Research Article

Some Extensions of Fixed Point Results over Quasi-$JS$-Spaces

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We introduce the notion of quasi-$JS$-metric space. After defining the basic topological properties of quasi-$JS$-metric space, we investigate fixed point of certain mapping in the frame of complete quasi-$JS$-metric space. Our results unify and cover several existing fixed point theorems in distinct structures (such as standard quasi-metric spaces, quasi-$b$-metric spaces, dislocated quasi-metric spaces, and quasi-modular spaces) in the literature.

1. Introduction and Preliminaries

Jleli and Samet [1] combined a number of existing fixed point results, by introducing a new distance (that includes, as particular cases, standard metric spaces, $b$-metric spaces, dislocated metric spaces, and modular spaces). In this paper, our aim is to refine the new distance by omitting a symmetry condition. Hence, our new approaches cover and combine several more interesting existing fixed point results (that includes, as particular cases, standard quasi-metric spaces, quasi-$b$-metric spaces, dislocated quasi-metric spaces, and quasi-modular spaces) including the results of Jleli and Samet [1].

For the sake of completeness, we collect some basic concepts and results from the literature. Let $\mathbb{N}_0$ denote the set $\mathbb{N} \cup \{0\}$ where $\mathbb{N}$ represent the set of all positive integers. Let $X$ be a nonempty set and let $\mathcal{D} : X \times X \to [0, \infty]$ be a given mapping. For every $x \in X$, define the sets

$$C_L(\mathcal{D}, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n \to \infty} \mathcal{D}(x, x_n) = 0 \right\},$$

$$C_R(\mathcal{D}, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n \to \infty} \mathcal{D}(x_n, x) = 0 \right\}.$$

(1)

Definition 1. We say that $\mathcal{D} : X \times X \to [0, \infty]$ is a quasi-$JS$-metric space on a nonempty set $X$ if it fulfills the following conditions:

(D_1) $\mathcal{D}(x, y) = \mathcal{D}(y, x) = 0 \Rightarrow x = y$, for every $x, y \in X$.

(D_2) There exists $C > 0$ such that

$$\mathcal{D}(x, y) \leq C \limsup_{n \to \infty} \mathcal{D}(x_n, y),$$

if $x, y \in X$, $\{x_n\} \in C_L(\mathcal{D}, X, x)$, then

$$\mathcal{D}(x, y) \leq C \limsup_{n \to \infty} \mathcal{D}(y, x_n),$$

if $x, y \in X$, $\{x_n\} \in C_R(\mathcal{D}, X, x)$, then (2)

In this case, the pair $(X, \mathcal{D})$ is called a quasi-$JS$-metric space.

Remark 2. If, in addition to the conditions in Definition 1, the equality,

(D_3) $\mathcal{D}(x, y) = \mathcal{D}(y, x)$,

is satisfied for each $x, y \in X$, then $(X, \mathcal{D})$ is called JS-metric space [1].

In what follows we shall define some basic topological notions for quasi-$JS$-metric space.

Definition 3. Let $(X, \mathcal{D})$ be a quasi-$JS$-metric space. Let $\{x_n\}$ be a sequence in $X$ and $x \in X$. Then,

(i) $\{x_n\}$ is said to be left $\mathcal{D}$-convergent to $x$ if $\{x_n\} \in C_L(\mathcal{D}, X, x)$; in this case $x$ is said to be a left $\mathcal{D}$-limit of $\{x_n\}$.
(ii) \( \{x_n\} \) is said to be right \( D \)-convergent to \( x \) if \( \{x_n\} \in C_R(D, X, x) \); in this case \( x \) is said to be a right \( D \)-limit of \( \{x_n\} \).

(iii) \( \{x_n\} \) is said to be \( D \)-convergent to \( x \) if \( x_n \) is both left and right \( D \)-convergent to \( x \); in this case \( x \) is said to be a \( D \)-limit of \( \{x_n\} \) (see [1]).

**Proposition 4.** The \( D \)-limit of any sequence in a quasi-JS-metric space is unique.

*Proof.* Let \( (X, D) \) be a quasi-JS-metric space. Let \( \{x_n\} \) be a sequence in \( X \). Assume that \( x, y \) are both \( D \)-limits of \( \{x_n\} \). On account of (D2) and regarding the definition of \( D \)-convergence, we find that

\[
D(y, x) \leq C \limsup_{n \to \infty} D(y, x_n) = 0,
\]

since \( \{x_n\} \in C_R(D, X, x), \{x_n\} \in C_L(D, X, y) \) and also

\[
D(x, y) \leq C \limsup_{n \to \infty} D(x, x_n) = 0,
\]

since \( \{x_n\} \in C_R(D, X, x), \{x_n\} \in C_L(D, X, y) \).

Thus, we have \( D(x, y) = 0 = D(y, x) \). By (D1), we find \( x = y \).

**Definition 5.** Let \( (X, D) \) be a quasi-JS-metric space. Let \( \{x_n\} \) be a sequence in \( X \).

(i) \( \{x_n\} \) is said to be left \( D \)-Cauchy sequence if

\[
\lim_{m,n \to \infty} D(x_m, x_n) = 0.
\]

(ii) \( \{x_n\} \) is said to be right \( D \)-Cauchy sequence if

\[
\lim_{m,n \to \infty} D(x_n, x_m) = 0.
\]

(iii) \( \{x_n\} \) is said to be \( D \)-Cauchy sequence if it is both left and right \( D \)-Cauchy sequence (see [1]).

**Definition 6.** Let \( (X, D) \) be a quasi-JS-metric space.

(i) \( X \) is said to be left \( D \)-complete if every left \( D \)-Cauchy sequence in \( X \) is left \( D \)-convergent to some element in \( X \).

(ii) \( X \) is said to be right \( D \)-complete if every right \( D \)-Cauchy sequence in \( X \) is right \( D \)-convergent to some element in \( X \).

(iii) \( X \) is said to be \( D \)-complete if and only if it is left and right \( D \)-complete, so that every \( D \)-Cauchy sequence in \( X \) is \( D \)-convergent to some element in \( X \) (see [1]).

**Example 7.** Let \( X = \mathbb{R} \cup \{\infty\} \) and define \( D : X \times X \to [0, \infty) \) by

\[
D(x, y) = \begin{cases} 
\infty & \text{if } x \text{ or } y = \infty \\
|x - y| & \text{otherwise}
\end{cases}
\]

Then, clearly, \( D \) satisfies (D1). Let \( x \in X \); we have two cases:

Case 1: if \( x = \infty \), then \( C_L(D, X, x) = C_R(D, X, x) = \emptyset \).

Case 2: if \( x < \infty \), let \( \{z_n\} \in C_L(D, X, x) \); then \( \lim_{n \to \infty} D(x, z_n) = 0 \) and so \( z_n < \infty \) except possibly for finite number of terms. Let \( m \) be the smallest natural number such that \( z_n < \infty \) for all \( n \geq m \). Now, let \( y \in X \); if \( y = \infty \), then

\[
D(x, y) = \infty = \limsup_{n \to \infty} D(z_n, y).
\]

On the other hand, if \( y < \infty \), then we get

\[
D(x, y) = |x - y| \leq |x - z_n| + |z_n - y| \quad \forall n \geq m.
\]

Thus, we find that

\[
D(x, y) = |x - y| \leq \limsup_{n \to \infty} D(z_n, y).
\]

Similarly, if \( \{x_n\} \in C_R(D, X, x) \), then for any \( y \in X \) we have

\[
D(y, x) \leq \limsup_{n \to \infty} D(y, x_n).
\]

Consequently, \( D \) satisfies condition (D2) with \( C = 1 \). Therefore, \( (X, D) \) is a quasi-JS-metric space. Now, let \( \{x_n\} \subseteq X \) be \( D \)-Cauchy sequence. Then there exists a smallest \( m \in \mathbb{N} \) such that \( x_n < \infty \) for all \( n \geq m \). As the restriction of \( D \) to \( \mathbb{R} \) is just the usual metric on \( \mathbb{R} \), \( \{x_n\} \) is \( D \)-convergent to some \( x \in X \). Thus, \( (X, D) \) is a complete quasi-JS-metric space.

**Definition 8.** Let \( X \) be a nonempty set. A mapping \( q : X \times X \to [0, \infty) \) is called quasi-metric on \( X \), if the following conditions are fulfilled:

\( (Q_1) \) for every \( x, y \in X \), we have \( q(x, y) = 0 = q(y, x) \) \( \iff x = y \);

\( (Q_2) \) for every \( x, y, z \in X \), we have \( q(x, y) \leq q(x, z) + q(z, y) \).

Here, the pair \( (X, q) \) is called quasi-metric space.

**Proposition 9.** Any quasi-metric space is a quasi-JS-metric space with \( C = 1 \).

*Proof.* Straightforward, so we omit it.

**Example 10.** Let \( X \) be a set and let \( f : X \to [0, \infty) \) be an arbitrary one to one function. Set

\[
q(x, y) = \max \{ f(y) - f(x), 0 \} \quad \forall x, y \in X.
\]

Then \( q \) is a quasi-metric space [2].

In 2012, Shah and Hussain [3] introduced the concept of quasi-\( b \)-metric spaces and verified some fixed point theorems in quasi-\( b \)-metric spaces.
**Definition 11.** Let $X$ be a nonempty set, let $s \geq 1$ be a given real number, and let $d : X \times X \to [0, \infty)$ be a mapping satisfying the following conditions:

(QBM$_1$) for every $x, y \in X$, we have $d(x, y) = 0 \iff x = y$;
(QBM$_2$) for every $x, y, z \in X$, we have $d(x, y) \leq s[d(x, z) + d(z, y)]$.

Then, $d$ is said to be a quasi-$b$-metric space, and $(X, d)$ is called quasi-$b$-metric space.

The following proposition followed immediately from the previous definition.

**Proposition 12.** Any quasi-$b$-metric space is a quasi-$JS$-metric space with $C = s$.

**Proof.** Let $(X, d)$ be a quasi-$b$-metric space. Since the first condition is straightforward, it is sufficient to show that $d$ fulfills the property $(D_2)$ of Definition 1. Let $x \in X$ and $\{x_n\} \in C, (d, X, x)$. For every $y \in X$, by the property (QBM$_2$), we have

$$d(x, y) \leq s \limsup_{n \to \infty} d(x_n, y).$$

for each $n$. Thus we have

$$d(x, y) \leq s \limsup_{n \to \infty} d(x_n, y).$$

Analogously, for the case $\{x_n\} \in C, (d, X, x)$, we derive that

$$d(y, x) \leq s \limsup_{n \to \infty} d(y, x_n).$$

In 2005, Zeyada et al. [4] introduced the concept of complete dislocated quasi-metric space and obtain some fixed point results on it.

**Definition 13.** Let $X$ be a nonempty set and $d : X \times X \to [0, \infty)$. $d$ is said to be dislocated quasi-metric (or quasi-metric-like) if it satisfies the following conditions for every $x, y, z \in X$:

(QML$_1$) $d(x, y) = 0 \iff d(y, x) = 0$;
(QML$_2$) $d(x, y) \leq d(x, z) + d(z, y)$.

In this case $(X, d)$ is called dislocated quasi-metric space (or quasi-metric-like space). If in addition $d$ satisfies

(QML$_3$) $d(x, y) = d(y, x)$ for every $x, y \in X$,

then it is called dislocated metric.

**Proposition 14.** Any dislocated quasi-metric space is a quasi-JS-metric space.

**Example 15.** Let $X = [0, 1]$ and define $q : X \times X \to [0, \infty)$ by

$$q(x, y) = |x - y| + x \quad \text{for each } x, y \in X.$$ (16)

Then $(X, q)$ is a dislocated quasi-metric.

In 1988, Kozlowski introduced the notion of modular spaces [5]; before we generalize this notion to the quasi form we need the following definitions.

**Definition 16.** Let $X$ be a linear space over $\mathbb{R}$. A function $\rho : X \to [0, \infty)$ is said to be quasi-modular if the following conditions hold:

$$(q\rho_1) \rho(x) = 0 \iff x = 0;$$

$$(q\rho_2) \text{ for every } x, y \in X, \text{ we have } \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y),$$

whenever $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. If in addition $\rho$ satisfies

$$\rho(-x) = \rho(x) \quad \text{for every } x \in X,$$

then $\rho$ is called modular on $X$.

**Definition 17.** Let $X$ be a linear space and let $\rho : X \to [0, \infty]$ be a quasi-modular space on $X$. The set

$$X_\rho = \left\{ x \in X : \lim_{\lambda \to 0} \rho(\lambda x) = 0 \right\}$$

is called a quasi-modular space.

The convergence in quasi-modular spaces is defined as follows.

**Definition 18.** Let $X_\rho$ be a quasi-modular space, let $\{x_n\}$ be a sequence in $X_\rho$, and $x \in X_\rho$.

(i) $\{x_n\}$ is said to be left $\rho$-convergent to $x \in X$ if $\lim_{n \to \infty} \rho(x_n - x) = 0$.

(ii) $\{x_n\}$ is said to be right $\rho$-convergent to $x \in X$ if $\lim_{n \to \infty} \rho(x_n - x) = 0$.

(iii) $\{x_n\}$ is said to be $\rho$-convergent to $x \in X$ if it is both left and right $\rho$-convergent to $x$.

**Definition 19.** Let $X_\rho$ be a quasi-modular space and let $\{x_n\}$ be a sequence in $X_\rho$.

(i) $\{x_n\}$ is said to be left $\rho$-Cauchy if $\lim_{m,n \to \infty} \rho(x_m - x_n) = 0$.

(ii) $\{x_n\}$ is said to be right $\rho$-Cauchy if $\lim_{m,n \to \infty} \rho(x_n - x_m) = 0$.

(iii) $\{x_n\}$ is said to be $\rho$-Cauchy if it is both left and right $\rho$-Cauchy.

**Definition 20.** (i) A quasi-modular space $X_\rho$ is said to be left (right) $\rho$-complete if every left (right) $\rho$-Cauchy sequence converges to some $x \in X_\rho$.

(ii) A quasi-modular space $X_\rho$ is said to be $\rho$-complete if and only if it is both left and right $\rho$-complete.
Definition 21. Let $X_\rho$ be a quasi-modular space.

(i) $\rho$ is said to have left Fatou property if for every $y \in X_\rho$

$$\rho(y-x) \leq \liminf_{n\to\infty} \rho(y-x_n),$$

whenever $\{x_n\} \subseteq X_\rho$ is left $\rho$-convergent to $x \in X_\rho$.

(ii) $\rho$ is said to have right Fatou property if for every $y \in X_\rho$

$$\rho(x-y) \leq \liminf_{n\to\infty} \rho(x_n-y),$$

whenever $\{x_n\} \subseteq X_\rho$ is right $\rho$-convergent to $x \in X_\rho$.

(iii) $\rho$ is said to have Fatou property if it has left and right Fatou property.

Proposition 22. Let $X_\rho$ be a quasi-modular space such that $\rho$ has Fatou property. Then $D_\rho$ is quasi-$JS$-metric on $X_\rho$ with $C = 1$.

Proof. Clearly, $D_\rho$ satisfies (D$_1$). Let us prove that $D_\rho$ satisfies (D$_2$). Let $x \in X_\rho$ and let $\{x_n\} \in C_L(\rho, X_\rho, x)$. As $\rho$ has Fatou property for any $y \in X_\rho$ we have

$$D_\rho(x,y) = \rho(x-y) \leq \liminf_{n\to\infty} \rho(x_n-y)$$

$$\leq \limsup_{n\to\infty} \rho(x_n-y)$$

$$= \limsup_{n\to\infty} D_\rho(x_n, y).$$

Similarly, if $\{x_n\} \in C_R(\rho, X_\rho, x)$, then for every $y \in X_\rho$ we have

$$D_\rho(y,x) = \rho(y-x) \leq \liminf_{n\to\infty} \rho(y-x_n)$$

$$\leq \limsup_{n\to\infty} \rho(y-x_n)$$

$$= \limsup_{n\to\infty} D_\rho(y, x_n).$$

Thus, $D_\rho$ is a quasi-$JS$-metric space on $X_\rho$.

Consequently, we have the following result.

Proposition 23. Let $X_\rho$ be a quasi-modular space where $\rho$ has Fatou property. Then

(i) a sequence $\{x_n\} \subseteq X_\rho$ is left $\rho$-Cauchy (right $\rho$-Cauchy or $\rho$-Cauchy) if and only if it is left $D_\rho$-Cauchy (right $D_\rho$-Cauchy or $D_\rho$-Cauchy);

(ii) $X_\rho$ is left $\rho$-complete (right $\rho$-complete or $\rho$-complete) if and only if it is left $D_\rho$-complete (right $D_\rho$-complete or $D_\rho$-complete).

2. The Banach Contraction Principle in a Quasi-$JS$-Metric Space

The Banach contraction principle was extended to a $JS$-metric space by Jleli and Samet in [1]. We shall extend this principle to a quasi-$JS$-metric space.

Definition 24. Let $(X, D)$ be a quasi-$JS$-metric space and let $f : X \to X$ be a function. We say that $f$ is $k$-contraction if

$$D(f(x), f(y)) \leq kD(x, y),$$

for every $x, y \in X$, where $k \in (0, 1)$.

Proposition 25. Let $(X, D)$ be a quasi-$JS$-metric space. Suppose that the function $f : X \to X$ is $k$-contraction for some $k \in (0, 1)$. Then any fixed point $\omega \in X$ of $f$ with $D(\omega, \omega) < \infty$ satisfies $D(\omega, \omega) = 0$.

Proof. Let $\omega \in X$ be a fixed point of $f$ with $D(\omega, \omega) < \infty$. Then, as $f$ is $k$-contraction, we have

$$D(\omega, \omega) = D(f(\omega), f(\omega)) \leq kD(\omega, \omega),$$

which is possible only if $D(\omega, \omega) = 0$.

For each $x \in X$ let us define

$$\delta(D, f, x) = \sup \{D(f^i(x), f^j(x)) : i, j \in \mathbb{N}_0\},$$

where $f^0(x) = x$.

The following theorem is an extension of Banach contraction principle in the context of a quasi-$JS$-metric space.

Theorem 26. Let $(X, D)$ be a $D$-complete quasi-$JS$-metric space and let $f : X \to X$ be a $k$-contraction mapping for some $k \in (0, 1)$. Suppose that there exists $x_0 \in X$ such that $D(f, x_0) < \infty$. Then, $f$ has a fixed point $\omega \in X$ and $\{f^n(x_0)\}$ is $D$-convergent to $\omega$. Moreover, if $\omega'$ is another fixed point of $f$ such that $D(\omega, \omega') < \infty$ and $D(\omega', \omega) < \infty$, then $\omega = \omega'$.

Proof. We shall prove that $\{f^n(x_0)\}$ is a $D$-Cauchy sequence. Let $n \in \mathbb{N}$, as $f$ is $k$-contraction, for each $i, j \in \mathbb{N}$ we have

$$D(f^{n+i}(x_0), f^{n+j}(x_0)) \leq kD(f^{n+i-1}(x_0), f^{n+j-1}(x_0)),\tag{28}$$

which implies that

$$\delta(D, f, f^n(x_0)) \leq k\delta(D, f, f^{n-1}(x_0)).\tag{29}$$

So, we obtain that

$$\delta(D, f, f^n(x_0)) \leq k^n\delta(D, f, x_0).\tag{30}$$
Taking (30) into account and regarding the definition of $\delta$, for every $m, n \in \mathbb{N}_0$, we have
\[ D(f^n(x_0), f^{nm}(x_0)) \leq k^n \delta(D, f, f^n(x_0)) \leq k^n \delta(D, f, x_0). \]
Using the fact that $\delta(D, f, x_0) < \infty$ and $k \in (0, 1)$ we have
\[ \lim_{m, n \to \infty} D(f^n(x_0), f^{nm}(x_0)) = 0, \]
which implies that $\{f^n(x_0)\}$ is right $D$-Cauchy.

Analogously, we have
\[ \begin{align*}
D(f^{nm}(x_0), f^n(x_0)) & \leq \delta(D, f, f^n(x_0)) \\
& \leq k^n \delta(D, f, x_0),
\end{align*} \]
which implies
\[ \lim_{m, n \to \infty} D(f^{nm}(x_0), f^n(x_0)) = 0. \]
Thus, $\{f^n(x_0)\}$ is left $D$-Cauchy and, hence, it is $D$-Cauchy sequence in $(X, D)$. By completeness of $(X, D)$ there exists $\omega \in X$ such that $\lim_{n \to \infty} f^n(x_0) = \omega$. Since $f$ is $k$-contraction, we have
\[ \begin{align*}
D(f^{n+1}(x_0), f(\omega)) & \leq k D(f^n(x_0), \omega) \\
D(f(\omega), f^{n+1}(x_0)) & \leq k D(\omega, f^n(x_0)).
\end{align*} \]
Hence $\lim_{n \to \infty} D(f^{n+1}(x_0), f(\omega)) = 0 = \lim_{n \to \infty} D(f(\omega), f^{n+1}(x_0)) = 0$. So $f(\omega)$ is another $D$-limit for the sequence $\{f^n(x_0)\}$. By the uniqueness of the limit in a quasi-$JS$-metric space (Proposition 4) we have $\omega = f(\omega)$. Now, if $\omega'$ is another fixed point of $f$ with $D(\omega, \omega') < \infty$, then, as $f$ is $k$-contraction, we have
\[ D(\omega, \omega') = D(f(\omega), f(\omega')) \leq k D(\omega, \omega'), \]
which implies that $D(\omega, \omega') = 0$. Similarly, using the fact that $\limsup_{n \to \infty} D(\omega, \omega) < \infty$, we can prove that $D(\omega', \omega) = 0$. Therefore, $\omega = \omega'$.

Since any quasi-metric space and any quasi-$b$-metric space is a quasi-$JS$-metric space, we derive the following results.

**Corollary 27.** Let $(X, q)$ be a complete quasi-metric space and let $f : X \to X$ be $k$-contraction mapping for some $k \in (0, 1)$. Suppose that there exists $x_0 \in X$ such that
\[ \sup \{q(f^i(x_0), f^j(x_0)) : i, j \in \mathbb{N}_0\} < \infty. \]
Then $f$ has a unique fixed point $\omega \in X$. Moreover, the sequence $\{f^n(x_0)\}$ converges to $\omega$.

**Corollary 28.** Let $(X, d)$ be a complete quasi-$b$-metric space and let $f : X \to X$ be $k$-contraction mapping for some $k \in (0, 1)$. Suppose that there exists $x_0 \in X$ such that
\[ \sup \{d(f^i(x_0), f^j(x_0)) : i, j \in \mathbb{N}_0\} < \infty. \]
Then $f$ has a unique fixed point $\omega \in X$. Moreover, the sequence $\{f^n(x_0)\}$ converges to $\omega$.

We can obtain a similar result in the context of complete dislocated quasi-metric spaces.

### 3. Ćirić Type Contraction in a Quasi-$JS$-Metric Space

In this section, we consider the existence and uniqueness of fixed point for Ćirić type contraction in the setting of quasi-$JS$-metric space.

**Definition 29.** Let $f : X \to X$ be a function and $k \in (0, 1)$. We say that $f$ is a generalized $k$-quasi-contraction mapping if it satisfies the following condition:
\[ (Q) : D(f(x), f(y)) \leq k \max \{D(x, y), D(y, f(y)), D(x, f(y)) \}, \]
for every $x, y \in X$. 

**Proposition 30.** Suppose that $f$ is a generalized $k$-quasi-contraction mapping for some $k \in (0, 1)$. If $f$ has a fixed point $\omega \in X$ with $D(\omega, \omega) < \infty$, then $D(\omega, \omega) = 0$.

**Theorem 31.** Let $(X, D)$ be a $D$-complete quasi-$JS$-metric space with constant $C$ and let $f : X \to X$ be a generalized $k$-quasi-contraction mapping for some $k \in (0, 1)$. Suppose that there exists $x_0 \in X$ such that $\delta(D, f, x_0) < \infty$. Then $\{f^n(x_0)\}$ converges to some $\omega \in X$. If $D(x_0, f(\omega)) < \infty$, $D(\omega, f(\omega)) < \infty$, and $C < 1$, then $\omega$ is a fixed point of $f$. Moreover, if $\omega'$ is another fixed point of $f$ with $D(\omega, \omega') < \infty$, $D(\omega', \omega') < \infty$, and $D(\omega, \omega') < \infty$, then $\omega = \omega'$.

**Proof.** Let $n \in \mathbb{N}$, since $f$ is generalized $k$-quasi-contraction mapping, for all $i, j \in \mathbb{N}_0$; we have
\[ \begin{align*}
D(f^n(x_0), f^{n+i}(x_0)) & \leq k^i \max \{D(f^{n-i}(x_0), f^{n-i+j}(x_0)) \}, \\
D(f^{n-i}(x_0), f^{n+i}(x_0)) \leq k \max \{D(f^{n-i-1}(x_0), f^{n-i+j}(x_0)) \}, \\
D(f^{n-i-1}(x_0), f^{n+i}(x_0)) \leq \max \{D(f^{n-i-2}(x_0), f^{n-i+j}(x_0)) \}, \\
D(f^{n-i-2}(x_0), f^{n+i}(x_0)) \leq \max \{D(f^{n-i-3}(x_0), f^{n-i+j}(x_0)) \}, \\
D(f^{n-i-3}(x_0), f^{n+i}(x_0)) \leq \max \{D(f^{n-i-4}(x_0), f^{n-i+j}(x_0)) \}.
\end{align*} \]
As
\[ \begin{align*}
D(f^n(x_0), f^{n+1}(x_0)) & \leq k^i \max \{D(f^{n-i}(x_0), f^{n-i+j}(x_0)) \}, \\\nD(f^{n-i}(x_0), f^{n+i}(x_0)) & \leq k \max \{D(f^{n-i-1}(x_0), f^{n-i+j}(x_0)) \}, \\
D(f^{n-i-1}(x_0), f^{n+i}(x_0)) & \leq \max \{D(f^{n-i-2}(x_0), f^{n-i+j}(x_0)) \}, \\
D(f^{n-i-2}(x_0), f^{n+i}(x_0)) & \leq \max \{D(f^{n-i-3}(x_0), f^{n-i+j}(x_0)) \}, \\
D(f^{n-i-3}(x_0), f^{n+i}(x_0)) & \leq \max \{D(f^{n-i-4}(x_0), f^{n-i+j}(x_0)) \}.
\end{align*} \]
we have
\[
\max \{D(f^{n-1+i}(x_0), f^{n-1+i}(x_0)), \\
D(f^{n-1+i}(x_0), f^{n+i}(x_0)), \\
D(f^{n-1+i}(x_0), f^{n+i}(x_0)), \\
D(f^{n-1+i}(x_0), f^{n+i}(x_0)), \\
D(f^{n-1+i}(x_0), f^{n+i}(x_0))\}
\]
for every $n \in \mathbb{N}_0$.

Hence for any $n \in \mathbb{N}_0$ we have
\[
\delta(D(f, f, f^n(x_0))) \leq k^n \delta(D(f, f, f(x_0))).
\]
Then for every $m, n \in \mathbb{N}_0$ we find
\[
D(f^n(x_0), f^{n+m}(x_0)) \leq \delta(D, f, f^n(x_0))
\]
\[
\leq k^n \delta(D, f, x_0),
\]
\[
D(f^{n+m}(x_0), f^n(x_0)) \leq \delta(D, f, f^n(x_0))
\]
\[
\leq k^n \delta(D, f, x_0).
\]

Since $\delta(D, f, x_0) < \infty$ and $k < 1$, we derive
\[
\lim_{m, n \to \infty} D(f^n(x_0), f^{n+m}(x_0)) = \lim_{m, n \to \infty} D(f^n(x_0), f^{n}(x_0)) = 0.
\]

This implies that $\{f^n(x_0)\}$ is both left and right $D$-Cauchy sequence and hence $D$-Cauchy sequence. By completeness of $X$ there exists some $\omega \in X$ such that $\{f^n(x_0)\}$ converges to $\omega$; that is, $\lim_{n \to \infty} D(f^n(x_0), \omega) = \lim_{n \to \infty} D(\omega, f^n(x_0)) = 0$.

Note that as in (44) for any $m, n \in \mathbb{N}_0$ we have
\[
D(f^n(x_0), f^m(x_0)) \leq \delta(D, f, f^n(x_0))
\]
\[
\leq k^n \delta(D, f, x_0),
\]
\[
D(f^m(x_0), f^n(x_0)) \leq \delta(D, f, f^n(x_0))
\]
\[
\leq k^n \delta(D, f, x_0).
\]

Now, assume that $D(\omega, f(\omega)) < \infty$ and $D(x_0, f(\omega)) < \infty$. Then, by using (46) and condition $(D_2)$ there exists $C > 0$ such that
\[
D(\omega, f^n(x_0)) \leq C \limsup_{m \to \infty} D(f^m(x_0), f^n(x_0))
\]
\[
\leq C k^n \delta(D, f, x_0),
\]
\[
D(f^n(x_0), \omega) \leq C \limsup_{m \to \infty} D(f^m(x_0), f^n(x_0))
\]
\[
\leq C k^n \delta(D, f, x_0)
\]

for every $n \in \mathbb{N}_0$.

On the other hand, as $f$ is generalized $k$-quasi-contraction mapping we have
\[
D(f(x_0), f(\omega)) \leq k \max \{D(x_0, \omega), D(x_0, f(x_0)),
D(\omega, f(\omega)), D(f(x_0), \omega), D(x_0, f(\omega))\}.
\]

By using (46) and (47) we have
\[
D(x_0, \omega) \leq Ck^0 \delta(D, f, x_0) = C \delta(D, f, x_0),
\]
\[
D(x_0, f(x_0)) \leq k^0 \delta(D, f, x_0) = \delta(D, f, x_0),
\]
\[
D(f(x_0), \omega) \leq Ck \delta(D, f, x_0) < C \delta(D, f, x_0).
\]

Hence, we derive
\[
D(f(x_0), f(\omega)) \leq \max \{kC \delta(D, f, x_0),
\]
\[
k \delta(D, f, x_0), k \delta(\omega, f(\omega)), k \delta(x_0, f(\omega))\}.
\]

Again by using the fact that $f$ is generalized $k$-quasi-contraction mapping and the technique used above, we observe that
\[
D(f^2(x_0), f(\omega)) \leq \max \{k^2 C \delta(D, f, x_0),
\]
\[
k^2 \delta(D, f, x_0), k \delta(\omega, f(\omega)), k^2 \delta(x_0, f(\omega))\}.
\]

By continuing in the same manner, we deduce
\[
D(f^n(x_0), f(\omega)) \leq \max \{k^n C \delta(D, f, x_0),
\]
\[
k^n \delta(D, f, x_0), k \delta(\omega, f(\omega)), k^n \delta(x_0, f(\omega))\},
\]

for every $n \in \mathbb{N}$. Now, as $D(x_0, f(\omega)) < \infty$ and $D(D, f, x_0) < \infty$, we have
\[
\limsup_{n \to \infty} D(f^n(x_0), f(\omega)) \leq k D(\omega, f(\omega)).
\]

Regarding the condition $(D_3)$ and the fact that $C < 1$ and $k < 1$, we get
\[
D(\omega, f(\omega)) \leq C \limsup_{n \to \infty} D(f^n(x_0), f(\omega))
\]
\[
\leq Ck D(\omega, f(\omega)) < D(\omega, f(\omega)).
\]

Thus, $D(\omega, f(\omega)) = 0$. By analogy, as in the above, we can conclude that
\[
D(f(\omega), \omega) = D(\omega, f(\omega)) = 0.
\]

Hence, we have $D(\omega, f(\omega)) = D(f(\omega), \omega) = 0$. Therefore, we get $\omega = f(\omega)$; that is, $\omega$ is a fixed point of $f$.

Now, if $\omega'$ is another fixed point of $f$ with $D(\omega', \omega) < \infty$, $D(\omega, \omega') < \infty$, and $D(\omega', \omega) < \infty$, then as $f$ is generalized $k$-quasi-contraction mapping we have
\[
D(\omega', \omega) = D(f(\omega'), f(\omega)) \leq k \cdot \max \{D(\omega', \omega), D(\omega', \omega'), D(\omega, \omega), D(\omega, \omega')\}.
\]
as $\mathcal{D}(f(\omega), \omega) = \mathcal{D}(\omega, \omega) < \infty$ and $\mathcal{D}(\omega', \omega') < \infty$. So, by Proposition 30, we have
\[ \mathcal{D}(\omega, \omega) = \mathcal{D}(\omega', \omega') = 0, \tag{57} \]
which implies that $\mathcal{D}(\omega', \omega) \leq k \max\{\mathcal{D}(\omega', \omega), \mathcal{D}(\omega, \omega')\}$. In a similar way, we derive that
\[ \mathcal{D}(\omega, \omega') \leq k \max\{ \mathcal{D}(\omega', \omega), \mathcal{D}(\omega, \omega') \}, \tag{58} \]
so that
\[ \max\{ \mathcal{D}(\omega', \omega), \mathcal{D}(\omega, \omega') \} \leq k \max\{ \mathcal{D}(\omega', \omega), \mathcal{D}(\omega, \omega') \}. \tag{59} \]
Since $\mathcal{D}(\omega, \omega') < \infty$, $\mathcal{D}(\omega', \omega) < \infty$, and $k \in (0, 1)$, we deduce that $\mathcal{D}(\omega, \omega') = \mathcal{D}(\omega, \omega') = 0$ which yields $\omega = \omega'$.

\section*{4. Fixed Point Theorems in Quasi-JS-Metric Space with Partial Order}

\textbf{Definition 32.} Let $(X, \mathcal{D})$ be a quasi-JS-metric space with partial order $\leq$ and let $f : X \rightarrow X$ be a mapping. We say that $f$ is weakly continuous if the following condition holds: if $\{x_n\} \subseteq X$ is $\mathcal{D}$-convergent to $x \in X$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{f(x_{n_k})\}$ is $\mathcal{D}$-convergent to $f(x)$ as $k \rightarrow \infty$.

\textbf{Definition 33.} Let $X$ be a nonempty set with partial order $\leq$. A mapping $f : X \rightarrow X$ is said to be nondecreasing if
\[ x \leq y \implies f(x) \leq f(y) \quad \text{where } x, y \in X. \tag{60} \]

\textbf{Definition 34.} The pair $(X, \leq)$ is said to be $\mathcal{D}$-left regular (resp., $\mathcal{D}$-right regular) if the following condition holds: for every sequence $\{x_n\} \subseteq X$ such that $x_{n+1} \leq x_n (x_n \leq x_{n+1})$ for each $n \in \mathbb{N}$, with $\{x_n\}$ being $\mathcal{D}$-convergent to $x \in X$; then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x \leq x_{n_k}$ ($x_{n_k} \leq x$) for every $k \in \mathbb{N}$.

The pair $(X, \leq)$ is said to be $\mathcal{D}$-regular if and only if it is left and right $\mathcal{D}$-regular.

\textbf{Definition 35.} A function $f : X \rightarrow X$ is said to be weakly $k$-contraction if $x \leq y$ or $y \leq x$ implies
\[ \mathcal{D}(f(x), f(y)) \leq k \mathcal{D}(x, y). \tag{61} \]
That is, whenever $x, y \in X$ are comparable condition (61) is satisfied.

\textbf{Theorem 36.} Let $(X, \mathcal{D})$ be a quasi-JS-metric space with partial order $\leq$ and let $f : X \rightarrow X$ be a function. Assume that the following conditions hold:

\begin{enumerate}[(i)]
  \item $(X, \mathcal{D})$ is $\mathcal{D}$-complete;
  \item $f$ is weakly continuous;
  \item $f$ is weakly $k$-contraction for some $k \in (0, 1)$;
  \item $f$ is nondecreasing;
  \item there exists $x_0 \in X$ such that $\mathcal{D}(f(x), f(x_0)) < \infty$ and $x_0 \leq f(x_0)$. \tag{62}
\end{enumerate}

Then $f$ has a fixed point $\omega \in X$ and $\{f^n(x_0)\}$ is $\mathcal{D}$-convergent to $\omega$. Moreover, if $\mathcal{D}(\omega, \omega) < \infty$, then $\mathcal{D}(\omega, \omega) = 0$.

\textbf{Proof.} Since $f$ is nondecreasing and $x_0 \leq f(x_0)$, then for each $n \in \mathbb{N}_0$ we have
\[ f^n(x_0) \leq f^{n+1}(x_0), \tag{63} \]
and by the transitivity of $\leq$, for every $p, q \in \mathbb{N}_0$, we have
\[ p \leq q \implies f^p(x_0) \leq f^q(x_0). \tag{64} \]
Therefore, for each $n \in \mathbb{N}_0$, $f^{n+i}(x_0)$ and $f^{n+i+1}(x_0)$ are always comparable. As $f$ is weak $k$-contraction for each $n \in \mathbb{N}_0$, $i, j \in \mathbb{N}_0$, we have
\[ \mathcal{D}\left(f^{n+i}(x_0), f^{n+i+1}(x_0)\right) \leq k \mathcal{D}\left(f^{n+i-1}(x_0), f^{n+i}(x_0)\right). \tag{65} \]
Hence
\[ \mathcal{D}(f^n(x_0), f^{n+m}(x_0)) \leq k^m \mathcal{D}(f(x_0), x_0) \tag{66} \]
Using the above inequality we have for every $m, n \in \mathbb{N}_0$
\[ \mathcal{D}(f^n(x_0), f^{n+m}(x_0)) \leq k^m \mathcal{D}(f(x_0), x_0), \tag{67} \]
which means that $\{f^n(x_0)\}$ is right and left $\mathcal{D}$-Cauchy and hence $\mathcal{D}$-Cauchy sequence. By the completeness of $(X, \mathcal{D})$ there exists $\omega \in X$ such that $\{f^n(x_0)\}$ is $\mathcal{D}$-convergent to $\omega$. Since $f$ is weakly continuous, there exists a subsequence $\{f^{n_k}(x_0)\}$ of $\{f^n(x_0)\}$ such that $\{f^{n_k+1}(x_0)\}$ is $\mathcal{D}$-convergent to $f(\omega)$ as $k \rightarrow \infty$. By the uniqueness of the limit in a quasi-JS-metric space we have $\omega = f(\omega)$ and $\omega$ is a fixed point of $f$.

Now, if $\mathcal{D}(\omega, \omega) < \infty$ then as $\omega \leq \omega$ and $f$ is weak $k$-contraction we have
\[ \mathcal{D}(\omega, \omega) = \mathcal{D}(f(\omega), f(\omega)) \leq k \mathcal{D}(\omega, \omega), \tag{69} \]
which is possible only if $\mathcal{D}(\omega, \omega) = 0$. □
The weak continuity condition of $f$ in the previous theorem can be replaced by the regularity of the pair $(X, \preceq)$ as in the following result.

**Theorem 37.** Let $(X, \mathcal{D})$ be a quasi-$JS$-metric space with partial order $\preceq$ and let $f : X \to X$ be a function. Assume that the following conditions hold:

(i) $(X, \mathcal{D})$ is $\mathcal{D}$-complete;

(ii) $(X, \preceq)$ is regular;

(iii) $f$ is weakly $k$-contraction for some $k \in (0, 1)$;

(iv) $f$ is nondecreasing;

(v) there exists $x_0 \in X$ such that $\delta(\mathcal{D}, f, x_0) < \infty$ and $x_0 \preceq f(x_0)$.

Then $f$ has a fixed point $\omega \in X$ and $\{f^n(x_0)\}$ is $\mathcal{D}$-convergent to $\omega$. Moreover, if $\mathcal{D}(\omega, \omega) < \infty$, then $\mathcal{D}(\omega, \omega) = 0$.

**Proof.** Following the steps of the previous proof we can prove that $\{f^n(x_0)\}$ is $\mathcal{D}$-convergent to $\omega \in X$. Moreover, we have

$$f^n(x_0) \leq f^{n+1}(x_0)$$

(70)

for every $n \in \mathbb{N}_0$. Since $(X, \preceq)$ is regular it is right regular and so there exists a subsequence $\{f^{n_k}(x_0)\}$ of $\{f^n(x_0)\}$ such that $f^{n_k}(x_0) \preceq \omega$ for each $k \in \mathbb{N}$. As $f$ is weakly $k$-contraction we have

$$\mathcal{D}\left(f^{n_k+1}(x_0), f(\omega)\right) \leq \mathcal{D}\left(f^{n_k}(x_0), \omega\right).$$

(71)

Using the inequality above, we get

$$\lim_{k \to \infty} \mathcal{D}\left(f^{n_k+1}(x_0), f(\omega)\right) = 0.$$ 

(72)

Similarly, we can prove that

$$\lim_{k \to \infty} \mathcal{D}\left(f(\omega), f^{n_k+1}(x_0)\right) = 0,$$

(73)

which implies that $f(\omega)$ is a $\mathcal{D}$-limit for the sequence $\{f^{n_k}(x_0)\}$. By Proposition 4, $f(\omega) = \omega$; and $\omega$ is a fixed point of $f$. As in the previous proof

$$\mathcal{D}(\omega, \omega) < \infty \implies \mathcal{D}(\omega, \omega) = 0.$$ 

(74)

Example 38. Let $(X, \mathcal{D})$ be the complete quasi-$JS$-metric space introduced in Example 7. Define the function $f : X \to X$ by

$$f(x) = \begin{cases} 
\infty & \text{if } x = \infty \\
x/2 & \text{otherwise.}
\end{cases}$$

(75)

Then clearly $f$ is a $k$-contraction mapping with $k = 1/2$. Let $x_0 = 0$ then $\delta(\mathcal{D}, f, x_0) < \infty$. So by Theorem 26 $f$ has a fixed point.