Research Article

Discussion on Some Recent Order-Theoretic Metrical Coincidence Theorems Involving Nonlinear Contractions

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We prove some coincidence theorems involving a pair of self-mappings f and g defined on an ordered metric space X wherein f is g -increasing φ -contractive mapping. In our results, neither the whole space X nor the range subspaces ($f(X)$ or $g(X)$) are required to be complete. Instead, we use the completeness of a subspace of X satisfying suitable conditions.

1. Introduction

The appearance of two noted papers by Goebel [1] and Jungck [2] containing two metrical coincidence theorems has attracted the attention of several researchers of this domain. The main difference between these two results lies in the fact that Goebel [1] employed the completeness of one of the range subspaces of the involved mappings while Jungck [2] assumed the completeness of entire metric space together with commutativity of the involved mappings. Specifically, Jungck's theorem ensures the uniqueness of common fixed point while Goebel's theorem is confined to mere coincidence points.

The origin of the order-theoretic metric fixed point theory can be traced back to Turinici [3, 4] which was later undertaken by Ran and Reurings [5] and Nieto and Rodríguez-López [6]. In recent years, the results of Nieto and Rodríguez-López [6] are generalized and extended by many authors (e.g., [7–20]). In order-theoretic results, the contractivity condition is weaker than the usual contractivity condition as it is merely required to be compatible with the involved partial ordering. The techniques involved in the proofs of such results are the combination of ideas used in the proof of contraction principle together with the one involved in monotone iterative technique.

Recently, Alam et al. [18] obtained order-theoretic versions of Goebel as well as Jungck coincidence theorems under Boyd-Wong type nonlinear contraction (cf. [21]).

Most recently, Alam et al. [19] weakened the relevant metrical notions, namely, completeness, continuity, g -continuity, and compatibility, by defining their order-theoretic analogues and utilized the same to enrich the results of Alam et al. [18]. In these two papers [18, 19], the authors assumed that either the whole space is complete or the range of any of the underlying mappings is complete.

The aim of this paper is to improve the coincidence theorems of Alam et al. [18, 19] without the completeness requirements of whole space (or range subspaces). Instead, we assume the completeness of a subspace satisfying suitable conditions. In this continuation, we also introduce the order-theoretic analogue of the notion of closed sets and utilize the same to derive consequences of our main results under completeness of whole space. Finally, as an application of one of our newly proved results, we indicate a result ensuring existence and uniqueness of solution of a first-order periodic boundary value problem.

2. Preliminaries

In this section, to make our exposition self-contained, we recall some basic definitions, relevant notions, and auxiliary results. Throughout this paper, $\mathbb{N}$ stands for the set of natural numbers and $\mathbb{N}_0$ for the set of whole numbers (i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$).

Definition 1 (see [22]). A set X together with a partial order $\leq$ (often denoted by $(X, \leq)$) is called an ordered set. As expected, $\geq$ denotes the dual order of $\leq$ (i.e., $x \geq y$ means $y \leq x$).

Definition 2 (see [22]). Two elements x and y of an ordered set $(X, \leq)$ are called comparable if either $x \leq y$ or $x \geq y$. For brevity, one denotes it by $x <> y$.

Clearly, the relation $<>$ is reflexive and symmetric, but not transitive in general (cf. [23]).

Definition 3 (see [22]). A subset E of an ordered set $(X, \leq)$ is called totally or linearly ordered if every pair of elements of E is comparable; that is,

$$x <> y \quad \forall x, y \in E. \quad (1)$$

Definition 4 (see [3]). A sequence $\{x_n\}$ in an ordered set $(X, \leq)$ is said to be

(i) increasing or ascending if, for any $m, n \in \mathbb{N}_0$,

$$m \leq n \implies x_m \leq x_n, \quad (2)$$

(ii) decreasing or descending if, for any $m, n \in \mathbb{N}_0$,

$$m \leq n \implies x_m \geq x_n, \quad (3)$$

(iii) monotone if it is either increasing or decreasing,

(iv) bounded above if there is an element $u \in X$ such that

$$x_n \leq u \quad \forall n \in \mathbb{N}_0 \quad (4)$$

so that u is an upper bound of $\{x_n\}$,

(v) bounded below if there is an element $l \in X$ such that

$$x_n \geq l \quad \forall n \in \mathbb{N}_0 \quad (5)$$

so that l is a lower bound of $\{x_n\}$.

Definition 5 (see [9]). Let f and g be two self-mappings on an ordered set $(X, \leq)$. One says that f is g -increasing if, for any $x, y \in X$,

$$g(x) \leq g(y) \implies f(x) \leq f(y). \quad (6)$$

Notice that, under the restriction $g = I$, the identity mapping on X , the notion of g -increasing mapping transforms to the notion of increasing mapping.

Definition 6 (see [1, 2, 24]). Let f and g be two self-mappings on a nonempty set X . Then

(i) an element $x \in X$ is called a coincidence point of f and g if

$$g(x) = f(x), \quad (7)$$

(ii) if $x \in X$ is a coincidence point of f and g and $\bar{x} \in X$ such that $\bar{x} = g(x) = f(x)$, then $\bar{x}$ is called a point of coincidence of f and g ,

(iii) if $x \in X$ is a coincidence point of f and g such that $x = g(x) = f(x)$, then x is called a common fixed point of f and g ,

(iv) f and g are called commuting if, for all $x \in X$,

$$g(fx) = f(gx), \quad (8)$$

(v) f and g are called weakly compatible (or partially commuting or coincidentally commuting) if f and g commute at their coincidence points; that is, for any $x \in X$,

$$g(x) = f(x) \implies g(fx) = f(gx). \quad (9)$$

Definition 7 (see [25, 26]). Let f and g be two self-mappings on a metric space (X, d) . One says that f and g are

(i) weakly commuting if, for all $x \in X$,

$$d(gfx, fgx) \leq d(gx, fx), \quad (10)$$

(ii) compatible if, for any sequence $\{x_n\} \subset X$ and for any $z \in X$,

$$\begin{aligned} \lim_{n \rightarrow \infty} g(x_n) = \lim_{n \rightarrow \infty} f(x_n) = z \implies \\ \lim_{n \rightarrow \infty} d(gfx_n, fgx_n) = 0. \end{aligned} \quad (11)$$

Definition 8 (see [27]). Let f and g be two self-mappings on a metric space (X, d) and $x \in X$. One says that f is g -continuous at x if, for any sequence $\{x_n\} \subset X$,

$$g(x_n) \xrightarrow{d} g(x) \implies f(x_n) \xrightarrow{d} f(x). \quad (12)$$

Moreover, f is called g -continuous if it is g -continuous at each point of X .

Notice that, particularly with $g = I$, the identity mapping on X , Definition 8 reduces to the definition of continuity.

Definition 9 (see [8]). A triplet $(X, d, \leq)$ is called an ordered metric space if (X, d) is a metric space and $(X, \leq)$ is an ordered set. Moreover, if the metric space (X, d) is complete, then $(X, d, \leq)$ is called an ordered complete metric space.

Definition 10. Let $(X, d, \leq)$ be an ordered metric space and Y a nonempty subset of X . Then d and $\leq$, respectively, induce a metric d_Y and a partial order $\leq_Y$ on Y so that

$$\begin{aligned} d_Y(x, y) &= d(x, y) \quad \forall x, y \in Y, \\ x \leq_Y y &\iff x \leq y \quad \forall x, y \in Y. \end{aligned} \quad (13)$$

Thus $(Y, d_Y, \leq_Y)$ is an ordered metric space, which is called a subspace of $(X, d, \leq)$.

Conventionally, we opt to refer to Y as a subspace of X rather than saying $(Y, d_Y, \leq_Y)$ is a subspace $(X, d, \leq)$ and continue to write d and $\leq$ instead of d_Y and $\leq_Y$, respectively.

Let $(X, d, \leq)$ be an ordered metric space and $\{x_n\}$ a sequence in X . We adopt the following notations:

- (i) If $\{x_n\}$ is increasing and $x_n \xrightarrow{d} x$, then we denote it symbolically by $x_n \uparrow x$.
- (ii) If $\{x_n\}$ is decreasing and $x_n \xrightarrow{d} x$, then we denote it symbolically by $x_n \downarrow x$.
- (iii) If $\{x_n\}$ is monotone and $x_n \xrightarrow{d} x$, then we denote it symbolically by $x_n \updownarrow x$.

Definition 11 (see [19]). An ordered metric space $(X, d, \leq)$ is called

- (i) $\overline{O}$ -complete if every increasing Cauchy sequence in X converges,
- (ii) $\underline{O}$ -complete if every decreasing Cauchy sequence in X converges,
- (iii) O -complete if every monotone Cauchy sequence in X converges.

Remark 12 (see [19]). In an ordered metric space, completeness $\Rightarrow O$ -completeness $\Rightarrow \overline{O}$ -completeness as well as $\underline{O}$ -completeness.

Definition 13. Let $(X, d, \leq)$ be an ordered metric space. A subset E of X is called

- (i) $\overline{O}$ -closed if, for any sequence $\{x_n\} \subset E$,

$$x_n \uparrow x \implies x \in E, \quad (14)$$

- (ii) $\underline{O}$ -closed if, for any sequence $\{x_n\} \subset E$,

$$x_n \downarrow x \implies x \in E, \quad (15)$$

- (iii) O -closed if, for any sequence $\{x_n\} \subset E$,

$$x_n \updownarrow x \implies x \in E. \quad (16)$$

Remark 14. In an ordered metric space, closedness $\Rightarrow O$ -closedness $\Rightarrow \overline{O}$ -closedness as well as $\underline{O}$ -closedness.

Proposition 15. (i) An $\overline{O}$ -complete subspace of an ordered metric space is $\overline{O}$ -closed. (ii) An $\underline{O}$ -complete subspace of an ordered metric space is $\underline{O}$ -closed. (iii) An O -complete subspace of an ordered metric space is O -closed.

Proof. We prove here only part (i). Proceeding on similar lines, one can prove part (ii) as well as part (iii). Let $(X, d, \leq)$ be an ordered metric space. Suppose that Y is an $\overline{O}$ -complete subspace of X . Take a sequence $\{x_n\} \subset Y$ such that $x_n \uparrow x \in X$. As each convergent sequence is Cauchy, $\{x_n\}$ is an increasing Cauchy sequence in Y . Hence, $\overline{O}$ -completeness of Y implies that the limit of $\{x_n\}$ must lie in Y ; that is, $x \in Y$. Therefore, Y is $\overline{O}$ -closed. $\square$

Proposition 16. (i) An $\overline{O}$ -closed subspace of an $\overline{O}$ -complete ordered metric space is $\overline{O}$ -complete. (ii) An $\underline{O}$ -closed subspace of an $\underline{O}$ -complete ordered metric space is $\underline{O}$ -complete. (iii) An O -closed subspace of an O -complete ordered metric space is O -complete.

Proof. We prove here only part (i). Proceeding on similar lines, one can prove part (ii) as well as part (iii). Let $(X, d, \leq)$ be an $\overline{O}$ -complete ordered metric space. Suppose that Y is an $\overline{O}$ -closed subspace of X . Take an increasing Cauchy sequence $\{x_n\}$ in Y . As X is $\overline{O}$ -complete, $\exists x \in X$ such that $x_n \xrightarrow{d} x$ and so $x_n \uparrow x$. Hence, $\overline{O}$ -closedness of Y implies that $x \in Y$. Therefore, Y is $\overline{O}$ -complete. $\square$

Definition 17 (see [19]). Let $(X, d, \leq)$ be an ordered metric space, $f : X \rightarrow X$ a mapping, and $x \in X$. Then, f is called

- (i) $\overline{O}$ -continuous at x if, for any sequence $\{x_n\} \subset X$,

$$x_n \uparrow x \implies f(x_n) \xrightarrow{d} f(x), \quad (17)$$

- (ii) $\underline{O}$ -continuous at x if, for any sequence $\{x_n\} \subset X$,

$$x_n \downarrow x \implies f(x_n) \xrightarrow{d} f(x), \quad (18)$$

- (iii) O -continuous at x if, for any sequence $\{x_n\} \subset X$,

$$x_n \updownarrow x \implies f(x_n) \xrightarrow{d} f(x). \quad (19)$$

Moreover, f is called O -continuous (resp., $\overline{O}$ -continuous, $\underline{O}$ -continuous) if it is O -continuous (resp., $\overline{O}$ -continuous, $\underline{O}$ -continuous) at each point of X .

Remark 18 (see [19]). In an ordered metric space, continuity $\Rightarrow O$ -continuity $\Rightarrow \overline{O}$ -continuity as well as $\underline{O}$ -continuity.

Definition 19 (see [19]). Let $(X, d, \leq)$ be an ordered metric space, f and g two self-mappings on X , and $x \in X$. Then, f is called

- (i) $(g, \overline{O})$ -continuous at x if, for any sequence $\{x_n\} \subset X$,

$$g(x_n) \uparrow g(x) \implies f(x_n) \xrightarrow{d} f(x), \quad (20)$$

- (ii) $(g, \underline{O})$ -continuous at x if, for any sequence $\{x_n\} \subset X$,

$$g(x_n) \downarrow g(x) \implies f(x_n) \xrightarrow{d} f(x), \quad (21)$$

- (iii) (g, O) -continuous at x if, for any sequence $\{x_n\} \subset X$,

$$g(x_n) \updownarrow g(x) \implies f(x_n) \xrightarrow{d} f(x). \quad (22)$$

Moreover, f is called (g, O) -continuous (resp., $(g, \overline{O})$ -continuous, $(g, \underline{O})$ -continuous) if it is (g, O) -continuous (resp., $(g, \overline{O})$ -continuous, $(g, \underline{O})$ -continuous) at each point of X .

Notice that, on setting $g = I$ (the identity mapping on X), Definition 19 reduces to Definition 17.

Remark 20 (see [19]). In an ordered metric space, g -continuity $\Rightarrow (g, O)$ -continuity $\Rightarrow (g, \overline{O})$ -continuity as well as $(g, \underline{O})$ -continuity.

Definition 21 (see [19]). Let $(X, d, \leq)$ be an ordered metric space and f and g two self-mappings on X . One says that f and g are

- (i) $\overline{O}$ -compatible if, for any sequence $\{x_n\} \subset X$ and for any $z \in X$,

$$\begin{aligned} g(x_n) \uparrow z, f(x_n) \uparrow z \implies \\ \lim_{n \rightarrow \infty} d(gfx_n, fgx_n) = 0, \end{aligned} \quad (23)$$

- (ii) $\underline{O}$ -compatible if, for any sequence $\{x_n\} \subset X$ and for any $z \in X$,

$$\begin{aligned} g(x_n) \downarrow z, f(x_n) \downarrow z \implies \\ \lim_{n \rightarrow \infty} d(gfx_n, fgx_n) = 0, \end{aligned} \quad (24)$$

- (iii) O -compatible if, for any sequence $\{x_n\} \subset X$ and for any $z \in X$,

$$\begin{aligned} g(x_n) \uparrow \downarrow z, f(x_n) \uparrow \downarrow z \implies \\ \lim_{n \rightarrow \infty} d(gfx_n, fgx_n) = 0. \end{aligned} \quad (25)$$

Here, it can be pointed out that the notion of O -compatibility is slightly weaker than the notion of O -compatibility defined by Luong and Thuan [28]. Luong and Thuan [28] assumed that only the sequence $\{gx_n\}$ is monotone but we assume that both $\{gx_n\}$ and $\{fx_n\}$ are monotone.

Remark 22 (see [19]). In an ordered metric space, commutativity $\Rightarrow$ weak commutativity $\Rightarrow$ compatibility $\Rightarrow O$ -compatibility $\Rightarrow \overline{O}$ -compatibility as well as $\underline{O}$ -compatibility $\Rightarrow$ weak compatibility.

In the following lines, we formulate some definitions using certain properties utilized by earlier authors especially from [6, 9, 29, 30] besides some other ones.

Definition 23 (see [18]). Let $(X, d, \leq)$ be an ordered metric space. One says that

- (i) $(X, d, \leq)$ has ICU (increasing-convergence-upper bound) property if every increasing convergent sequence $\{x_n\}$ in X is bounded above by its limit (as an upper bound); that is,

$$x_n \uparrow x \implies x_n \leq x \quad \forall n \in \mathbb{N}_0, \quad (26)$$

- (ii) $(X, d, \leq)$ has DCL (decreasing-convergence-lower bound) property if every decreasing convergent sequence $\{x_n\}$ in X is bounded below by its limit (as a lower bound); that is,

$$x_n \downarrow x \implies x_n \geq x \quad \forall n \in \mathbb{N}_0, \quad (27)$$

- (iii) $(X, d, \leq)$ has MCB (monotone-convergence-boundedness) property if it has both ICU and DCL property.

Definition 24 (see [18]). Let $(X, d, \leq)$ be an ordered metric space and g a self-mapping on X . One says that

- (i) $(X, d, \leq)$ has g -ICU property if g -image of every increasing convergent sequence $\{x_n\}$ in X is bounded above by g -image of its limit (as an upper bound); that is,

$$x_n \uparrow x \implies g(x_n) \leq g(x) \quad \forall n \in \mathbb{N}_0, \quad (28)$$

- (ii) $(X, d, \leq)$ has g -DCL property if g -image of every decreasing convergent sequence $\{x_n\}$ in X is bounded below by g -image of its limit (as a lower bound); that is,

$$x_n \downarrow x \implies g(x_n) \geq g(x) \quad \forall n \in \mathbb{N}_0, \quad (29)$$

- (iii) $(X, d, \leq)$ has g -MCB property if it has both g -ICU and g -DCL property.

Notice that, under the restriction $g = I$, the identity mapping on X , Definition 24 reduces to Definition 23.

Now, we recall the following notions which are relatively weaker than their corresponding earlier definitions.

Definition 25 (see [19]). Let $(X, d, \leq)$ be an ordered metric space. One says that

- (i) $(X, d, \leq)$ has ICC (increasing-convergence-comparable) property if every increasing convergent sequence $\{x_n\}$ in X has a subsequence $\{x_{n_k}\}$ such that every term of $\{x_{n_k}\}$ is comparable with the limit of $\{x_n\}$; that is,

$$\begin{aligned} x_n \uparrow x \implies \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \\ \text{with } x_{n_k} <> x \quad \forall k \in \mathbb{N}_0, \end{aligned} \quad (30)$$

- (ii) $(X, d, \leq)$ has DCC (decreasing-convergence-comparable) property if every decreasing convergent sequence $\{x_n\}$ in X has a subsequence $\{x_{n_k}\}$ such that every term of $\{x_{n_k}\}$ is comparable with the limit of $\{x_n\}$; that is,

$$\begin{aligned} x_n \downarrow x \implies \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \\ \text{with } x_{n_k} <> x \quad \forall k \in \mathbb{N}_0, \end{aligned} \quad (31)$$

- (iii) $(X, d, \leq)$ has MCC (monotone-convergence-comparable) property if every monotone convergent sequence $\{x_n\}$ in X has a subsequence $\{x_{n_k}\}$ such that every term of $\{x_{n_k}\}$ is comparable with the limit of $\{x_n\}$; that is,

$$\begin{aligned} x_n \uparrow \downarrow x \implies \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \\ \text{with } x_{n_k} <> x \quad \forall k \in \mathbb{N}_0. \end{aligned} \quad (32)$$

Remark 26 (see [19]). In an ordered metric space, we have the following:

ICU property $\Rightarrow$ ICC property.

DCL property $\Rightarrow$ DCC property.

MCB property $\Rightarrow$ MCC property $\Rightarrow$ ICC property as well as DCC property.

Definition 27 (see [19]). Let $(X, d, \leq)$ be an ordered metric space and g a self-mapping on X . One says that

- (i) $(X, d, \leq)$ has g -ICC property if every increasing convergent sequence $\{x_n\}$ in X has a subsequence $\{x_{n_k}\}$ such that every term of $\{gx_{n_k}\}$ is comparable with g -image of the limit of $\{x_n\}$; that is,

$$x_n \uparrow x \implies \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \quad (33)$$

$$\text{with } g(x_{n_k}) <> g(x) \quad \forall k \in \mathbb{N}_0,$$

- (ii) $(X, d, \leq)$ has g -DCC property if each decreasing convergent sequence $\{x_n\}$ in X has a subsequence $\{x_{n_k}\}$ such that every term of $\{gx_{n_k}\}$ is comparable with g -image of the limit of $\{x_n\}$; that is,

$$x_n \downarrow x \implies \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \quad (34)$$

$$\text{with } g(x_{n_k}) <> g(x) \quad \forall k \in \mathbb{N}_0,$$

- (iii) $(X, d, \leq)$ has g -MCC property if each monotone convergent sequence $\{x_n\}$ in X has a subsequence $\{x_{n_k}\}$ such that every term of $\{gx_{n_k}\}$ is comparable with g -image of the limit of $\{x_n\}$; that is,

$$x_n \uparrow \downarrow x \implies \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \quad (35)$$

$$\text{with } g(x_{n_k}) <> g(x) \quad \forall k \in \mathbb{N}_0.$$

Observe that under the restriction $g = I$, the identity mapping on X , Definition 27 reduces to Definition 25.

Remark 28 (see [19]). In an ordered metric space, we have the following:

g -ICU property $\Rightarrow g$ -ICC property.

g -DCL property $\Rightarrow g$ -DCC property.

g -MCB property $\Rightarrow g$ -MCC property $\Rightarrow g$ -ICC property as well as g -DCC property.

Definition 29 (see [21, 31]). One denotes by Ω the family of functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ satisfying

- (a) $\varphi(t) < t$ for each $t > 0$,
 (b) $\limsup_{r \rightarrow t^+} \varphi(r) < t$ for each $t > 0$.

Finally, we record the following results in the form of lemmas needed in our results.

Lemma 30 (see [18]). Let $\varphi \in \Omega$. If $\{a_n\} \subset (0, \infty)$ is a sequence such that $a_{n+1} \leq \varphi(a_n) \quad \forall n \in \mathbb{N}_0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 31 (see [14, 32, 33]). Let (X, d) be a metric space and $\{x_n\}$ a sequence in X . If $\{x_n\}$ is not Cauchy, then there exist $\epsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that

- (i) $k \leq m_k < n_k \quad \forall k \in \mathbb{N}$,
 (ii) $d(x_{m_k}, x_{n_k}) \geq \epsilon$,
 (iii) $d(x_{m_k}, x_{p_k}) < \epsilon \quad \forall p_k \in \{m_k + 1, m_k + 2, \dots, n_k - 2, n_k - 1\}$.

Moreover, suppose that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$; then,

- (iv) $\lim_{k \rightarrow \infty} d(x_{m_k}, x_{n_k}) = \epsilon$,
 (v) $\lim_{k \rightarrow \infty} d(x_{m_k+1}, x_{n_k+1}) = \epsilon$.

Lemma 32 (see [18]). Let f and g be two self-mappings defined on an ordered set $(X, \leq)$. If f is g -increasing and $g(x) = g(y)$, then $f(x) = f(y)$.

Lemma 33 (see [34]). Let X be a nonempty set and g a self-mapping on X . Then, there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \rightarrow X$ is one-one.

3. Main Results

Now, we prove one of our main results on coincidence points which runs as follows.

Theorem 34. Let $(X, d, \leq)$ be an ordered metric space and Y an $\overline{O}$ -complete subspace of X . Let f and g be two self-mappings on X . Suppose that the following conditions hold:

- (a) $f(X) \subseteq g(X) \cap Y$.
 (b) f is g -increasing.
 (c) There exists $x_0 \in X$ such that $g(x_0) \leq f(x_0)$.
 (d) There exists $\varphi \in \Omega$ such that

$$d(fx, fy) \leq \varphi(d(gx, gy)) \quad (36)$$

$$\forall x, y \in X \text{ with } g(x) <> g(y).$$

- (e) (e1) f and g are $\overline{O}$ -compatible.

- (e2) g is $\overline{O}$ -continuous.

- (e3) Either f is $\overline{O}$ -continuous or $(Y, d, \leq)$ has g -ICC property.

Or alternately

- (e') (e'1) $Y \subseteq g(X)$.

- (e'2) Either f is $(g, \overline{O})$ -continuous or f and g are continuous or $(Y, d, \leq)$ has ICC property.

Then, f and g have a coincidence point.

Proof. Firstly, we notice that assumption (a) is equivalent to saying that $f(X) \subseteq g(X)$ and $f(X) \subseteq Y$. Now, in view of assumption (c), if $g(x_0) = f(x_0)$, then x_0 is a coincidence point of f and g and hence we are through. Otherwise, using assumption $f(X) \subseteq g(X)$, we construct a sequence $\{x_n\} \subset X$ of joint iteration of f and g based at point x_0 ; that is,

$$g(x_{n+1}) = f(x_n) \quad \forall n \in \mathbb{N}_0. \quad (37)$$

Now, we assert that $\{gx_n\}$ is an increasing sequence; that is,

$$g(x_n) \leq g(x_{n+1}) \quad \forall n \in \mathbb{N}_0. \quad (38)$$

We prove this fact by mathematical induction. On using (37) with $n = 0$ and assumption (c), we have

$$g(x_0) \leq f(x_0) = g(x_1) \quad (39)$$

so that (38) holds for $n = 0$. Suppose that (38) holds for $n = r > 0$; that is,

$$g(x_r) \leq g(x_{r+1}), \quad (40)$$

which on using (37) and assumption (b) gives rise to

$$g(x_{r+1}) = f(x_r) \leq f(x_{r+1}) = g(x_{r+2}); \quad (41)$$

that is, (38) holds for $n = r + 1$. Hence, by induction, (38) holds for all $n \in \mathbb{N}_0$.

In view of (37) and (38), the sequence $\{fx_n\}$ is also an increasing sequence; that is,

$$f(x_n) \leq f(x_{n+1}) \quad \forall n \in \mathbb{N}_0. \quad (42)$$

If $g(x_{n_0}) = g(x_{n_0+1})$ for some $n_0 \in \mathbb{N}$, then, using (37), we have $g(x_{n_0}) = f(x_{n_0})$; that is, x_{n_0} is a coincidence point of f and g so that we are through. On the other hand, if $g(x_n) \neq g(x_{n+1})$ for each $n \in \mathbb{N}_0$, we can define a sequence $\{d_n\}_{n=0}^\infty \subset (0, \infty)$, where

$$d_n := d(gx_n, gx_{n+1}). \quad (43)$$

On using (37), (38), (43), and assumption (d), we obtain

$$\begin{aligned} d_{n+1} &= d(gx_{n+1}, gx_{n+2}) = d(fx_n, fx_{n+1}) \\ &\leq \varphi(d(gx_n, gx_{n+1})) = \varphi(d_n) \end{aligned} \quad (44)$$

so that

$$d_{n+1} \leq \varphi(d_n). \quad (45)$$

Hence, by Lemma 30, we obtain

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} d(gx_n, gx_{n+1}) = 0. \quad (46)$$

Next, we show that $\{gx_n\}$ is a Cauchy sequence. On the contrary, suppose that $\{gx_n\}$ is not Cauchy; then, owing to Lemma 31, there exist $\epsilon > 0$ and two subsequences $\{gx_{n_k}\}$ and $\{gx_{m_k}\}$ of $\{gx_n\}$ such that $k \leq m_k < n_k$, $d(gx_{m_k}, gx_{n_k}) \geq \epsilon$, and $d(gx_{m_k}, gx_{p_k}) < \epsilon$, where $p_k \in \{m_k + 1, m_k + 2, \dots, n_k - 2, n_k - 1\}$. Further, in view of (46), Lemma 31 assures us that

$$\lim_{k \rightarrow \infty} d(gx_{m_k}, gx_{n_k}) = \lim_{k \rightarrow \infty} d(gx_{m_k+1}, gx_{n_k+1}) = \epsilon. \quad (47)$$

Denote $r_k := d(gx_{m_k}, gx_{n_k})$. As $m_k < n_k$, due to (38), we have $g(x_{m_k}) \leq g(x_{n_k})$. On using (37) and assumption (c), we obtain

$$\begin{aligned} d(gx_{m_k+1}, gx_{n_k+1}) &= d(fx_{m_k}, fx_{n_k}) \\ &\leq \varphi(d(gx_{m_k}, gx_{n_k})) = \varphi(r_k) \end{aligned} \quad (48)$$

so that

$$d(gx_{m_k+1}, gx_{n_k+1}) \leq \varphi(r_k). \quad (49)$$

On taking limit superior as $k \rightarrow \infty$ in (49) and using (47) and the definition of Ω , we have

$$\begin{aligned} \epsilon &= \limsup_{k \rightarrow \infty} d(gx_{m_k+1}, gx_{n_k+1}) \leq \limsup_{k \rightarrow \infty} \varphi(r_k) \\ &= \limsup_{r_k \rightarrow \epsilon^+} \varphi(r_k) < \epsilon, \end{aligned} \quad (50)$$

which is a contradiction so that the sequence $\{gx_n\}$ is Cauchy.

Owing to (37), $\{gx_n\} \subset f(X) \subseteq Y$ so that $\{gx_n\}$ is an increasing Cauchy sequence in Y . As Y is $\overline{O}$ -complete, there exists $z \in Y$ such that $\lim_{n \rightarrow \infty} g(x_n) = z$, which together with (38) gives rise to

$$g(x_n) \uparrow z. \quad (51)$$

On using (37), (42), and (51), we obtain

$$f(x_n) \uparrow z. \quad (52)$$

Now, using assumptions (e) and (e'), we accomplish the rest of the proof. Assume that (e) holds. Using assumption (e2) (i.e., $\overline{O}$ -continuity of g) in (51) and (52), we have

$$\lim_{n \rightarrow \infty} g(gx_n) = g(z), \quad (53)$$

$$\lim_{n \rightarrow \infty} g(fx_n) = g(z). \quad (54)$$

On using (51), (52), and assumption (e1) (i.e., $\overline{O}$ -compatibility of f and g), we obtain

$$\lim_{n \rightarrow \infty} d(gfx_n, fgx_n) = 0. \quad (55)$$

Now, we show that z is a coincidence point of f and g . To accomplish this, we use assumption (e3). Suppose that f is $\overline{O}$ -continuous. On using (51) and $\overline{O}$ -continuity of f , we obtain

$$\lim_{n \rightarrow \infty} f(gx_n) = f(z). \quad (56)$$

On using (54), (55), (56), and continuity of d , we obtain

$$\begin{aligned} d(gz, fz) &= d\left(\lim_{n \rightarrow \infty} gfx_n, \lim_{n \rightarrow \infty} fgx_n\right) \\ &= \lim_{n \rightarrow \infty} d(gfx_n, fgx_n) = 0 \end{aligned} \quad (57)$$

so that

$$g(z) = f(z). \quad (58)$$

Thus, $z \in X$ is a coincidence point of f and g and hence we are through. Alternately, suppose that $(Y, d, \leq)$ has g -ICC property; then, due to availability of (51), there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that

$$g(gx_{n_k}) <_> g(z) \quad \forall k \in \mathbb{N}_0. \quad (59)$$

On using (59) and assumption (d), we obtain

$$d(fgx_{n_k}, fz) \leq \varphi(d(ggx_{n_k}, gz)) \quad \forall k \in \mathbb{N}_0. \quad (60)$$

Now, we assert that

$$d(fgx_{n_k}, fz) \leq d(ggx_{n_k}, gz) \quad \forall k \in \mathbb{N}. \quad (61)$$

On account of two different possibilities arising here, we consider a partition $\{\mathbb{N}^0, \mathbb{N}^+\}$ of $\mathbb{N}$; that is, $\mathbb{N}^0 \cup \mathbb{N}^+ = \mathbb{N}$ and $\mathbb{N}^0 \cap \mathbb{N}^+ = \emptyset$ verifying that

$$(i) \quad d(ggx_{n_k}, gz) = 0 \quad \forall k \in \mathbb{N}^0,$$

$$(ii) \quad d(ggx_{n_k}, gz) > 0 \quad \forall k \in \mathbb{N}^+.$$

In case (i), on using Lemma 32, we get $d(fgx_{n_k}, fz) = 0 \quad \forall k \in \mathbb{N}^0$ and hence (61) holds for all $k \in \mathbb{N}^0$. In case (ii), owing to the definition of Ω , we have $d(fgx_{n_k}, fz) \leq \varphi(d(ggx_{n_k}, gz)) < d(ggx_{n_k}, gz) \quad \forall k \in \mathbb{N}^+$ and hence (61) holds for all $k \in \mathbb{N}^+$. Thus, (61) holds for all $k \in \mathbb{N}$.

On using triangular inequality, (53), (54), (55), and (61), we get

$$\begin{aligned} d(gz, fz) &\leq d(gz, gfx_{n_k}) + d(gfx_{n_k}, fgx_{n_k}) \\ &\quad + d(fgx_{n_k}, fz) \\ &\leq d(gz, gfx_{n_k}) + d(gfx_{n_k}, fgx_{n_k}) \\ &\quad + d(ggx_{n_k}, gz) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty \end{aligned} \quad (62)$$

so that

$$g(z) = f(z). \quad (63)$$

Thus, $z \in X$ is a coincidence point of f and g .

Now, assume that (e') holds. Owing to assumption (e'1) (i.e., $Y \subseteq g(X)$), we can find some $u \in X$ such that $z = g(u)$. Hence, (51) and (52), respectively, reduce to

$$g(x_n) \uparrow g(u), \quad (64)$$

$$f(x_n) \uparrow g(u). \quad (65)$$

Now, we show that u is a coincidence point of f and g . To accomplish this, we use assumption (e'2). Firstly, suppose that f is $(g, \overline{O})$ -continuous; then using (64), we get

$$\lim_{n \rightarrow \infty} f(x_n) = f(u). \quad (66)$$

On using (65) and (66), we get

$$g(u) = f(u). \quad (67)$$

Hence, we are done. Secondly, suppose that f and g are continuous. Owing to Lemma 33, there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \rightarrow X$ is one-one. Without loss of generality, we are able to choose $E \subseteq X$ such that $u \in E$. Now, define $T : g(E) \rightarrow g(X)$ by

$$T(ge) = f(e) \quad \forall g(e) \in g(E) \text{ where } e \in E. \quad (68)$$

As $g : E \rightarrow X$ is one-one and $f(X) \subseteq g(X)$, T is well defined. As f and g are continuous, it follows that T is continuous. Since $\{x_n\} \subset X$ and $g(X) = g(E)$, there exists $\{e_n\} \subset E$ such that $g(x_n) = g(e_n) \quad \forall n \in \mathbb{N}_0$. By using Lemma 33, we get $f(x_n) = f(e_n) \quad \forall n \in \mathbb{N}_0$. On using (64) and (65), we get

$$\lim_{n \rightarrow \infty} g(e_n) = \lim_{n \rightarrow \infty} f(e_n) = g(u). \quad (69)$$

Making use of (68), (69), and continuity of T , we get

$$\begin{aligned} f(u) &= T(gu) = T\left(\lim_{n \rightarrow \infty} ge_n\right) = \lim_{n \rightarrow \infty} T(ge_n) \\ &= \lim_{n \rightarrow \infty} f(e_n) = g(u). \end{aligned} \quad (70)$$

Thus, $u \in X$ is a coincidence point of f and g and hence we are through. Finally, suppose that $(Y, d, \leq)$ has the ICC property. Then, due to availability of (64), there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that

$$g(x_{n_k}) <_> g(u) \quad \forall k \in \mathbb{N}_0. \quad (71)$$

On using (37), (71), and assumption (d), we obtain

$$\begin{aligned} d(gx_{n_k+1}, fu) &= d(fx_{n_k}, fu) \leq \varphi(d(gx_{n_k}, gu)) \\ &\quad \forall k \in \mathbb{N}_0. \end{aligned} \quad (72)$$

We assert that

$$d(gx_{n_k+1}, fu) \leq d(gx_{n_k}, gu) \quad \forall k \in \mathbb{N}. \quad (73)$$

On account of two different possibilities arising here, we consider a partition $\{\mathbb{N}^0, \mathbb{N}^+\}$ of $\mathbb{N}$; that is, $\mathbb{N}^0 \cup \mathbb{N}^+ = \mathbb{N}$ and $\mathbb{N}^0 \cap \mathbb{N}^+ = \emptyset$ verifying that

$$(i) \quad d(gx_{n_k}, gu) = 0 \quad \forall k \in \mathbb{N}^0,$$

$$(ii) \quad d(gx_{n_k}, gu) > 0 \quad \forall k \in \mathbb{N}^+.$$

In case (i), on using Lemma 33, we get $d(fx_{n_k}, fu) = 0 \quad \forall k \in \mathbb{N}^0$, which, in view of (37), gives rise to $d(gx_{n_k+1}, fu) = 0 \quad \forall k \in \mathbb{N}^0$ and hence (73) holds for all $k \in \mathbb{N}^0$. In case (ii), by the definition of Ω , we have $d(gx_{n_k+1}, fu) \leq \varphi(d(gx_{n_k}, gu)) < d(gx_{n_k}, gu) \quad \forall k \in \mathbb{N}^+$ and hence (73) holds for all $n \in \mathbb{N}^+$. Thus, (73) holds for all $k \in \mathbb{N}$.

On using (64), (73), and continuity of d , we get

$$\begin{aligned} d(gu, fu) &= d\left(\lim_{n \rightarrow \infty} gx_{n+1}, fu\right) \\ &= \lim_{k \rightarrow \infty} d(gx_{n_k+1}, fu) \leq \lim_{k \rightarrow \infty} d(gx_{n_k}, gu) \\ &= 0 \end{aligned} \quad (74)$$

so that

$$g(u) = f(u). \quad (75)$$

Hence, u is a coincidence point of f and g . This completes the proof. $\square$

Now, as a consequence, we particularize Theorem 34 by assuming the $\overline{O}$ -completeness of whole space X .

Corollary 35. *Let $(X, d, \leq)$ be an $\overline{O}$ -complete ordered metric space. Let f and g be two self-mappings on X . Suppose that the following conditions hold:*

- (a) $f(X) \subseteq g(X)$.
- (b) f is g -increasing.
- (c) There exists $x_0 \in X$ such that $g(x_0) \leq f(x_0)$.
- (d) There exists $\varphi \in \Omega$ such that

$$d(fx, fy) \leq \varphi(d(gx, gy)) \quad (76)$$

$$\forall x, y \in X \text{ with } g(x) <_> g(y).$$

- (e) (e1) f and g are $\overline{O}$ -compatible.
- (e2) g is $\overline{O}$ -continuous.
- (e3) Either f is $\overline{O}$ -continuous or $(X, d, \leq)$ has g -ICC property.

Or alternately

- (e') (e'1) there exists an $\overline{O}$ -closed subspace Y of X such that $f(X) \subseteq Y \subseteq g(X)$.
- (e'2) Either f is $(g, \overline{O})$ -continuous or f and g are continuous or $(Y, d, \leq)$ has ICC property.

Then, f and g have a coincidence point.

Proof. The result corresponding to part (e) follows easily on setting $Y = X$ in Theorem 34 while the same (result) in the presence of part (e') follows using Proposition 16. $\square$

Remark 36. If g is onto in Corollary 35, then we can drop assumption (a) as in this case it trivially holds. Also, we can remove assumption (e'1) as it trivially holds for $Y = g(X) = X$ using Proposition 15. Whenever f is onto, owing to assumption (a), g must be onto and hence earlier conclusion is immediate.

Next, we present a dual result corresponding to Theorem 37.

Theorem 37. *Theorem 34 (similarly Corollary 35) remains true if certain involved terms, namely, $\overline{O}$ -complete, $\overline{O}$ -closed, $\overline{O}$ -compatible, $\overline{O}$ -continuous, $(g, \overline{O})$ -continuous, ICC property, and g -ICC property, are, respectively, replaced*

by $\underline{O}$ -complete, $\underline{O}$ -closed, $\underline{O}$ -compatible, $\underline{O}$ -continuous, $(g, \underline{O})$ -continuous, DCC property, and g -DCC property provided assumption (c) is replaced by the following condition:

- (c)' there exists $x_0 \in X$ such that $g(x_0) \geq f(x_0)$.

Proof. The scheme of the proof is similar to the one adopted in the proof of Theorem 34 (or Corollary 35). In this case, instead of (38) and (42), we, respectively, infer that the sequences $\{gx_n\}$ and $\{fx_n\}$ are decreasing so that we have to use $\underline{O}$ -analogues of completeness, closedness, compatibility, continuity, and g -continuity besides DCC property and g -DCC property instead of ICC property and g -ICC property. Therefore, proceeding on the lines of the proof of Theorem 34 (or Corollary 35), the rest of the proof of this theorem can be completed. $\square$

Now, combining Theorems 34 and 37 and making use of Remarks 12, 14, 18, 20, 22, 26, and 28, we obtain the following result.

Theorem 38. *Theorem 34 (similarly Corollary 35) remains true if certain involved terms, namely, $\overline{O}$ -complete, $\overline{O}$ -closed, $\overline{O}$ -compatible, $\overline{O}$ -continuous, $(g, \overline{O})$ -continuous, ICC property, and g -ICC property, are, respectively, replaced by $\underline{O}$ -complete, $\underline{O}$ -closed, $\underline{O}$ -compatible, $\underline{O}$ -continuous, $(g, \underline{O})$ -continuous, MCC property, and g -MCC property provided assumption (c) is replaced by the following condition:*

- (c)'' there exists $x_0 \in X$ such that $g(x_0) <_> f(x_0)$.

On setting $\varphi(t) = \alpha t$ with $\alpha \in [0, 1)$, in the previous results, we get the corresponding results for linear contractions in the form of the following consequence.

Corollary 39. *Theorem 34 (similarly, Theorems 37 and 38) remains true if one replaces condition (d) by the following condition:*

- (d)' there exists $\alpha \in [0, 1)$ such that

$$d(fx, fy) \leq \alpha d(gx, gy) \quad (77)$$

$$\forall x, y \in X \text{ with } g(x) <_> g(y).$$

The following consequences of our main results are apparent in view of Remarks 26 and 28.

Corollary 40. *Theorem 34 remains true if the term ICC property (resp., g -ICC property) is replaced by ICU property (resp., g -ICU property).*

Corollary 41. *Theorem 37 remains true if the term DCC property (resp., g -DCC property) is replaced by DCL property (resp., g -DCL property).*

Corollary 42. *Theorem 38 remains true if the term MCC property (resp., g -MCC property) is replaced by MCB property (resp., g -MCB property).*

On using Remarks 12, 14, 18, 20, and 22, we obtain the more natural versions of our main results in the form of the following consequence.

Corollary 43. *Theorem 34 (resp., Theorems 37 and 38) remains true if the usual metrical notions, namely, completeness, closedness, compatibility (or commutativity/weak commutativity), continuity, and g -continuity, are used instead of their respective $\overline{O}$ -analogues (resp., $\underline{O}$ -analogues and O -analogues).*

4. Uniqueness Results

In this section, we prove certain results ensuring the uniqueness of a point of coincidence, coincidence point, and common fixed point corresponding to earlier established results.

Definition 44 (see [19]). Let $(X, \preceq)$ be an ordered set and f and g two self-mappings on X . One says that $(X, \preceq)$ is (f, g) -directed if for each pair $x, y \in X$, $\exists z \in X$, such that $f(x) \prec g(z)$ and $f(y) \prec g(z)$. In case $g = I$ (where I denotes identity mapping on X), $(X, \preceq)$ is called f -directed.

In particular, for $f = g = I$, Definition 44 coincides with the usual notion of directed set suggested by Jleli et al. [14]. In other words, we use the term directed set instead of (I, I) -directed set.

The following notion is inspired by Turinici [23].

Definition 45 (see [19]). Let $(X, \preceq)$ be an ordered set, $E \subseteq X$, and $a, b \in E$. A subset $\{e_1, e_2, \dots, e_k\}$ of E is called $\prec$ -chain between a and b in E if

- (i) $k \geq 2$,
- (ii) $e_1 = a$ and $e_k = b$,
- (iii) $e_1 \prec e_2 \prec \dots \prec e_{k-1} \prec e_k$.

We denote by $C(a, b, \prec, E)$ the class of all $\prec$ -chains between a and b in E . In particular, for $E = X$, we write $C(x, y, \prec)$ instead of $C(x, y, \prec, X)$.

Theorem 46. *In addition to the hypotheses of Theorem 34 (similarly Theorems 37 and 38), suppose that the following condition holds:*

- (u_1) $C(fx, fy, \prec, gX)$ is nonempty, for each $x, y \in X$.

Then, f and g have a unique point of coincidence.

Corollary 47. *Theorem 46 remains true if one replaces condition (u_1) by one of the following conditions:*

- (u_1^1) $(fX, \preceq)$ is totally ordered.
- (u_1^2) $(X, \preceq)$ is (f, g) -directed.

Theorem 48. *In addition to the hypotheses of Theorem 46, suppose that the following condition holds:*

- (u_2) one of f and g is one-one.

Then, f and g have a unique coincidence point.

Theorem 49. *In addition to the hypotheses embodied in condition (e') of Theorem 46, suppose that the following condition holds:*

- ($e'3$) f and g are weakly compatible.

Then, f and g have a unique common fixed point.

We skip the proofs of the above results as the same can be completed on the lines of the proofs of the relevant results of Alam et al. [19].

5. Fixed Point Theorems

On setting $g = I$, the identity mapping on X , in earlier results, we deduce the following corresponding fixed point theorems.

Theorem 50. *Let $(X, d, \preceq)$ be an ordered metric space and f a self-mapping on X . Let Y be an $\overline{O}$ -complete subspace of X such that $f(X) \subseteq Y$. Suppose that the following conditions hold:*

- (i) f is increasing.
- (ii) Either f is $\overline{O}$ -continuous or $(Y, d, \preceq)$ has ICC property.
- (iii) There exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$.
- (iv) There exists $\varphi \in \Omega$ such that

$$d(fx, fy) \leq \varphi(d(x, y)) \quad \forall x, y \in X \text{ with } x \prec y. \quad (78)$$

Then, f has a fixed point.

Theorem 51. *Theorem 50 remains true if certain involved terms, namely, $\overline{O}$ -complete, $\overline{O}$ -continuous, and ICC property, are, respectively, replaced by $\underline{O}$ -complete, $\underline{O}$ -continuous, and DCC property provided assumption (iii) is replaced by the following condition:*

- (iii)' there exists $x_0 \in X$ such that $x_0 \geq f(x_0)$.

Theorem 52. *Theorem 50 remains true if certain involved terms, namely, $\overline{O}$ -complete, $\overline{O}$ -continuous, and ICC property, are, respectively, replaced by O -complete, O -continuous, and MCC property provided assumption (iii) is replaced by the following condition:*

- (iii)'' there exists $x_0 \in X$ such that $x_0 \prec f(x_0)$.

Corollary 53. *Theorem 50 (similarly Theorems 51 and 52) remains true if condition (iv) is replaced by the following condition:*

- (iv)' there exists $\alpha \in [0, 1)$ such that

$$d(fx, fy) \leq \alpha d(x, y) \quad \forall x, y \in X \text{ with } x \prec y. \quad (79)$$

Corollary 54. *Theorem 50 remains true if the term ICC property is replaced by ICU property.*

Corollary 55. *Theorem 51 remains true if the term DCC property is replaced by DCL property.*

Corollary 56. *Theorem 52 remains true if the term MCC property is replaced by MCB property.*

Corollary 57. *Theorem 50 (resp., Theorems 51 and 52) remains true if the usual metrical notions, namely, completeness and continuity, are used instead of their respective $\overline{O}$ -analogues (resp., $\underline{O}$ -analogues and O -analogues).*

Theorem 58. *In addition to the hypotheses of Theorem 50 (similarly Theorems 51 and 52), suppose that the following condition holds:*

(u) $C(fx, fy, \langle \rangle)$ is nonempty for each $x, y \in X$.

Then, f has a unique fixed point.

Corollary 59. *Theorem 58 remains true if one replaces condition (u) by one of the following conditions:*

(u¹) $(fX, \leq)$ is totally ordered.

(u²) $(X, \leq)$ is f -directed.

Notice that all the above-mentioned results in this section enrich and sharpen the fixed point theorems of Kutbi et al. [20].

Finally, inspired by [6], we present an example, where Theorem 58 can be applied. We study the existence and uniqueness of solution for the following first-order periodic boundary value problem:

$$\begin{aligned} u'(t) &= f(t, u(t)), \quad t \in I = [0, T] \\ u(0) &= u(T), \end{aligned} \quad (80)$$

where $T > 0$ and $f: I \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

Let $\mathcal{C}(I)$ denote the space of all continuous functions defined on I . Now, we recall the following definitions.

Definition 60 (see [6]). A function $\alpha \in \mathcal{C}^1(I)$ is called a lower solution of (80), if

$$\begin{aligned} \alpha'(t) &\leq f(t, \alpha(t)), \quad t \in I \\ \alpha(0) &\leq \alpha(T). \end{aligned} \quad (81)$$

Definition 61 (see [6]). A function $\alpha \in \mathcal{C}^1(I)$ is called an upper solution of (80), if

$$\begin{aligned} \alpha'(t) &\geq f(t, \alpha(t)), \quad t \in I \\ \alpha(0) &\geq \alpha(T). \end{aligned} \quad (82)$$

Let $\mathfrak{F}$ denote the family of functions $\phi: [0, \infty) \rightarrow [0, \infty)$ satisfying the following conditions:

- (i) ϕ is continuous and increasing.
- (ii) $\phi(t) < t$ for each $t > 0$.

Typical examples of $\mathfrak{F}$ are $\phi(t) = \alpha \cdot t$, $0 \leq \alpha < 1$, $\phi(t) = t/(1+t)$, and $\phi(t) = \ln(1+t)$. Also, clearly, $\mathfrak{F} \subset \Omega$.

Now, we record the following result regarding the existence and uniqueness of a solution of problem (80) in the presence of a lower solution or an upper solution.

Theorem 62. *In addition to problem (80), suppose that there exists $\lambda > 0$ and $\phi \in \mathfrak{F}$ such that for all $x, y \in \mathbb{R}$ with $x \leq y$*

$$0 \leq f(t, y) + \lambda y - [f(t, x) + \lambda x] \leq \lambda \phi(y - x). \quad (83)$$

Then, the existence of a lower solution or an upper solution of problem (80) provides the existence and uniqueness of a solution of problem (80).

The proof of Theorem 62 can be completed on the lines of the proof of Theorem 11 due to Kutbi et al. [20], wherein the authors used Theorem 58 (with $Y = X$).

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed equally. Thus, formally, all the authors read and approved the final paper.

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