Research Article

Existence of Positive Solution to System of Nonlinear Third-Order Three-Point BVPs

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We are concerned with the following system of third-order three-point boundary value problems: $u'''(t) + f(t, v(t)) = 0$, $t \in (0, 1)$, $v'''(t) + g(t, u(t)) = 0$, $t \in (0, 1)$, $u(0) = u''(0) = 0$, $u'(1) = \alpha u(\eta)$, $v(0) = v''(0) = 0$, and $v'(1) = \alpha v(\eta)$, where $0 < \eta < 1$ and $0 < \alpha < 1/\eta$. By imposing some suitable conditions on f and g , we obtain the existence of at least one positive solution to the above system. The main tool used is the theory of the fixed-point index.

1. Introduction

Third-order differential equations arise from a variety of different areas of applied mathematics and physics, for example, in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves, or gravity driven flows, and so on [1].

Recently, there are a lot of papers concerning the existence of positive solutions to third-order three-point boundary value problems (BVPs for short); see [2–13] and the references therein. Yet, only in a few papers has the problem of existence of positive solutions to systems of third-order three-point BVPs been considered.

It is worth mentioning that there are some excellent works on systems of second-order or higher-order multipoint BVPs; see [14–18].

In this paper, we study the existence of positive solution for the following system of nonlinear third-order three-point BVPs:

$$u'''(t) + f(t, v(t)) = 0, \quad t \in (0, 1),$$

$$v'''(t) + g(t, u(t)) = 0, \quad t \in (0, 1),$$

$$u(0) = u''(0) = 0,$$

$$u'(1) = \alpha u(\eta),$$

$$v(0) = v''(0) = 0,$$

$$v'(1) = \alpha v(\eta).$$

(1)

Throughout this paper, we always assume that the following conditions are fulfilled:

$$(H_1) \quad 0 < \eta < 1 \text{ and } 0 < \alpha < 1/\eta.$$

$$(H_2) \quad f, g \in C([0, 1] \times [0, \infty), [0, \infty)) \text{ and } f(t, 0) \equiv 0, \\ g(t, 0) \equiv 0, \quad t \in [0, 1].$$

If $(u, v) \in C^3([0, 1], [0, \infty)) \times C^3([0, 1], [0, \infty))$ satisfies the differential equations and boundary conditions in system (1), then (u, v) is said to be a solution of system (1). If (u, v) is a solution of system (1) and $u(t) > 0$, $v(t) > 0$, $t \in (0, 1)$, then (u, v) is said to be a positive solution of system (1).

To end this section, we state the following results on the theory of the fixed-point index [19].

Let E be a real Banach space, $P \subset E$ a cone, and θ the zero element in E . For $\rho > 0$, we denote

$$B_\rho = \{u \in E : \|u\| < \rho\}. \quad (2)$$

Theorem 1. Let $A : \overline{B_\rho} \cap P \rightarrow P$ be a completely continuous operator. If there exists $u_0 \in P \setminus \{\theta\}$ such that

$$u - Au \neq \lambda u_0, \quad u \in \partial B_\rho \cap P, \quad \lambda \geq 0, \quad (3)$$

then $i(A, B_\rho \cap P, P) = 0$.

Theorem 2. Let $A : \overline{B_\rho} \cap P \rightarrow P$ be a completely continuous operator which has no fixed point on $\partial B_\rho \cap P$. If $\|Au\| \leq \|u\|$ for all $u \in \partial B_\rho \cap P$, then $i(A, B_\rho \cap P, P) = 1$.

2. Preliminaries

Let $E = C[0, 1]$ be equipped with the maximum norm. Then E is a Banach space.

Lemma 3 (see [7]). For any $y \in E$, the BVP

$$\begin{aligned} u'''(t) + y(t) &= 0, \quad t \in (0, 1), \\ u(0) &= u''(0) = 0, \\ u'(1) &= \alpha u(\eta) \end{aligned} \quad (4)$$

has a unique solution

$$u(t) = \int_0^1 G(t, s) y(s) ds, \quad t \in [0, 1], \quad (5)$$

where

$$G(t, s) = \frac{1}{2(1-\alpha\eta)} \begin{cases} 2t - (1-\alpha\eta)(t^2 + s^2) - \alpha t(\eta^2 + s^2), & s \leq \min\{\eta, t\}, \\ 2t - t^2 - s^2 + \alpha\eta(t-s)^2, & \eta \leq s \leq t, \\ 2t - 2ts(1-\alpha\eta) - \alpha t(\eta^2 + s^2), & t \leq s \leq \eta, \\ 2t(1-s), & s \geq \max\{\eta, t\}. \end{cases} \quad (6)$$

For convenience, we denote

$$\begin{aligned} a &= \frac{1+\alpha\eta}{1-\alpha\eta}, \\ \gamma &= \frac{\eta^2}{2(1+\alpha\eta)} \min\{1, \alpha\}. \end{aligned} \quad (7)$$

Obviously, $a > 0$ and $0 < \gamma < 1$.

Lemma 4 (see [7]). For any $(t, s) \in [0, 1] \times [0, 1]$, $0 \leq G(t, s) \leq a(1-s)$.

Lemma 5. For any $(t, s) \in [\eta, 1] \times [0, 1]$, $G(t, s) \geq \gamma a(1-s)$.

Proof. Since $G(t, 1) = 0$, we only need to consider $(t, s) \in [\eta, 1] \times [0, 1]$.

If $s \leq \eta \leq t$, then

$$\begin{aligned} \frac{G(t, s)}{a(1-s)} &= \frac{2t - (1-\alpha\eta)(t^2 + s^2) - \alpha t(\eta^2 + s^2)}{2(1+\alpha\eta)(1-s)} \\ &> \frac{2t - (1-\alpha\eta)(t + ts) - \alpha t(\eta + \eta s)}{2(1+\alpha\eta)(1-s)} \\ &= \frac{t}{2(1+\alpha\eta)} \geq \frac{\eta}{2(1+\alpha\eta)} > \frac{\eta^2}{2(1+\alpha\eta)} \\ &\geq \gamma. \end{aligned} \quad (8)$$

If $\eta \leq s \leq t$, then

$$\begin{aligned} \frac{G(t, s)}{a(1-s)} &= \frac{2t - t^2 - s^2 + \alpha\eta(t-s)^2}{2(1+\alpha\eta)(1-s)} \\ &\geq \frac{2t - 2t(1-\alpha\eta) - 2\alpha\eta ts}{2(1+\alpha\eta)(1-s)} \geq \frac{\alpha\eta^2}{1+\alpha\eta} \\ &> \frac{\alpha\eta^2}{2(1+\alpha\eta)} \geq \gamma. \end{aligned} \quad (9)$$

If $\eta \leq t \leq s$, then

$$\frac{G(t, s)}{a(1-s)} = \frac{t}{1+\alpha\eta} \geq \frac{\eta}{1+\alpha\eta} > \frac{\eta^2}{2(1+\alpha\eta)} \geq \gamma. \quad (10)$$

Therefore, for any $(t, s) \in [\eta, 1] \times [0, 1]$, $G(t, s) \geq \gamma a(1-s)$. $\square$

Lemma 6. Let $y \in E$ and $y(t) \geq 0$, $t \in [0, 1]$. Then the unique solution u of BVP (4) satisfies

$$\begin{aligned} u(t) &\geq 0, \quad t \in [0, 1], \\ \min_{t \in [\eta, 1]} u(t) &\geq \eta \|u\|. \end{aligned} \quad (11)$$

Proof. Since $y(t) \geq 0$, $t \in [0, 1]$, it follows from Lemma 4 that $u(t) \geq 0$ for $t \in [0, 1]$. In view of $u'''(t) = -y(t) \leq 0$, $t \in (0, 1)$, we have $u''(t) \leq u''(0) = 0$, $t \in (0, 1)$. This shows that $u'(t) \geq u'(1) = \alpha u(\eta) \geq 0$, $t \in (0, 1)$, which indicates that

$$\min_{t \in [\eta, 1]} u(t) = u(\eta), \quad (12)$$

$$\|u\| = u(1).$$

On the other hand, it follows from the fact $u''(t) \leq 0$ for $t \in (0, 1)$ that u is concave down on $[0, 1]$, which together with $u(0) = 0$ implies that $u(\eta) \geq \eta u(1)$. In view of (12), we get that $\min_{t \in [\eta, 1]} u(t) \geq \eta \|u\|$. $\square$

Lemma 7. There exists $x_0 \in (0, (4\sqrt{3}/9)\pi)$ such that

$$\begin{aligned} x_0 e^{-x_0} + \alpha e^{-\eta x_0} + 2x_0 e^{(1/2)x_0} \cos \frac{\sqrt{3}}{2} x_0 \\ - 2\alpha e^{(1/2)\eta x_0} \sin \left(\frac{\pi}{6} + \frac{\sqrt{3}}{2} \eta x_0 \right) = 0. \end{aligned} \quad (13)$$

Proof. Let

$$\begin{aligned} \varphi(x) &= x e^{-x} + \alpha e^{-\eta x} + 2x e^{(1/2)x} \cos \frac{\sqrt{3}}{2} x \\ &\quad - 2\alpha e^{(1/2)\eta x} \sin \left(\frac{\pi}{6} + \frac{\sqrt{3}}{2} \eta x \right), \end{aligned} \quad (14)$$

$$x \in \left[0, \frac{4\sqrt{3}}{9} \pi \right].$$

Then it is obvious that

$$\begin{aligned} \varphi(0) &= 0, \\ \varphi'(0) &= 3(1-\alpha\eta) > 0. \end{aligned} \quad (15)$$

This indicates that there exists $x^* \in (0, (4\sqrt{3}/9)\pi)$ such that $\varphi(x^*) > 0$, which together with the fact that

$$\begin{aligned} \varphi\left(\frac{4\sqrt{3}}{9}\pi\right) &= \frac{4\sqrt{3}}{9}\pi e^{-(4\sqrt{3}/9)\pi} + \alpha e^{-(4\sqrt{3}/9)\pi\eta} - \frac{4\sqrt{3}}{9} \\ &\cdot \pi e^{(2\sqrt{3}/9)\pi} - 2\alpha e^{(2\sqrt{3}/9)\pi\eta} \sin\left(\frac{\pi}{6} + \frac{2\pi}{3}\eta\right) < \frac{4\sqrt{3}}{9} \\ &\cdot \pi e^{-(4\sqrt{3}/9)\pi} + \alpha e^{-(4\sqrt{3}/9)\pi\eta} - \frac{4\sqrt{3}}{9}\pi e^{(2\sqrt{3}/9)\pi} \\ &- \alpha e^{(2\sqrt{3}/9)\pi\eta} = -\left[\frac{4\sqrt{3}}{9}\pi\left(e^{(2\sqrt{3}/9)\pi} - e^{-(4\sqrt{3}/9)\pi}\right) \right. \\ &\left. + \alpha\left(e^{(2\sqrt{3}/9)\pi\eta} - e^{-(4\sqrt{3}/9)\pi\eta}\right)\right] < 0 \end{aligned} \quad (16)$$

implies that there exists $x_0 \in (x^*, (4\sqrt{3}/9)\pi) \subset (0, (4\sqrt{3}/9)\pi)$ such that $\varphi(x_0) = 0$; that is, (13) is satisfied. $\square$

In the remainder of this paper, we always assume that x_0 is defined as in Lemma 7 and $u_0(t) = -e^{-x_0 t} + 2e^{(1/2)x_0 t} \sin(\pi/6 + (\sqrt{3}/2)x_0 t)$, $t \in [0, 1]$.

Corollary 8. x_0^3 is an eigenvalue of the eigenvalue problem

$$\begin{aligned} u'''(t) + \lambda u(t) &= 0, \quad t \in (0, 1), \\ u(0) &= u''(0) = 0, \\ u'(1) &= \alpha u(\eta) \end{aligned} \quad (17)$$

and u_0 is an eigenfunction corresponding to the eigenvalue x_0^3 .

Proof. By Lemma 7, it is obvious. $\square$

Let

$$\begin{aligned} P &= \{u \in E : u(t) \geq 0, \quad t \in [0, 1]\}, \\ P_0 &= \left\{u \in P : \min_{t \in [\eta, 1]} u(t) \geq \eta \|u\|\right\}. \end{aligned} \quad (18)$$

Then P and P_0 are cones in E . Now, we define a linear operator B as follows:

$$(Bu)(t) = \int_0^1 G(t, s) u(s) ds, \quad u \in P, \quad t \in [0, 1]. \quad (19)$$

Lemma 9. Consider $B : P \rightarrow P_0$.

Proof. In view of Lemmas 3, 4, and 6, it is not difficult to verify that $B(P) \subset P_0$. $\square$

Lemma 10. Consider $u_0 \in P$ and

$$u_0(t) = x_0^3 (Bu_0)(t), \quad t \in [0, 1]. \quad (20)$$

Proof. Obviously, u_0 is continuous and $u_0(t) \geq e^{-x_0 t} (e^{(3/2)x_0 t} - 1) \geq 0$, $t \in [0, 1]$, which indicates that $u_0 \in P$. And it follows from Lemma 3 and Corollary 8 that (20) is satisfied. $\square$

3. Main Results

Obviously, (u, v) is a solution of system (1) if and only if $(u, v) \in C([0, 1], [0, \infty)) \times C([0, 1], [0, \infty))$ is a solution of the following system:

$$\begin{aligned} u(t) &= \int_0^1 G(t, s) f(s, v(s)) ds, \quad t \in [0, 1], \\ v(t) &= \int_0^1 G(t, s) g(s, u(s)) ds, \quad t \in [0, 1]. \end{aligned} \quad (21)$$

Moreover, system (21) can be written as the integral equation:

$$u(t) = \int_0^1 G(t, s) f\left(s, \int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau\right) ds, \quad t \in [0, 1]. \quad (22)$$

If we define an operator A on P by

$$\begin{aligned} (Au)(t) &= \int_0^1 G(t, s) f\left(s, \int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau\right) ds, \\ &t \in [0, 1], \end{aligned} \quad (23)$$

then it is easy to verify that $A : P \rightarrow P$ is completely continuous. Moreover, if $u \in P$ is a fixed point of A and $v(t) = \int_0^1 G(t, s) g(s, u(s)) ds$, $t \in [0, 1]$, then (u, v) is a solution of system (1).

Theorem 11. Assume that the following conditions are fulfilled:

(A₁) There exists a constant $p \in (0, 1]$ such that

$$\begin{aligned} (1) \quad \lim_{x \rightarrow \infty} \frac{f(t, x)}{x^p} &> 0, \\ (2) \quad \lim_{x \rightarrow \infty} \frac{g(t, x)}{x^{1/p}} &= \infty \end{aligned} \quad (24)$$

uniformly on $[0, 1]$.

(A₂) There exists a constant $q \in (0, \infty)$ such that

$$\begin{aligned} (1) \quad \lim_{x \rightarrow 0^+} \frac{f(t, x)}{x^q} &< \infty, \\ (2) \quad \lim_{x \rightarrow 0^+} \frac{g(t, x)}{x^{1/q}} &= 0 \end{aligned} \quad (25)$$

uniformly on $[0, 1]$.

Then system (1) has at least one positive solution.

Proof. First, let

$$M = \{u \in P : u - Au = \lambda u_0, \quad \lambda \geq 0\}. \quad (26)$$

Then we may assert that M is a bounded subset of E .

In fact, if $u \in M$, then there exists $\lambda \geq 0$ such that $u = Au + \lambda u_0$, which together with Lemma 10 implies that

$$\begin{aligned} u(t) &= (Au)(t) + \lambda u_0(t) \\ &= \int_0^1 G(t, s) f\left(s, \int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau\right) ds \\ &\quad + \lambda x_0^3 (Bu_0)(t) \\ &= \int_0^1 G(t, s) (Fu)(s) ds + \lambda x_0^3 (Bu_0)(t) \\ &= B[(Fu)(t) + \lambda x_0^3 u_0(t)], \quad t \in [0, 1], \end{aligned} \quad (27)$$

where $F : P \rightarrow P$ is defined as follows:

$$(Fu)(t) = f\left(t, \int_0^1 G(t, s) g(s, u(s)) ds\right), \quad (28)$$

$$t \in [0, 1].$$

In view of (27) and Lemma 9, we have $u \in P_0$. This implies that

$$\|u\| \leq \frac{1}{\eta} \min_{t \in [\eta, 1]} u(t). \quad (29)$$

On the other hand, from (1) of (A_1) , we know that there exist positive constants C_1 and C_2 such that

$$f(t, x) \geq C_1 x^p - C_2, \quad (t, x) \in [0, 1] \times [0, \infty), \quad (30)$$

which together with Jensen inequality and Lemma 4 implies that

$$\begin{aligned} (Au)(t) &= \int_0^1 G(t, s) \\ &\quad \cdot f\left(s, \int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau\right) ds \\ &\geq \int_0^1 G(t, s) \\ &\quad \cdot \left[C_1 \left(\int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau\right)^p - C_2\right] ds \\ &\geq C_1 \int_0^1 G(t, s) \int_0^1 (G(s, \tau))^p (g(\tau, u(\tau)))^p d\tau ds \\ &\quad - C_2 \int_0^1 G(t, s) ds \geq C_1 \int_0^1 G(t, s) \\ &\quad \cdot \int_0^1 (G(s, \tau))^p (g(\tau, u(\tau)))^p d\tau ds - C_3, \end{aligned} \quad (31)$$

$$t \in [0, 1],$$

where $C_3 = (1/2)aC_2$.

From (2) of (A_1) , we know that there exists $C_4 > 0$ such that

$$(g(t, x))^p \geq \frac{2}{C_1 \gamma^p a^{p-1} \eta m_1^2} x - C_4, \quad (32)$$

$$(t, x) \in [0, 1] \times [0, \infty),$$

where $m_1 = \min_{t \in [\eta, 1]} \int_\eta^1 G(t, s) ds > 0$.

So, it follows from (31), (32), and Lemmas 4 and 5 that

$$\begin{aligned} u(t) &= (Au)(t) + \lambda u_0(t) \geq (Au)(t) \geq C_1 \int_0^1 G(t, s) \\ &\quad \cdot \int_0^1 (G(s, \tau))^p (g(\tau, u(\tau)))^p d\tau ds - C_3 \\ &\geq C_1 \int_\eta^1 G(t, s) \int_0^1 (G(s, \tau))^p (g(\tau, u(\tau)))^p d\tau ds \\ &\quad - C_3 \geq C_1 \gamma^p a^p \int_\eta^1 G(t, s) \\ &\quad \cdot \int_0^1 (1 - \tau)^p (g(\tau, u(\tau)))^p d\tau ds - C_3 \\ &\geq C_1 \gamma^p a^p \int_\eta^1 G(t, s) \int_0^1 (1 - \tau) (g(\tau, u(\tau)))^p d\tau ds \\ &\quad - C_3 \geq C_1 \gamma^p a^{p-1} \int_\eta^1 G(t, s) \\ &\quad \cdot \int_0^1 G(s, \tau) (g(\tau, u(\tau)))^p d\tau ds - C_3 \\ &\geq C_1 \gamma^p a^{p-1} \int_\eta^1 G(t, s) \\ &\quad \cdot \int_0^1 G(s, \tau) \left[\frac{2}{C_1 \gamma^p a^{p-1} \eta m_1^2} u(\tau) - C_4 \right] d\tau ds \\ &\quad - C_3 = \frac{2}{\eta m_1^2} \int_\eta^1 G(t, s) \int_0^1 G(s, \tau) u(\tau) d\tau ds \\ &\quad - C_1 C_4 \gamma^p a^{p-1} \int_\eta^1 G(t, s) \int_0^1 G(s, \tau) d\tau ds - C_3 \\ &= \frac{2}{\eta m_1^2} \int_\eta^1 G(t, s) (Bu)(s) ds \\ &\quad - \left[C_1 C_4 \gamma^p a^{p-1} \int_\eta^1 G(t, s) \int_0^1 G(s, \tau) d\tau ds + C_3 \right] \\ &\geq \frac{2}{\eta m_1^2} \min_{s \in [\eta, 1]} (Bu)(s) \int_\eta^1 G(t, s) ds \end{aligned}$$

$$\begin{aligned}
& - \left[C_1 C_4 \gamma^p a^p \int_{\eta}^1 (1-s) \int_0^1 G(s, \tau) d\tau ds + C_3 \right] \\
& \geq \frac{2}{\eta m_1} \min_{s \in [\eta, 1]} (Bu)(s) - C_5, \quad t \in [\eta, 1],
\end{aligned} \tag{33}$$

where $C_5 = C_1 C_4 \gamma^p a^p \int_{\eta}^1 (1-s) \int_0^1 G(s, \tau) d\tau ds + C_3 > 0$.
 Since $B(u) \in P_0$, we have

$$\begin{aligned}
\min_{s \in [\eta, 1]} (Bu)(s) & \geq \eta \|Bu\| \geq \eta \int_0^1 G(\eta, \tau) u(\tau) d\tau \\
& \geq \eta \int_{\eta}^1 G(\eta, \tau) u(\tau) d\tau \\
& \geq \eta \min_{\tau \in [\eta, 1]} u(\tau) \int_{\eta}^1 G(\eta, \tau) d\tau \\
& \geq \eta m_1 \min_{\tau \in [\eta, 1]} u(\tau).
\end{aligned} \tag{34}$$

In view of (33) and (34), we get

$$u(t) \geq 2 \min_{\tau \in [\eta, 1]} u(\tau) - C_5, \quad t \in [\eta, 1]. \tag{35}$$

So,

$$\min_{t \in [\eta, 1]} u(t) \geq 2 \min_{t \in [\eta, 1]} u(t) - C_5, \tag{36}$$

and so,

$$\min_{t \in [\eta, 1]} u(t) \leq C_5, \tag{37}$$

which together with (29) indicates that

$$\|u\| \leq \frac{C_5}{\eta}. \tag{38}$$

This shows that M is a bounded subset of E . Therefore, there exists a sufficiently larger $R > 1$ such that

$$u - Au \neq \lambda u_0, \quad u \in \partial B_R \cap P, \quad \lambda \geq 0. \tag{39}$$

So, it follows from Theorem 1 that

$$i(A, B_R \cap P, P) = 0. \tag{40}$$

Next, from (1) of (A_2) , we have

$$b := \sup \left\{ \frac{f(t, x)}{x^q} : (t, x) \in [0, 1] \times (0, 1] \right\} < \infty, \tag{41}$$

which together with $f(t, 0) \equiv 0, t \in [0, 1]$, implies that

$$f(t, x) \leq bx^q, \quad (t, x) \in [0, 1] \times [0, 1]. \tag{42}$$

Let

$$\varepsilon = \min \left\{ \frac{1}{m_2}, \left(\frac{1}{(b+1)m_2^{q+1}} \right)^{1/q} \right\}, \tag{43}$$

where $m_2 = \max_{t \in [0, 1]} \int_0^1 G(t, s) ds > 0$. Then from (2) of (A_2) and $g(t, 0) \equiv 0, t \in [0, 1]$, we know that there exists $r \in (0, 1]$ such that

$$g(t, x) \leq \varepsilon x^{1/q}, \quad (t, x) \in [0, 1] \times [0, r]. \tag{44}$$

By (44), for any $u \in \partial B_r \cap P$, we have

$$\begin{aligned}
\int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau & \leq \varepsilon \int_0^1 G(s, \tau) (u(\tau))^{1/q} d\tau \\
& \leq \varepsilon m_2 \|u\|^{1/q} (\leq 1), \quad s \in [0, 1],
\end{aligned} \tag{45}$$

which together with (42) shows that

$$\begin{aligned}
(Au)(t) & = \int_0^1 G(t, s) f\left(s, \int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau\right) ds \\
& \leq b \int_0^1 G(t, s) \left(\int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau \right)^q ds \\
& \leq b \varepsilon^q m_2^q \|u\| \int_0^1 G(t, s) ds \leq b \varepsilon^q m_2^{q+1} \|u\| \\
& < (b+1) \varepsilon^q m_2^{q+1} \|u\| \leq \|u\|, \quad t \in [0, 1].
\end{aligned} \tag{46}$$

This indicates that $\|Au\| < \|u\|, u \in \partial B_r \cap P$. So, it follows from Theorem 2 that

$$i(A, B_r \cap P, P) = 1. \tag{47}$$

In view of (40) and (47), we get

$$\begin{aligned}
i(A, (B_R \setminus \overline{B_r}) \cap P, P) & \\
= i(A, B_R \cap P, P) - i(A, B_r \cap P, P) & = -1.
\end{aligned} \tag{48}$$

Therefore, A has at least one fixed point $u \in P$ and $r < \|u\| < R$. Let

$$v(t) = \int_0^1 G(t, s) g(s, u(s)) ds, \quad t \in [0, 1]. \tag{49}$$

Then (u, v) is a solution of system (1).

Thirdly, we prove $u(t) > 0$ for $t \in (0, 1)$.

Suppose on the contrary that there exists $t_0 \in (0, 1)$ such that $u(t_0) = 0$. Since $u'''(t) = -f(t, v(t)) \leq 0, t \in (0, 1)$, we know that $u''(t) \leq u''(0) = 0, t \in (0, 1)$. So, $u(t)$ is concave down on $[0, 1]$, which together with $u(0) = u(t_0) = 0$ implies that $u(1) = 0$. At the same time, it follows from $u''(t) \leq 0, t \in (0, 1)$, that $u'(t) \geq u'(1) = \alpha u(\eta) \geq 0, t \in (0, 1)$. So, $0 = u(0) \leq u(t) \leq u(1) = 0, t \in [0, 1]$; that is, $u(t) \equiv 0, t \in [0, 1]$, which contradicts with the fact that $\|u\| > r > 0$. Therefore, $u(t) > 0, t \in (0, 1)$.

Finally, we verify $v(t) > 0$ for $t \in (0, 1)$.

First, we may assert that $\|v\| > 0$. Indeed, if $v(t) \equiv 0, t \in [0, 1]$, in view of $f(t, 0) \equiv 0, t \in [0, 1]$, then we get

$$\begin{aligned}
u(t) & = \int_0^1 G(t, s) f(s, v(s)) ds \\
& = \int_0^1 G(t, s) f(s, 0) ds = 0, \quad t \in [0, 1],
\end{aligned} \tag{50}$$

which is a contradiction. This shows that $\|v\| > 0$. So, by using the similar method to prove $u(t) > 0$ for $t \in (0, 1)$, we may obtain $v(t) > 0$, $t \in (0, 1)$.

To sum it up, (u, v) is a positive solution of system (1). $\square$

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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