{L^2(\Delta;V)}^2 \\ &\leq \frac{\|A\|^2 c_a \|A^{-1}\|^2}{\hat{c}_a} \left\| \sum_{n=N}^M g_n v_n \right\|_{L^2(\Delta;V')}^2, \end{aligned} \quad (25)$$

$\forall N, M \in \mathbf{N}, N \leq M.$

Lemma 7. (a) If $g \in L^2(\Delta; V_1)$, then

$$\sum_{n=1}^{\infty} \lambda_n^4 \|g_n\|_{L^2(\Delta)}^2 \leq \|g\|_{L^2(\Delta;V_1)}^2, \quad g_n = (g, v_n), \quad n \in \mathbf{N}. \quad (26)$$

(b) Suppose that $g_n \in L^2(\Delta)$, $n \in \mathbf{N}$. The series $\sum_{n=1}^{\infty} \lambda_n^4 \|g_n\|_{L^2(\Delta)}^2$ converges if and only if the series $\sum_{n=1}^{\infty} g_n v_n$ converges in the space $L^2(\Delta; V_1)$.

Proof. (a) Since $\{v_k\}_{k \in \mathbf{N}}$ are eigenvectors of operator A and $\{v_k\}_{k \in \mathbf{N}}$ is orthonormal in H we have $(Ag, v_k) = a(g, v_k) = a(v_k, g) = \lambda_k^2 g_k$ for all $k \in \mathbf{N}$, and hence

$$\begin{aligned} &\int_{\Delta} \left(Ag - \sum_{n=1}^N g_n A v_n, Ag - \sum_{k=1}^N g_k A v_k \right) d\Delta \\ &= \int_{\Delta} \left(Ag - \sum_{n=1}^N g_n \lambda_n^2 v_n, Ag - \sum_{k=1}^N g_k \lambda_k^2 v_k \right) d\Delta \end{aligned}$$

$$= \int_{\Delta} (Ag, Ag) d\Delta - \int_{\Delta} \sum_{n=1}^N \lambda_n^4 g_n^2 d\Delta \geq 0 \quad (27)$$

for all $N \in \mathbf{N}$, and tending $N \rightarrow \infty$ from the definition of the norm $\|\cdot\|_1$ we obtain the required inequality.

(b) The equivalence of the convergences of the series directly follows from the following inequalities:

$$\begin{aligned} \left\| \sum_{n=N}^M g_n v_n \right\|_{L^2(\Delta;V_1)}^2 &= \int_{\Delta} \left(\sum_{n=N}^M g_n A v_n, \sum_{k=N}^M g_k A v_k \right) d\Delta \\ &= \int_{\Delta} \left(\sum_{n=N}^M g_n \lambda_n^2 v_n, \sum_{k=N}^M g_k \lambda_k^2 v_k \right) d\Delta \\ &= \sum_{n=N}^M \lambda_n^4 \|g_n\|_{L^2(\Delta)}^2, \quad \forall N, M \in \mathbf{N}, N \leq M. \end{aligned} \quad (28)$$

□

Lemma 8. (a) If $g \in L^2(\Delta; V'_1)$, then

$$\sum_{n=1}^{\infty} \frac{\|g_n\|_{L^2(\Delta)}^2}{\lambda_n^4} \leq \|(A_1 A)^{-1}\|^2 \|g\|_{L^2(\Delta;V'_1)}^2, \quad (29)$$

$$g_n = \langle g, v_n \rangle_1, \quad n \in \mathbf{N}.$$

(b) Suppose that $g_n \in L^2(\Delta)$, $n \in \mathbf{N}$. The series $\sum_{n=1}^{\infty} \|g_n\|_{L^2(\Delta)}^2 / \lambda_n^4$ converges if and only if the series $\sum_{n=1}^{\infty} g_n v_n$ converges in the space $L^2(\Delta; V'_1)$.

Proof. (a) Note that the operator $B = A_1 A : V_1 \rightarrow V'_1$ is linear and continuous, and applying (14) we obtain that B satisfies the following coerciveness condition:

$$\langle Bz, z \rangle_1 = \langle A_1 A z, z \rangle_1 = (Az, Az) = \|z\|_1^2 \quad \forall z \in V_1. \quad (30)$$

Consequently, the operator $B \in L(V_1; V'_1)$ is invertible and $B^{-1} \in L(V'_1; V_1)$ [47].

Let us define $\hat{g} = B^{-1}g \in L^2(\Delta; V_1)$. Applying part (a) of Lemma 7 we have $\sum_{n=1}^{\infty} \lambda_n^4 \|\hat{g}_n\|_{L^2(\Delta)}^2 \leq \|\hat{g}\|_{L^2(\Delta;V_1)}^2$ and from equality (14) we obtain

$$\begin{aligned} \hat{g}_n &= (\hat{g}, v_n) = (v_n, \hat{g}) = \frac{a(v_n, \hat{g})}{\lambda_n^2} = \frac{a(\hat{g}, v_n)}{\lambda_n^2} \\ &= \frac{(A\hat{g}, v_n)}{\lambda_n^2} = \frac{(A\hat{g}, A v_n)}{\lambda_n^4} = \frac{\langle A_1 A \hat{g}, v_n \rangle_1}{\lambda_n^4} \\ &= \frac{\langle g, v_n \rangle_1}{\lambda_n^4} = \frac{g_n}{\lambda_n^4} \end{aligned} \quad (31)$$

for all $n \in \mathbf{N}$. Consequently,

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_n^4 \|\hat{g}_n\|_{L^2(\Delta)}^2 &= \sum_{n=1}^{\infty} \frac{\|g_n\|_{L^2(\Delta)}^2}{\lambda_n^4} \leq \|(A_1 A)^{-1}\|^2 \|g\|_{L^2(\Delta;V_1)}^2 \\ &\leq \|(A_1 A)^{-1}\|^2 \|g\|_{L^2(\Delta;V'_1)}^2. \end{aligned} \quad (32)$$

(b) Since $Bv_n = A_1Av_n = \lambda_n^2A_1v_n = \lambda_n^2Av_n = \lambda_n^4v_n$, we have $v_n/\lambda_n^4 = B^{-1}v_n$ and the equivalence of the convergences of the series directly follows from the following inequalities:

$$\begin{aligned}
\left\| \sum_{n=N}^M g_n v_n \right\|_{L^2(\Delta; V_1')}^2 &= \left\| BB^{-1} \sum_{n=N}^M g_n v_n \right\|_{L^2(\Delta; V_1')}^2 \\
&\leq \|B\|^2 \left\| \sum_{n=N}^M g_n B^{-1} v_n \right\|_{L^2(\Delta; V_1)}^2 \\
&= \|B\|^2 \left\| \sum_{n=N}^M g_n \frac{v_n}{\lambda_n^4} \right\|_{L^2(\Delta; V_1)}^2 \\
&= \|B\|^2 \left\| A \sum_{n=N}^M g_n \frac{v_n}{\lambda_n^4} \right\|_{L^2(\Delta; H)}^2 \quad (33) \\
&= \|B\|^2 \left\| \sum_{n=N}^M g_n \frac{v_n}{\lambda_n^2} \right\|_{L^2(\Delta; H)}^2 \\
&= \|B\|^2 \int_{\Delta} \sum_{n=N}^M \frac{g_n^2}{\lambda_n^4} d\Delta \\
&\leq \|B\|^2 \|B^{-1}\|^2 \left\| \sum_{n=N}^M g_n v_n \right\|_{L^2(\Delta; V_1')}^2
\end{aligned}$$

for all $N, M \in \mathbb{N}$, $N \leq M$. $\square$

3. Main Result

In the present paper we investigate nonlocal problem (1), (2) applying the following variational formulation: find a vector-function $u \in W$, $d(\text{tr}_{t_2=0}u)/dt_1 \in L^2(0, T_1; V_1')$, $d(\text{tr}_{t_1=0}u)/dt_2 \in L^2(0, T_2; V_1')$, which satisfies equation

$$\frac{\partial}{\partial t_1}(u, v) + \frac{\partial}{\partial t_2}(u, v) + a(u, v) = \langle f, v \rangle, \quad \forall v \in V, \quad (34)$$

in the sense of distributions on ω , and initial conditions

$$\begin{aligned}
(\text{tr}_{t_1=0}u)(t_2) &= \varphi(t_2), \quad 0 \leq t_2 \leq T_2, \\
(\text{tr}_{t_2=0}u)(t_1) &= \alpha(\text{tr}_{t_2=\xi}u)(t_1) + \psi(t_1), \quad (35) \\
0 &\leq t_1 \leq T_1,
\end{aligned}$$

in the spaces $L^2(0, T_2; H)$ and $L^2(0, T_1; H)$, respectively. Applying Theorem 3 and by taking account of the density of linear combinations of products ϕv , $\phi \in D(\omega)$, $v \in V$,

in $L^2(\omega; V)$ we obtain that (35) is equivalent to (1), which is considered as equation in the space $L^2(\omega; V')$.

Let us prove the following lemmas, which will be used to obtain the existence of solution of problem (34), (35). We denote by $\omega_1 = \{(t_1, t_2) \in \omega; t_2 > t_1\}$ and $\omega_2 = \{(t_1, t_2) \in \omega; t_1 > t_2\}$ the parts of the rectangle ω above and below the bisector $\{(t_1, t_2); t_1 = t_2\}$, respectively; $\bar{\omega}_1$ and $\bar{\omega}_2$ are closures of ω_1 and ω_2 .

Lemma 9. *If $g \in L^2(0, T_2)$ and $\tilde{g} \in L^2(0, T_1)$, then the following estimates are valid*

$$\begin{aligned}
\int_{\omega_1} e^{-2\lambda_n^2 t_1} g^2(t_2 - t_1) dt_1 dt_2 &\leq \frac{1}{2\lambda_n^2} \int_0^{T_2} g^2(s) ds, \\
\int_{\omega_2} e^{-2\lambda_n^2 t_2} g^2(t_1 - t_2) dt_1 dt_2 &\leq \frac{1}{2\lambda_n^2} \int_0^{T_1} g^2(s) ds. \quad (36)
\end{aligned}$$

Proof. Taking into account the definition of the sets ω_1 and ω_2 we obtain

$$\begin{aligned}
&\int_{\omega_1} e^{-2\lambda_n^2 t_1} g^2(t_2 - t_1) dt_1 dt_2 \\
&\leq \int_0^{T_2} \int_0^{t_2} e^{-2\lambda_n^2(t_2-s)} g^2(s) ds dt_2 \\
&= \int_0^{T_2} \int_s^{T_2} e^{-2\lambda_n^2(t_2-s)} g^2(s) ds dt_2 \\
&= \int_0^{T_2} \frac{1 - e^{-2\lambda_n^2(T_2-s)}}{2\lambda_n^2} g^2(s) ds, \quad (37)
\end{aligned}$$

which implies the first estimate. Likewise we obtain the second estimate. $\square$

Lemma 10. *If $g \in H^1(\omega)$, then the following estimates are valid*

$$\begin{aligned}
&\int_{\omega_1} \left(\int_0^{t_1} e^{-\lambda_n^2(t_1-\tau)} g(\tau, \tau + t_2 - t_1) d\tau \right)^2 dt_1 dt_2 \\
&\leq \frac{c}{\lambda_n^4} \|g\|_{H^1(\omega)}^2, \\
&\int_{\omega_2} \left(\int_0^{t_2} e^{-\lambda_n^2(t_2-\tau)} g(\tau + t_1 - t_2, \tau) d\tau \right)^2 dt_1 dt_2 \\
&\leq \frac{c}{\lambda_n^4} \|g\|_{H^1(\omega)}^2. \quad (38)
\end{aligned}$$

Proof. Assuming that $g \in C^1(\bar{\omega})$, $\bar{\omega} = [0, T_1] \times [0, T_2]$, we have

$$\begin{aligned}
&\int_{\omega_1} \left(\int_0^{t_1} e^{-\lambda_n^2(t_1-\tau)} g(\tau, \tau + t_2 - t_1) d\tau \right)^2 dt_1 dt_2 \\
&= \int_{\omega_1} \left(\frac{g(t_1, t_2)}{\lambda_n^2} - \frac{e^{-\lambda_n^2 t_1} g(0, t_2 - t_1)}{\lambda_n^2} - \int_0^{t_1} \frac{e^{-\lambda_n^2(t_1-\tau)}}{\lambda_n^2} \left(\frac{\partial g}{\partial \tau_1}(\tau, \tau + t_2 - t_1) + \frac{\partial g}{\partial \tau_2}(\tau, \tau + t_2 - t_1) \right) d\tau \right)^2 dt_1 dt_2
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{4}{\lambda_n^4} \int_{\omega_1} g^2(t_1, t_2) dt_1 dt_2 + \frac{4}{\lambda_n^4} \int_0^{T_2} \int_0^{t_2} g^2(0, s) ds dt_2 \\
&+ \frac{4T_1 T_2}{\lambda_n^4} \left[\int_{\omega_1} \left(\frac{\partial g}{\partial \tau_1}(t_1, t_2) \right)^2 dt_1 dt_2 + \int_{\omega_1} \left(\frac{\partial g}{\partial \tau_2}(t_1, t_2) \right)^2 dt_1 dt_2 \right].
\end{aligned} \tag{39}$$

From continuity of the trace operator $H^1(\omega) \rightarrow L^2(\partial\omega)$ we obtain

$$\begin{aligned}
\int_0^{T_2} \int_0^{t_2} g^2(0, s) ds dt_2 &\leq T_2 \int_0^{T_2} g^2(0, s) ds \\
&\leq T_2 c \|g\|_{H^1(\omega)}^2.
\end{aligned} \tag{40}$$

The latter inequality implies the first estimate of the lemma for functions from $C^1(\bar{\omega})$ and since $C^1(\bar{\omega})$ is dense in $H^1(\omega)$, by passing to the limit we obtain the estimate for functions from $H^1(\omega)$. Likewise we obtain the second estimate. $\square$

Lemma 11. *If $g \in H^1(\omega)$, then the following estimates are valid*

$$\begin{aligned}
&\int_0^{\xi_{T_1}} \left(\int_0^{t_1} e^{-\lambda_n^2(t_1-\tau)} g(\tau, \tau + \xi - t_1) d\tau \right)^2 dt_1 \\
&= \int_0^{\xi_{T_1}} \left(\frac{g(t_1, \xi)}{\lambda_n^2} - \frac{e^{-\lambda_n^2 t_1} g(0, \xi - t_1)}{\lambda_n^2} - \int_0^{t_1} \frac{e^{-\lambda_n^2(t_1-\tau)}}{\lambda_n^2} \left(\frac{\partial g}{\partial \tau_1}(\tau, \tau + \xi - t_1) + \frac{\partial g}{\partial \tau_2}(\tau, \tau + \xi - t_1) \right) d\tau \right)^2 dt_1 \leq \frac{4}{\lambda_n^4} \\
&\cdot \int_0^{\xi_{T_1}} g^2(t_1, \xi) dt_1 + \frac{4}{\lambda_n^4} \int_0^\xi g^2(0, s) ds + \frac{4T_1}{\lambda_n^4} \int_{\omega_1} \left[\left(\frac{\partial g}{\partial \tau_1} \right)^2 + \left(\frac{\partial g}{\partial \tau_2} \right)^2 \right] dt_1 dt_2.
\end{aligned} \tag{42}$$

The continuity of the trace operator $H^1(\omega_\xi) \rightarrow L^2(\partial\omega_\xi)$, $\omega_\xi = (0, T_1) \times (0, \xi)$, implies that

$$\begin{aligned}
&\max \left\{ \int_0^{\xi_{T_1}} g^2(t_1, \xi) dt_1, \int_0^\xi g^2(0, s) ds \right\} \\
&\leq c \|g\|_{H^1(\omega_\xi)}^2 \leq c \|g\|_{H^1(\omega)}^2.
\end{aligned} \tag{43}$$

From the latter inequality we obtain the first estimate of the lemma for functions from $C^1(\bar{\omega})$ and since $C^1(\bar{\omega})$ is dense in $H^1(\omega)$, by passing to the limit we obtain the estimate for functions from $H^1(\omega)$. Likewise we obtain the second estimate. $\square$

The following existence and uniqueness theorem is valid for nonlocal problem (34), (35).

Theorem 12. *If $\varphi \in L^2(0, T_2; H)$, $\varphi' \in L^2(0, T_2; V')$, $\psi \in L^2(0, T_1; H)$, $\psi' \in L^2(0, T_1; V')$, φ and ψ satisfy compatibility condition $\varphi(0) = \alpha\varphi(\xi) + \psi(0)$ and*

$$\begin{aligned}
&\int_0^{\xi_{T_1}} \left(\int_0^{t_1} e^{-\lambda_n^2(t_1-\tau)} g(\tau, \tau + \xi - t_1) d\tau \right)^2 dt_1 \\
&\leq \frac{c}{\lambda_n^4} \|g\|_{H^1(\omega)}^2, \\
&\int_{\xi_{T_1}}^{T_1} \left(\int_0^\xi e^{-\lambda_n^2(\xi-\tau)} g(\tau + t_1 - \xi, \tau) d\tau \right)^2 dt_1 \\
&\leq \frac{c}{\lambda_n^4} \|g\|_{H^1(\omega)}^2,
\end{aligned} \tag{41}$$

where $\xi_{T_1} = \min\{\xi, T_1\}$.

Proof. Assuming that $g \in C^1(\bar{\omega})$ we get

f, $\partial f / \partial t_1, \partial f / \partial t_2 \in L^2(\omega; V')$, then problem (34), (35) has a unique solution and the following estimate is valid:

$$\begin{aligned}
\|u\|_W &\leq c \left(\|\varphi\|_{L^2(0, T_2; H)} + \|\varphi'\|_{L^2(0, T_2; V')} + \|\psi\|_{L^2(0, T_1; H)} \right. \\
&+ \|\psi'\|_{L^2(0, T_1; V')} + \|f\|_{L^2(\omega; V')} + \left\| \frac{\partial f}{\partial t_1} \right\|_{L^2(\omega; V')} \\
&\left. + \left\| \frac{\partial f}{\partial t_2} \right\|_{L^2(\omega; V')} \right).
\end{aligned} \tag{44}$$

Proof. Applying embedding theorem [46] we have that $\varphi \in C([0, T_2]; V')$, $\psi \in C([0, T_1]; V')$ and the compatibility condition is the equality in V' . First, we prove the existence of a solution. Let us consider the sequence of approximate solutions $\omega_N(t_1, t_2) = \sum_{n=1}^N u_n(t_1, t_2) v_n$, where u_n is a solution of the following nonlocal problem for the first-order partial differential equation:

$$\frac{\partial u_n}{\partial t_1} + \frac{\partial u_n}{\partial t_2} + \lambda_n^2 u_n = f_n, \quad (t_1, t_2) \in \omega, \tag{45}$$

$$\begin{aligned}
u_n(0, t_2) &= \varphi_n(t_2), \quad 0 \leq t_2 \leq T_2, \\
u_n(t_1, 0) &= \alpha u_n(t_1, \xi) + \psi_n(t_1), \\
0 &\leq t_1 \leq T_1,
\end{aligned} \tag{46}$$

where $\varphi_n = (\varphi, v_n)$, $\psi_n = (\psi, v_n)$, $f_n = \langle f, v_n \rangle$, $n \in \mathbf{N}$. Note that $\varphi_n \in H^1(0, T_2)$, $\psi_n \in H^1(0, T_1)$, $f_n \in H^1(\omega)$ and hence $\varphi_n \in C[0, T_2]$, $\psi_n \in C[0, T_1]$.

Let us denote by g_{1n} and g_{2n} the following functions:

$$\begin{aligned}
g_{1n}(t_1, t_2) &= \int_0^{t_1} e^{-\lambda_n^2(t_1-\tau)} f_n(\tau, \tau + t_2 - t_1) d\tau, \\
&\quad (t_1, t_2) \in \bar{\omega}_1, \\
g_{2n}(t_1, t_2) &= \int_0^{t_2} e^{-\lambda_n^2(t_2-\tau)} f_n(\tau + t_1 - t_2, \tau) d\tau, \\
&\quad (t_1, t_2) \in \bar{\omega}_2.
\end{aligned} \tag{47}$$

Functions g_{1n} and g_{2n} are defined for all $(t_1, t_2) \in \bar{\omega}_1$ and $(t_1, t_2) \in \bar{\omega}_2$, respectively, because the trace of f_n on each line

$$l_{t_1 t_2} = \begin{cases} (\tau, \tau + t_2 - t_1); & 0 < \tau < t_1 \text{ for } t_2 \geq t_1, \\ (\tau + t_1 - t_2, \tau); & 0 < \tau < t_2 \text{ for } t_1 \geq t_2, \end{cases} \tag{48}$$

which is parallel to the bisector $\{(t_1, t_2); t_1 = t_2\}$, which belongs to $L^2(l_{t_1 t_2})$.

Note that $g_{1n} \in L^2(\omega_1)$ and $g_{2n} \in L^2(\omega_2)$. Their generalized derivative is given by

$$\begin{aligned}
\frac{\partial g_{1n}}{\partial t_1} &= f_n(t_1, t_2) \\
&\quad - \lambda_n^2 \int_0^{t_1} e^{-\lambda_n^2(t_1-\tau)} f_n(\tau, \tau + t_2 - t_1) d\tau \\
&\quad - \int_0^{t_1} e^{-\lambda_n^2(t_1-\tau)} \frac{\partial f_n}{\partial \tau_2}(\tau, \tau + t_2 - t_1) d\tau, \\
\frac{\partial g_{1n}}{\partial t_2} &= \int_0^{t_1} e^{-\lambda_n^2(t_1-\tau)} \frac{\partial f_n}{\partial \tau_2}(\tau, \tau + t_2 - t_1) d\tau
\end{aligned} \tag{49}$$

for almost all $(t_1, t_2) \in \omega_1$,

$$\begin{aligned}
\frac{\partial g_{2n}}{\partial t_1} &= \int_0^{t_2} e^{-\lambda_n^2(t_2-\tau)} \frac{\partial f_n}{\partial \tau_1}(\tau + t_1 - t_2, \tau) d\tau, \\
\frac{\partial g_{2n}}{\partial t_2} &= f_n(t_1, t_2) \\
&\quad - \lambda_n^2 \int_0^{t_2} e^{-\lambda_n^2(t_2-\tau)} f_n(\tau + t_1 - t_2, \tau) d\tau \\
&\quad + \int_0^{t_2} e^{-\lambda_n^2(t_2-\tau)} \frac{\partial f_n}{\partial \tau_1}(\tau + t_1 - t_2, \tau) d\tau
\end{aligned} \tag{50}$$

for almost all $(t_1, t_2) \in \omega_2$. Applying Lemma 10 we obtain

$$\begin{aligned}
\left\| \frac{\partial g_{1n}}{\partial t_1} \right\|_{L^2(\omega_1)}^2 &\leq 3 \|f_n\|_{L^2(\omega_1)}^2 + 3c \|f_n\|_{H^1(\omega)}^2 \\
&\quad + T_1 \int_0^{T_2} \int_0^{\min\{t_2, T_1\}} \int_0^{t_1} \left(\frac{\partial f_n}{\partial \tau_2} \right)^2(\tau, \tau + t_2 - t_1) d\tau dt_1 dt_2 \\
&\leq 3(c+1) \|f_n\|_{H^1(\omega)}^2 + T_1 T_2 \int_{\omega_1} \left(\frac{\partial f_n}{\partial \tau_2} \right)^2 dt_1 dt_2 \\
&\leq c \|f_n\|_{H^1(\omega)}^2,
\end{aligned} \tag{51}$$

$$\begin{aligned}
\left\| \frac{\partial g_{1n}}{\partial t_2} \right\|_{L^2(\omega_1)}^2 &\leq T_1 \int_0^{T_2} \int_0^{\min\{t_2, T_1\}} \int_0^{t_1} \left(\frac{\partial f_n}{\partial \tau_2} \right)^2(\tau, \tau + t_2 - t_1) d\tau dt_1 dt_2 \\
&\leq T_1 T_2 \int_{\omega_1} \left(\frac{\partial f_n}{\partial \tau_2} \right)^2 dt_1 dt_2 \leq T_1 T_2 \|f_n\|_{H^1(\omega)}^2,
\end{aligned} \tag{52}$$

$$\begin{aligned}
\left\| \frac{\partial g_{2n}}{\partial t_1} \right\|_{L^2(\omega_2)}^2 &\leq T_2 \int_0^{T_1} \int_0^{\min\{t_1, T_2\}} \int_0^{t_2} \left(\frac{\partial f_n}{\partial \tau_1} \right)^2(\tau + t_1 - t_2, \tau) d\tau dt_2 dt_1 \\
&\leq T_1 T_2 \int_{\omega_2} \left(\frac{\partial f_n}{\partial \tau_1} \right)^2 dt_1 dt_2 \leq T_1 T_2 \|f_n\|_{H^1(\omega)}^2,
\end{aligned} \tag{53}$$

$$\begin{aligned}
\left\| \frac{\partial g_{2n}}{\partial t_2} \right\|_{L^2(\omega_2)}^2 &\leq 3 \|f_n\|_{L^2(\omega_2)}^2 + 3c \|f_n\|_{H^1(\omega)}^2 \\
&\quad + T_2 \int_0^{T_1} \int_0^{\min\{t_1, T_2\}} \int_0^{t_2} \left(\frac{\partial f_n}{\partial \tau_1} \right)^2(\tau + t_1 - t_2, \tau) d\tau dt_2 dt_1 \\
&\leq 3(c+1) \|f_n\|_{H^1(\omega)}^2 + T_2 T_1 \int_{\omega_2} \left(\frac{\partial f_n}{\partial \tau_1} \right)^2 dt_1 dt_2 \\
&\leq c \|f_n\|_{H^1(\omega)}^2.
\end{aligned} \tag{54}$$

Therefore, $g_{1n} \in H^1(\omega_1)$ and $g_{2n} \in H^1(\omega_2)$.

Applying Lemma 11 and continuity of the trace operator $H^1(\omega_\xi) \rightarrow L^2(\partial\omega_\xi)$ we obtain

$$\begin{aligned}
\|g_{1n}(t_1, \xi)\|_{L^2(0, \xi_{T_1})}^2 &\leq \frac{c}{\lambda_n^4} \|f_n\|_{H^1(\omega)}^2, \\
\|g_{2n}(t_1, \xi)\|_{L^2(\xi, T_1)}^2 &\leq \frac{c}{\lambda_n^4} \|f_n\|_{H^1(\omega)}^2 \quad \text{for } T_1 > \xi,
\end{aligned}$$

$$\begin{aligned}
\left\| \frac{dg_{1n}(t_1, \xi)}{dt_1} \right\|_{L^2(0, \xi_{T_1})}^2 &\leq 3 \|f_n(t_1, \xi)\|_{L^2(0, \xi_{T_1})}^2 \\
&\quad + 3\lambda_n^4 \int_0^{\xi_{T_1}} \left(\int_0^{t_1} e^{-\lambda_n^2(t_1-\tau)} f_n(\tau, \tau + \xi - t_1) d\tau \right)^2 dt_1 \\
&\quad + 3T_1 \int_0^{\xi_{T_1}} \int_0^{t_1} \left(\frac{\partial f_n}{\partial \tau_2} \right)^2(\tau, \tau + \xi - t_1) d\tau dt_1 \\
&\leq c \|f_n\|_{H^1(\omega)}^2,
\end{aligned}$$

$$\begin{aligned}
& \left\| \frac{dg_{2n}(t_1, \xi)}{dt_1} \right\|_{L^2(\xi, T_1)}^2 \\
& \leq \xi \int_{\xi}^{T_1} \int_0^{\xi} \left(\frac{\partial f_n}{\partial \tau_1} \right)^2 (\tau + t_1 - \xi, \tau) d\tau dt_1 \\
& \leq \xi \|f_n\|_{H^1(\omega)}^2 \quad \text{for } T_1 > \xi.
\end{aligned} \tag{55}$$

Hence, $g_{1n}(t_1, \xi) \in H^1(0, \xi_{T_1})$ and $g_{2n}(t_1, \xi) \in H^1(\xi, T_1)$ for $T_1 > \xi$, and from embedding theorem we obtain that $g_{1n}(t_1, \xi) \in C[0, \xi_{T_1}]$, $g_{2n}(t_1, \xi) \in C[\xi, T_1]$ for $T_1 > \xi$.

Note that the classical problem for (45) with initial conditions $u_n(0, t_2) = \varphi_n(t_2)$ and $u_n(t_1, 0) = d_n(t_1)$, $d_n \in H^1(0, T_1)$, $\varphi_n(0) = d_n(0)$, has a unique solution $u_n \in H^1(\omega)$, $du_n(t_1, 0)/dt_1 \in L^2(0, T_1)$, $du_n(0, t_2)/dt_2 \in L^2(0, T_2)$, which is given by

$$\begin{aligned}
& u_n(t_1, t_2) \\
& = \begin{cases} e^{-\lambda_n^2 t_1} \varphi_n(t_2 - t_1) + g_{1n}(t_1, t_2), & (t_1, t_2) \in \omega_1, \\ e^{-\lambda_n^2 t_2} d_n(t_1 - t_2) + g_{2n}(t_1, t_2), & (t_1, t_2) \in \omega_2. \end{cases} \tag{56}
\end{aligned}$$

From nonlocal conditions (46) for function d_n we have

$$\begin{aligned}
& d_n(t_1) \\
& = \begin{cases} \alpha \left(e^{-\lambda_n^2 t_1} \varphi_n(\xi - t_1) + g_{1n}(t_1, \xi) \right) + \psi_n(t_1), & t_1 \in [0, \xi_{T_1}], \\ \alpha \left(e^{-\lambda_n^2 \xi} d_n(t_1 - \xi) + g_{2n}(t_1, \xi) \right) + \psi_n(t_1), & t_1 \in (\xi_{T_1}, T_1], \end{cases} \tag{57}
\end{aligned}$$

from which we obtain expression of d_n for $t_1 > \xi$, $T_1 > \xi$,

$$\begin{aligned}
& d_n(t_1) \\
& = \left(\alpha e^{-\lambda_n^2 \xi} \right)^k d_n(t_1 - k\xi) \\
& \quad + \sum_{i=0}^{k-1} \left(\alpha e^{-\lambda_n^2 \xi} \right)^i \left(\alpha g_{2n}(t_1 - i\xi, \xi) + \psi_n(t_1 - i\xi) \right)
\end{aligned} \tag{58}$$

for $k\xi < t_1 \leq \min\{(k+1)\xi, T_1\}$, $k = 1, \dots, K$, where $K = [T_1/\xi]$, if $T_1/\xi \notin \mathbf{N}$, and $K = T_1/\xi - 1$, if $T_1/\xi \in \mathbf{N}$; $[x]$ denotes the integer part of $x \in \mathbf{R}$.

Since functions φ_n , ψ_n , $g_{1n}(t_1, \xi)$, and $g_{2n}(t_1, \xi)$ are continuous, we have $d_n \in C[0, \xi_{T_1}]$. Let us assume that $T_1 > \xi$ and then $\xi_{T_1} = \xi$. Note that $g_{1n}(0, \xi) = 0$, $g_{1n}(\xi, \xi) = g_{2n}(\xi, \xi)$ and from compatibility condition we have $\varphi_n(0) = \alpha\varphi_n(\xi) + \psi_n(0)$. Hence, we get

$$\begin{aligned}
& d_n(\xi) = \alpha \left(e^{-\lambda_n^2 \xi} \varphi_n(0) + g_{1n}(\xi, \xi) \right) + \psi_n(\xi) \\
& = \alpha \left(e^{-\lambda_n^2 \xi} (\alpha\varphi_n(\xi) + \psi_n(0)) + g_{2n}(\xi, \xi) \right) \\
& \quad + \psi_n(\xi)
\end{aligned}$$

$$= \alpha \left(e^{-\lambda_n^2 \xi} d_n(0) + g_{2n}(\xi, \xi) \right) + \psi_n(\xi), \tag{59}$$

which imply continuity of d_n at point ξ . Moreover, we have

$$\begin{aligned}
& d_n((k+1)\xi) = \left(\alpha e^{-\lambda_n^2 \xi} \right)^k d_n(\xi) + \sum_{i=0}^{k-1} \left(\alpha e^{-\lambda_n^2 \xi} \right)^i \\
& \quad \cdot \left(\alpha g_{2n}((k-i+1)\xi, \xi) + \psi_n((k-i+1)\xi) \right) \\
& = \left(\alpha e^{-\lambda_n^2 \xi} \right)^{k+1} d_n(0) + \left(\alpha e^{-\lambda_n^2 \xi} \right)^k \left(\alpha g_{2n}(\xi, \xi) \right. \\
& \quad \left. + \psi_n(\xi) \right) + \sum_{i=0}^{k-1} \left(\alpha e^{-\lambda_n^2 \xi} \right)^i \left(\alpha g_{2n}((k-i+1)\xi, \xi) \right. \\
& \quad \left. + \psi_n((k-i+1)\xi) \right) = \left(\alpha e^{-\lambda_n^2 \xi} \right)^{k+1} d_n(0) \\
& \quad + \sum_{i=0}^k \left(\alpha e^{-\lambda_n^2 \xi} \right)^i \\
& \quad \cdot \left(\alpha g_{2n}((k-i+1)\xi, \xi) + \psi_n((k-i+1)\xi) \right).
\end{aligned} \tag{60}$$

Consequently, $d_n \in C[0, T_1]$ and its generalized derivative d'_n belongs to $L^2(0, T_1)$, since

$$\begin{aligned}
& d'_n(t_1) = \alpha \left(-\lambda_n^2 e^{-\lambda_n^2 t_1} \varphi_n(\xi - t_1) - e^{-\lambda_n^2 t_1} \varphi'_n(\xi - t_1) \right. \\
& \quad \left. + \frac{dg_{1n}(t_1, \xi)}{dt_1} \right) + \psi'_n(t_1)
\end{aligned} \tag{61}$$

for almost all $t_1 \in (0, \xi_{T_1})$ and

$$\begin{aligned}
& d'_n(t_1) = \left(\alpha e^{-\lambda_n^2 \xi} \right)^k d'_n(t_1 - k\xi) \\
& \quad + \sum_{i=0}^{k-1} \left(\alpha e^{-\lambda_n^2 \xi} \right)^i \left(\alpha \frac{dg_{2n}(t_1 - i\xi, \xi)}{dt_1} + \psi'_n(t_1 - i\xi) \right)
\end{aligned} \tag{62}$$

for $T_1 > \xi$ and for almost all $t_1 \in (k\xi, \min\{(k+1)\xi, T_1\})$, $k = 1, \dots, K$.

Since $g_{1n}(0, t_2) = 0$ and $g_{1n}(0, t_2) = 0$, from construction of d_n it follows that u_n satisfies conditions (46).

Note that the trace on the bisector l_{TT} , $T = \min\{T_1, T_2\}$, of restriction of function u_n on ω_1 equals the trace on the bisector l_{TT} of restriction of function u_n on ω_2 , because $g_{1n}(t, t) = g_{2n}(t, t)$, $0 < t \leq T$, $\varphi_n(0) = \alpha\varphi_n(\xi) + \psi_n(0)$, and

$$\begin{aligned}
& u_n(t, t) = e^{-\lambda_n^2 t} \varphi_n(0) + g_{1n}(t, t) \\
& = e^{-\lambda_n^2 t} (\alpha\varphi_n(\xi) + \psi_n(0)) + g_{2n}(t, t) \\
& = e^{-\lambda_n^2 t} d_n(0) + g_{2n}(t, t).
\end{aligned} \tag{63}$$

The generalized derivatives of function u_n are given by

$$\begin{aligned} \frac{\partial u_n}{\partial t_1} &= \begin{cases} -\lambda_n^2 e^{-\lambda_n^2 t_1} \varphi_n(t_2 - t_1) - e^{-\lambda_n^2 t_1} \varphi'_n(t_2 - t_1) + \frac{\partial g_{1n}}{\partial t_1}(t_1, t_2), & (t_1, t_2) \in \omega_1, \\ e^{-\lambda_n^2 t_2} d'_n(t_1 - t_2) + \frac{\partial g_{2n}}{\partial t_1}(t_1, t_2), & (t_1, t_2) \in \omega_2, \end{cases} \\ \frac{\partial u_n}{\partial t_2} &= \begin{cases} e^{-\lambda_n^2 t_1} \varphi'_n(t_2 - t_1) + \frac{\partial g_{1n}}{\partial t_2}(t_1, t_2), & (t_1, t_2) \in \omega_1, \\ -\lambda_n^2 e^{-\lambda_n^2 t_2} d_n(t_1 - t_2) - e^{-\lambda_n^2 t_2} d'_n(t_1 - t_2) + \frac{\partial g_{2n}}{\partial t_2}(t_1, t_2), & (t_1, t_2) \in \omega_2. \end{cases} \end{aligned} \quad (64)$$

Therefore, we have $\partial u_n / \partial t_1, \partial u_n / \partial t_2 \in L^2(\omega)$ and u_n satisfies (45) in $L^2(\omega)$.

Now we obtain estimates for u_n , which permit one to prove the convergence of the sequence of approximate solutions $(w_N)_{N \geq 1}$.

Applying Lemmas 9 and 10 we obtain

$$\begin{aligned} \|u_n\|_{L^2(\omega)}^2 &\leq 2 \int_{\omega_1} e^{-2\lambda_n^2 t_1} \varphi_n^2(t_2 - t_1) dt_1 dt_2 \\ &\quad + 2 \int_{\omega_1} g_{1n}^2(t_1, t_2) dt_1 dt_2 \\ &\quad + 2 \int_{\omega_2} e^{-2\lambda_n^2 t_2} d_n^2(t_1 - t_2) dt_1 dt_2 \\ &\quad + 2 \int_{\omega_2} g_{2n}^2(t_1, t_2) dt_1 dt_2 \\ &\leq \frac{1}{\lambda_n^2} \|\varphi_n\|_{L^2(0, T_2)}^2 + \frac{c}{\lambda_n^4} \|f_n\|_{H^1(\omega)}^2 \\ &\quad + \frac{1}{\lambda_n^2} \|d_n\|_{L^2(0, T_1)}^2. \end{aligned} \quad (65)$$

To obtain estimate for d_n without loss of generality we assume that $T_1 > \xi$. Applying Lemmas 9–11 we get

$$\begin{aligned} \|d_n\|_{L^2(0, T_1)}^2 &= \|d_n\|_{L^2(0, \xi)}^2 \\ &\quad + \sum_{k=1}^K \int_{k\xi}^{\min\{(k+1)\xi, T_1\}} \left(\left(\alpha e^{-\lambda_n^2 \xi} \right)^k d_n(t_1 - k\xi) \right. \\ &\quad \left. + \sum_{i=0}^{k-1} \left(\alpha e^{-\lambda_n^2 \xi} \right)^i (\alpha g_{2n}(t_1 - i\xi, \xi) + \psi_n(t_1 - i\xi)) \right)^2 dt_1 \\ &\leq \int_0^\xi d_n^2(t_1) dt_1 + K(2K+1) \max\{\alpha^{2K}, \alpha^2\} \\ &\quad \cdot \int_0^\xi d_n^2(s) ds + K(2K+1) \max\{\alpha^{2K}, \alpha^2\} \int_\xi^{T_1} g_{2n}^2(t_1, \\ &\quad \xi) dt_1 + K(2K+1) \max\{1, \alpha^{2(K-1)}\} \int_\xi^{T_1} \psi_n^2(s) ds \end{aligned}$$

$$\begin{aligned} &\leq (1 + K(2K+1) \max\{\alpha^{2K}, \alpha^2\}) \left(3\alpha^2 \int_0^\xi \varphi_n^2(\xi \right. \\ &\quad \left. - t_1) dt_1 + 3\alpha^2 \int_0^\xi g_{1n}^2(t_1, \xi) dt_1 \right. \\ &\quad \left. + 3 \int_0^\xi \psi_n^2(t_1) dt_1 \right) + K(2K+1) \max\{\alpha^{2K}, \alpha^2\} \\ &\quad \cdot \frac{c}{\lambda_n^4} \|f_n\|_{H^1(\omega)}^2 + K(2K+1) \max\{1, \alpha^{2(K-1)}\} \\ &\quad \cdot \int_\xi^{T_1} \psi_n^2(s) ds \leq c \left(\|\varphi_n\|_{L^2(0, T_2)}^2 + \|\psi_n\|_{L^2(0, T_1)}^2 \right. \\ &\quad \left. + \frac{1}{\lambda_n^4} \|f_n\|_{H^1(\omega)}^2 \right). \end{aligned} \quad (66)$$

From the latter estimate we have

$$\begin{aligned} \|u_n\|_{L^2(\omega)}^2 &\leq c \left(\frac{\|\varphi_n\|_{L^2(0, T_2)}^2}{\lambda_n^2} + \frac{\|\psi_n\|_{L^2(0, T_1)}^2}{\lambda_n^2} + \frac{\|f_n\|_{H^1(\omega)}^2}{\lambda_n^4} \right). \end{aligned} \quad (67)$$

Applying expressions for $\partial u_n / \partial t_1$ and $\partial u_n / \partial t_2$, Lemma 9, and estimates (51)–(54) we obtain

$$\begin{aligned} \left\| \frac{\partial u_n}{\partial t_1} \right\|_{L^2(\omega)}^2 &\leq 3\lambda_n^4 \left\| e^{-\lambda_n^2 t_1} \varphi_n(t_2 - t_1) \right\|_{L^2(\omega_1)}^2 \\ &\quad + 3 \left\| e^{-\lambda_n^2 t_1} \varphi'_n(t_2 - t_1) \right\|_{L^2(\omega_1)}^2 \\ &\quad + 3 \left\| \frac{\partial g_{1n}}{\partial t_1} \right\|_{L^2(\omega_1)}^2 \\ &\quad + 2 \left\| e^{-\lambda_n^2 t_2} d'_n(t_1 - t_2) \right\|_{L^2(\omega_2)}^2 \\ &\quad + 2 \left\| \frac{\partial g_{2n}}{\partial t_1} \right\|_{L^2(\omega_2)}^2 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{3\lambda_n^2}{2} \|\varphi_n\|_{L^2(0,T_2)}^2 + \frac{3}{2\lambda_n^2} \|\varphi'_n\|_{L^2(0,T_2)}^2 \\
&\quad + \frac{1}{\lambda_n^2} \|d'_n\|_{L^2(0,T_1)}^2 + 5c \|f_n\|_{H^1(\omega)}^2.
\end{aligned} \tag{68}$$

In order to estimate d'_n without loss of generality we assume that $T_1 > \xi$. Applying Lemmas 9–11 we have

$$\begin{aligned}
\|d'_n\|_{L^2(0,T_1)}^2 &= \|d'_n\|_{L^2(0,\xi)}^2 + \sum_{k=1}^K \int_{k\xi}^{\min\{(k+1)\xi, T_1\}} \left((ae^{-\lambda_n^2 \xi})^k \right. \\
&\quad \cdot d'_n(t_1 - k\xi) \\
&\quad \left. + \sum_{i=0}^{k-1} (ae^{-\lambda_n^2 \xi})^i \left(\alpha \frac{\partial g_{2n}}{\partial t_1}(t_1 - i\xi, \xi) + \psi'_n(t_1 - i\xi) \right) \right)^2 dt_1 \\
&\leq \|d'_n\|_{L^2(0,\xi)}^2 + K(2K+1) \max\{\alpha^{2K}, \alpha^2\} \|d'_n\|_{L^2(0,\xi)}^2 \\
&\quad + K(2K+1) \max\{\alpha^{2K}, \alpha^2\} \int_{\xi}^{T_1} \left(\frac{dg_{2n}(t_1, \xi)}{dt_1} \right)^2 dt_1 \\
&\quad + K(2K+1) \max\{1, \alpha^{2(K-1)}\} \|\psi'_n\|_{L^2(\xi, T_1)}^2 \leq (1 + K(2K+1)) \tag{69} \\
&\quad \cdot \max\{\alpha^{2K}, \alpha^2\} \left(4\alpha^2 \lambda_n^4 \int_0^{\xi} \varphi_n^2(\xi - t_1) dt_1 \right. \\
&\quad \left. + 4\alpha^2 \int_0^{\xi} (\varphi'_n(\xi - t_1))^2 dt_1 + 4\alpha^2 \int_0^{\xi} \left(\frac{dg_{1n}(t_1, \xi)}{dt_1} \right)^2 dt_1 \right. \\
&\quad \left. + 4 \int_0^{\xi} (\psi'_n)^2(t_1) dt_1 \right) + K(2K+1) \max\{\alpha^{2K}, \alpha^2\} \\
&\quad \cdot \xi \|f_n\|_{H^1(\omega)}^2 + K(2K+1) \max\{1, \alpha^{2(K-1)}\} \|\psi'_n\|_{L^2(\xi, T_1)}^2 \\
&\leq c \left(\lambda_n^4 \|\varphi_n\|_{L^2(0,T_2)}^2 + \|\varphi'_n\|_{L^2(0,T_2)}^2 + \|\psi'_n\|_{L^2(0,T_1)}^2 + \|f_n\|_{H^1(\omega)}^2 \right).
\end{aligned}$$

From the estimate for $\|\partial u_n / \partial t_1\|_{L^2(\omega)}^2$ and (69) we get

$$\begin{aligned}
\left\| \frac{\partial u_n}{\partial t_1} \right\|_{L^2(\omega)}^2 &\leq c \left(\lambda_n^2 \|\varphi_n\|_{L^2(0,T_2)}^2 + \frac{1}{\lambda_n^2} \|\varphi'_n\|_{L^2(0,T_2)}^2 \right. \\
&\quad \left. + \frac{1}{\lambda_n^2} \|\psi'_n\|_{L^2(0,T_1)}^2 + \|f_n\|_{H^1(\omega)}^2 \right).
\end{aligned} \tag{70}$$

By using Lemma 9 and estimates (52), (54), (66), and (69) we obtain

$$\begin{aligned}
\left\| \frac{\partial u_n}{\partial t_2} \right\|_{L^2(\omega)}^2 &\leq 2 \left\| e^{-\lambda_n^2 t_1} \varphi'_n(t_2 - t_1) \right\|_{L^2(\omega_1)}^2 \\
&\quad + 2 \left\| \frac{\partial g_{1n}}{\partial t_2} \right\|_{L^2(\omega_1)}^2 + 3\lambda_n^4 \left\| e^{-\lambda_n^2 t_2} d_n(t_1 - t_2) \right\|_{L^2(\omega_2)}^2 \\
&\quad + 3 \left\| e^{-\lambda_n^2 t_2} d'_n(t_1 - t_2) \right\|_{L^2(\omega_2)}^2 + 3 \left\| \frac{\partial g_{2n}}{\partial t_2} \right\|_{L^2(\omega_2)}^2 \\
&\leq \frac{1}{\lambda_n^2} \|\varphi'_n\|_{L^2(0,T_2)}^2 + c \|f_n\|_{H^1(\omega)}^2 + \frac{3\lambda_n^2}{2} \|d_n\|_{L^2(0,T_1)}^2
\end{aligned}$$

$$\begin{aligned}
&+ \frac{3 \|d'_n\|_{L^2(0,T_1)}^2}{2\lambda_n^2} \leq c \left(\lambda_n^2 \|\varphi_n\|_{L^2(0,T_2)}^2 + \frac{\|\varphi'_n\|_{L^2(0,T_2)}^2}{\lambda_n^2} \right. \\
&\quad \left. + \lambda_n^2 \|\psi_n\|_{L^2(0,T_1)}^2 + \frac{1}{\lambda_n^2} \|\psi'_n\|_{L^2(0,T_1)}^2 + \|f_n\|_{H^1(\omega)}^2 \right).
\end{aligned} \tag{71}$$

Applying parts (a) of Lemmas 4, 6, and 8 and inequalities (67), (70), and (71) we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} \lambda_n^2 \|u_n\|_{L^2(\omega)}^2 &\leq c \left(\|\varphi\|_{L^2(0,T_2;H)}^2 + \|\psi\|_{L^2(0,T_1;H)}^2 \right. \\
&\quad \left. + \|f\|_{L^2(\omega;V')}^2 \right), \\
\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \left\| \frac{\partial u_n}{\partial t_i} \right\|_{L^2(\omega)}^2 &\leq c \left(\|\varphi\|_{L^2(0,T_2;H)}^2 + \|\varphi'\|_{L^2(0,T_2;V_1')}^2 \right. \\
&\quad + \|\psi\|_{L^2(0,T_1;H)}^2 + \|\psi'\|_{L^2(0,T_1;V_1')}^2 + \|f\|_{L^2(\omega;V')}^2 \\
&\quad \left. + \left\| \frac{\partial f}{\partial t_1} \right\|_{L^2(\omega;V')}^2 + \left\| \frac{\partial f}{\partial t_2} \right\|_{L^2(\omega;V')}^2 \right), \quad i = 1, 2.
\end{aligned} \tag{72}$$

Consequently, from parts (b) of Lemmas 5 and 6 we get the sequence $(w_N)_{N \geq 1}$ and hence the series

$$u(t_1, t_2) = \sum_{n=1}^{\infty} u_n(t_1, t_2) v_n \tag{73}$$

converges in the space W . Since the trace operators $\text{tr}_{t_1=0}$, $\text{tr}_{t_2=0}$, and $\text{tr}_{t_2=\xi}$ are continuous, from (46) we obtain that u satisfies conditions (35), and, by (72), the estimate (44) is valid. Moreover,

$$\begin{aligned}
\text{tr}_{t_1=0} u &= \lim_{N \rightarrow \infty} \sum_{n=1}^N u_n(0, t_2) v_n \in L^2(0, T_2; H) \\
&\subset D'(0, T_2; H),
\end{aligned} \tag{74}$$

$$\begin{aligned}
\text{tr}_{t_2=0} u &= \lim_{N \rightarrow \infty} \sum_{n=1}^N u_n(t_1, 0) v_n \in L^2(0, T_1; H) \\
&\subset D'(0, T_1; H),
\end{aligned}$$

and since the operator of differentiation is continuous on the space of distributions we have

$$\begin{aligned}
\frac{d(\text{tr}_{t_1=0} u)}{dt_2} &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{d(u_n(0, t_2))}{dt_2} v_n \\
&\in D'(0, T_2; H), \\
\frac{d(\text{tr}_{t_2=0} u)}{dt_1} &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{d(u_n(t_1, 0))}{dt_1} v_n \\
&\in D'(0, T_1; H).
\end{aligned} \tag{75}$$

Applying part (a) of Lemma 8 we obtain

$$\begin{aligned} \sum_{n=1}^N \frac{1}{\lambda_n^4} \left\| \frac{d(u_n(0, t_2))}{dt_2} \right\|_{L^2(0, T_2)}^2 &= \sum_{n=1}^N \frac{\|\varphi'_n\|_{L^2(0, T_2)}^2}{\lambda_n^4} \\ &\leq \|(A_1 A)^{-1}\|^2 \|\varphi'\|_{L^2(0, T_2; V'_1)}^2, \end{aligned} \quad (76)$$

and by using parts (a) of Lemmas 4–6 and (69) we get

$$\begin{aligned} \sum_{n=1}^N \frac{1}{\lambda_n^4} \left\| \frac{d(u_n(t_1, 0))}{dt_1} \right\|_{L^2(0, T_2)}^2 &= \sum_{n=1}^N \frac{\|d'_n\|_{L^2(0, T_1)}^2}{\lambda_n^4} \\ &\leq c \left(\|\varphi_n\|_{L^2(0, T_2)}^2 + \frac{\|\varphi'_n\|_{L^2(0, T_2)}^2}{\lambda_n^4} + \frac{\|\psi'_n\|_{L^2(0, T_1)}^2}{\lambda_n^4} \right. \\ &\quad \left. + \frac{\|f_n\|_{H^1(\omega)}^2}{\lambda_n^4} \right) \leq c \left(\|\varphi\|_{L^2(0, T_2; H)}^2 + \|\varphi'\|_{L^2(0, T_2; V'_1)}^2 \right. \\ &\quad \left. + \|\psi'\|_{L^2(0, T_1; V'_1)}^2 + \|f\|_{L^2(\omega; V')} + \left\| \frac{\partial f}{\partial t_1} \right\|_{L^2(\omega; V')} \right. \\ &\quad \left. + \left\| \frac{\partial f}{\partial t_2} \right\|_{L^2(\omega; V')} \right). \end{aligned} \quad (77)$$

Consequently, from part (b) of Lemma 8 we obtain

$$\begin{aligned} \frac{d(\text{tr}_{t_1=0} u)}{dt_2} &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{d(u_n(0, t_2))}{dt_2} v_n \\ &\in L^2(0, T_2; V'_1), \\ \frac{d(\text{tr}_{t_2=0} u)}{dt_1} &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{d(u_n(t_1, 0))}{dt_1} v_n \\ &\in L^2(0, T_1; V'_1). \end{aligned} \quad (78)$$

Let us show that $u \in W$, $d(\text{tr}_{t_2=0} u)/dt_1 \in L^2(0, T_1; V'_1)$, $d(\text{tr}_{t_1=0} u)/dt_2 \in L^2(0, T_2; V'_1)$ is a solution of (34). Indeed, since the system $\{v_k\}_{k \in \mathbb{N}}$ is complete in V , it follows that for each $v \in V$ there exists a sequence $(z_k)_{k \geq 1}$, where z_k is a linear combination of $v_1, \dots, v_k$, such that $z_k \rightarrow v$ in V as $k \rightarrow \infty$. By taking account of the properties of eigenvectors $\{v_k\}_{k \in \mathbb{N}}$, from (45) for each $\psi \in D(\omega)$ and $v \in V$ we have

$$\begin{aligned} - \int_{\omega} (w_N, z_N) \frac{\partial \psi}{\partial t_1} dt_1 dt_2 - \int_{\omega} (w_N, z_N) \frac{\partial \psi}{\partial t_2} dt_1 dt_2 \\ + \int_{\omega} a(w_N, z_N) \psi dt_1 dt_2 = 0, \quad \forall N \in \mathbb{N}. \end{aligned} \quad (79)$$

By passing to the limit in the last relation, we find that u satisfies (34). So, the existence of solution is proved.

To complete the proof, we prove the uniqueness of solution. Note that the classical problem for (34) with initial conditions

$$\begin{aligned} (\text{tr}_{t_1=0} u)(t_2) &= \tilde{\varphi}(t_2), \quad 0 \leq t_2 \leq T_2, \\ (\text{tr}_{t_2=0} u)(t_1) &= \tilde{\psi}(t_1), \quad 0 \leq t_1 \leq T_1, \end{aligned} \quad (80)$$

has at most one solution. Indeed, if $\tilde{u}$ is a solution of the homogeneous classical problem (34), (80), that is, $\tilde{\varphi} \equiv 0$, $\tilde{\psi} \equiv 0$ and $f \equiv 0$, then, by multiplying (1), which is equivalent to (34), by u , integrating on $(0, T_1) \times (0, T_2)$, and applying Theorem 2 we get

$$\begin{aligned} \frac{1}{2} \int_0^{T_2} \|(\text{tr}_{t_1=T_1} \tilde{u})(t_2)\|_H^2 dt_2 \\ + \frac{1}{2} \int_0^{T_1} \|(\text{tr}_{t_2=T_2} \tilde{u})(t_1)\|_H^2 dt_1 \\ + \int_0^{T_1} \int_0^{T_2} a(\tilde{u}, \tilde{u}) dt_2 dt_1 = 0, \end{aligned} \quad (81)$$

and, consequently, $\tilde{u} \equiv 0$ in $(0, T_1) \times (0, T_2)$.

Since problem (34), (35) for $\alpha = 0$ is a classical problem, from the proof of the existence it follows that the classical problem for (34) with initial conditions (80), $\tilde{\varphi} \in L^2(0, T_2; H)$, $\tilde{\varphi}' \in L^2(0, T_2; V'_1)$, $\tilde{\psi} \in L^2(0, T_1; H)$, $\tilde{\psi}' \in L^2(0, T_1; V'_1)$, $\tilde{\varphi}(0) = \tilde{\psi}(0)$, has a unique solution $u \in W$, $\text{tr}_{t_1=0} u \in L^2(0, T_2; H)$, $d(\text{tr}_{t_1=0} u)/dt_2 \in L^2(0, T_2; V'_1)$, $\text{tr}_{t_2=0} u \in L^2(0, T_1; H)$, $d(\text{tr}_{t_2=0} u)/dt_1 \in L^2(0, T_1; V'_1)$, which can be represented in the form of convergent in W series (73), where u_n is given by (56). Therefore, the uniqueness of the solution of nonlocal problem (34), (35) follows from the uniqueness of the solution of problem (45), (46), which completes the proof. $\square$

4. Application

In this section, we consider an application of Theorem 12 obtained for abstract nonlocal problem to nonclassical problem for ultraparabolic partial differential equation with nonlocal initial condition. Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded domain with Lipschitz boundary Γ [47]. By $H^k(\Omega)$ we denote the Sobolev space of functions in $L^2(\Omega)$ whose generalized partial derivatives up to the order $k \in \mathbb{N}$ belong to $L^2(\Omega)$. We denote the closure in $H^k(\Omega)$ of the set $D(\Omega)$ by $H_0^k(\Omega)$ and its dual space by $H^{-k}(\Omega)$. We consider the second-order elliptic operator:

$$A \equiv - \sum_{q,j=1}^n \frac{\partial}{\partial x_q} \left(a_{qj}(x) \frac{\partial}{\partial x_j} \right) + a_0(x), \quad (82)$$

where $a_{qj}(x), a_0(x) \in L^\infty(\Omega)$, $q, j = 1, \dots, n$, and the following inequalities are valid

$$a_0(x) \geq 0,$$

$$\sum_{q,j=1}^n a_{qj}(x) \eta_j \eta_q \geq c_A \sum_{j=1}^n (\eta_j)^2, \quad c_A = \text{const} > 0, \quad (83)$$

for almost all $x \in \Omega$, η_j ($j = 1, \dots, n$) are arbitrary real numbers. Partial derivatives in the definition of A are treated as generalized derivatives with respect to the corresponding variables. Note that the first-order partial derivatives of a function in $L^2(\Omega)$ belongs to the space $H^{-1}(\Omega)$. Since the multiplication of a function in the space $L^2(\Omega)$ by a function in $L^\infty(\Omega)$ leaves it in $L^2(\Omega)$, for each function $v \in H_0^1(\Omega)$ we have $Av \in H^{-1}(\Omega)$. The elliptic operator A is a continuous operator from $H_0^1(\Omega)$ to the space $H^{-1}(\Omega)$, which, by virtue of (83) and Poincaré inequality, satisfies the following self-adjointness and coerciveness conditions:

$$\begin{aligned} \langle Av, w \rangle_* &= \langle Aw, v \rangle_*, \\ \langle Av, v \rangle_* &\geq \alpha \|v\|_{H^1(\Omega)}^2, \\ \forall v, w &\in H_0^1(\Omega), \end{aligned} \quad (84)$$

where $\alpha = \text{const} > 0$ and $\langle \cdot, \cdot \rangle_*$ is the duality relation between the spaces $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. We denote by $D(A) = \{v \in H_0^1(\Omega); Av \in L^2(\Omega)\}$ the domain of operator $A : D(A) \rightarrow L^2(\Omega)$. The bilinear form $a(v, w) = \langle Av, w \rangle_*$ corresponding to the operator A is of the form

$$a(v, w) = \int_{\Omega} \left(\sum_{q,j=1}^n a_{qj}(x) \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_q} + a_0(x) vw \right) dx \quad (85)$$

and satisfies conditions (11). Since the continuous embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$ is compact, it follows from general theorem for self-adjoint coercive operators mapping a Hilbert space into its dual [47] that there exists a system of eigenfunctions $\{v_k\}_{k=1}^\infty$ of the operator $A : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ corresponding to the eigenvalues $\{\lambda_k\}_{k=1}^\infty$, such that $\{v_k\}_{k=1}^\infty$ is orthonormal in $L^2(\Omega)$ and complete in $H_0^1(\Omega)$, and $0 < \lambda_1 \leq \lambda_2 \leq \dots$, $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$.

Let us consider nonclassical problem for ultraparabolic partial differential equation

$$\frac{\partial u}{\partial t_1} + \frac{\partial u}{\partial t_2} + Au = f \quad \text{in } \Omega \times (0, T_1) \times (0, T_2), \quad (86)$$

with nonlocal initial and homogeneous boundary conditions

$$\begin{aligned} u(x, 0, t_2) &= \varphi(x, t_2), \quad (x, t_2) \in \Omega \times (0, T_2), \\ u(x, t_1, 0) &= \alpha u(x, t_1, \xi) + \psi(x, t_1), \\ (x, t_1) &\in \Omega \times (0, T_1), \end{aligned} \quad (87)$$

$$u(x, t_1, t_2) = 0, \quad (x, t_1, t_2) \in \Gamma \times (0, T_1) \times (0, T_2), \quad (88)$$

where $\alpha \in \mathbf{R}$. We identify functions defined on $\Omega \times \omega$ with vector-functions defined on ω and ranging in the corresponding function spaces defined on Ω . Hence, the nonlocal in time problem (86)–(88) can be stated as the problem of finding a vector-function $u \in L^2(\omega; H_0^1(\Omega))$, $\partial u / \partial t_1, \partial u / \partial t_2 \in L^2(\omega; H^{-1}(\Omega))$, $d(\text{tr}_{t_2=0} u) / dt_1 \in L^2(0, T_1; (D(A))')$, $d(\text{tr}_{t_1=0} u) / dt_2 \in L^2(0, T_2; (D(A))')$, which satisfies (86) in

the space $D'(\omega; H^{-1}(\Omega))$ of distributions on ω ranging in $H^{-1}(\Omega)$, where the partial derivatives $\partial u / \partial t_1$ and $\partial u / \partial t_2$ are treated as the first-order generalized derivatives $\partial u / \partial t_1, \partial u / \partial t_2 \in D'(\omega; H_0^1(\Omega))$. The unknown function u satisfies conditions (87) in the spaces $L^2(0, T_2; L^2(\Omega))$ and $L^2(0, T_1; L^2(\Omega))$, respectively. The boundary conditions (88) are valid, because the trace of function from $H_0^1(\Omega)$ vanishes on the boundary of the domain Ω . Thus, nonlocal in time problem (86)–(88) is a particular case of the nonclassical problem (34), (35) for $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, and $V_1 = D(A)$. The following existence and uniqueness theorem is valid for nonlocal in time problem (86)–(88).

Theorem 13. *If $\varphi \in L^2(0, T_2; L^2(\Omega))$, $\varphi' \in L^2(0, T_2; (D(A))')$, $\psi \in L^2(0, T_1; L^2(\Omega))$, $\psi' \in L^2(0, T_1; (D(A))')$, φ and ψ satisfy compatibility condition $\langle \varphi(0), v \rangle_* = \alpha \langle \varphi(\xi), v \rangle_* + \langle \psi(0), v \rangle_*$ for all $v \in H_0^1(\Omega)$ and $f, \partial f / \partial t_1, \partial f / \partial t_2 \in L^2(\omega; H^{-1}(\Omega))$, then nonlocal problem (86)–(88) has a unique solution $u \in L^2(\omega; H_0^1(\Omega))$, $\partial u / \partial t_1, \partial u / \partial t_2 \in L^2(\omega; H^{-1}(\Omega))$, $d(\text{tr}_{t_2=0} u) / dt_1 \in L^2(0, T_1; (D(A))')$, $d(\text{tr}_{t_1=0} u) / dt_2 \in L^2(0, T_2; (D(A))')$, and the following estimate is valid:*

$$\begin{aligned} &\|u\|_{L^2(\omega; H_0^1(\Omega))}^2 + \sum_{i=1}^2 \left\| \frac{\partial u}{\partial t_i} \right\|_{L^2(\omega; H^{-1}(\Omega))}^2 \\ &\leq c \left(\|\varphi\|_{L^2(0, T_2; L^2(\Omega))}^2 + \|\varphi'\|_{L^2(0, T_2; (D(A))')}^2 \right. \\ &\quad + \|\psi\|_{L^2(0, T_1; L^2(\Omega))}^2 + \|\psi'\|_{L^2(0, T_1; (D(A))')}^2 \\ &\quad + \|f\|_{L^2(\omega; H^{-1}(\Omega))}^2 + \left\| \frac{\partial f}{\partial t_1} \right\|_{L^2(\omega; H^{-1}(\Omega))}^2 \\ &\quad \left. + \left\| \frac{\partial f}{\partial t_2} \right\|_{L^2(\omega; H^{-1}(\Omega))}^2 \right). \end{aligned} \quad (89)$$

5. Conclusions

In this paper, we investigated the existence and uniqueness of solution of nonclassical problem for ultraparabolic equation with two time variables and nonlocal initial condition, which connect values of the unknown vector-function at the initial and some later moment of time. The considered problem is a generalization of the classical problem for ultraparabolic equation and the obtained results give new well-posedness result for the classical problem, as well as for nonlocal one, in suitable function spaces with minimal regularity that is necessary to define the traces on the boundary of the time domain. Note that applying the results obtained in this paper and the presented method of investigation one can study various initial-boundary value problems for ultraparabolic equations with several time variables and more general nonlocal initial conditions. The proof of the existence of solution of the nonclassical problem is constructive and can be used to obtain approximate solutions of the nonlocal problem.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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