Research Article

A New Approach to the Study of Fixed Point Theorems with w -Distances via R -Functions

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We introduce some new generalization of fixed point theorems in complete metric spaces endowed with w -distances via R -functions. Our results extend many of known fixed point theorems such as Reich type contraction, Geraghty contraction, Meir-Keeler contraction, and $\mathcal{F}$ -contraction. In addition, the result and corollaries show that our approach has a constructive attitude and many known and unknown results can be constructed in such way.

1. Introduction

Recently, extensive researches have been performed on various topics in different spaces equipped w -distances, for example, fuzzy spaces [1], polish spaces [2], quasimetric spaces [3], metric spaces endowed with a graph [4], TVS-Cone metric spaces [5–8], partially ordered metric spaces [9–15], and other related papers such as [16, 17].

In 2014, Du and Khojasteh [18] introduced a very strong class of mappings called *manageable function* of which many of known contractions in the set of multivalued mappings have closely related to such class and obtain from them.

Moreover, the notion of *simulation function* was introduced by Khojasteh et al. [19]. They introduced $\mathcal{F}$ -contraction with respect to simulation functions. The $\mathcal{F}$ -contraction generalizes the Banach contraction and unifies several known types of contractions in complete metric space (X, d) involving the combination of $d(Tx, Ty)$ and $d(x, y)$ and satisfies some particular conditions in such spaces.

Very recently, in order to extend the previous families of functions, Roldán López de Hierro and Shahzad [20] introduced a new generalized class of mappings called R -functions and a new contraction called R -contraction and

show that every Geraghty contraction, Meir-Keeler contraction, $\mathcal{F}$ -contraction, $\widehat{Man}(\mathbb{R})$ -contraction, and many of other known contractions are special case of R -contraction.

In this work, we introduce some new generalization of fixed point theorems in complete metric spaces endowed with w -distances via R -functions and many of known fixed point theorems such as Reich type contraction, Geraghty contraction [9], Meir-Keeler contraction [21], and $\mathcal{F}$ -contraction [19] which in such spaces are concluded as apparent corollaries.

2. Preliminaries

In the following, we assert some notions which are needed in continuance.

Definition 1 (see [22]). Let (X, d) be a metric space. The mapping $p : X \times X \rightarrow [0, 1]$ is said to be w -distance on X , if it satisfies the following axioms:

- (1) $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$;
- (2) for any $x \in X$, $p(x, \cdot) : X \rightarrow [0, 1]$ is lower semi-continuous;

(3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ implies $d(x, y) < \epsilon$.

Denote $\text{ran}(p) = \{p(x, y) : x, y \in X\}$.

Lemma 2 (see [22]). *Let (X, d) be a metric space and p be w -distance on X . Assume that $\{x_n\}, \{y_n\}$ are sequences in X and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1)$ which are convergent to 0 and $x, y \in X$. Then the following holds:*

- (i) if $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$, for any $n \in \mathbb{N}$, then $y = z$;
- (ii) if $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$, for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z ;
- (iii) if $p(x_n, x_m) \leq \alpha_n$, for any $m, n \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;
- (iv) if $p(y, x_n) \leq \alpha_n$, for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

Definition 3 (see [23]). A symmetric w -distance on a metric space (X, d) is w -distance p on (X, d) such that

$$(SY): q(x, y) = q(y, x) \text{ for all } x, y \in X.$$

If p is w -distance satisfying (SY), we then say that it is a symmetric w -distance on (X, d) .

Definition 4 (see [24]). A function $\phi : [0, \infty) \rightarrow [0, \infty)$ is called L -function if

- (a) $\phi(0) = 0$,
- (b) $\phi(t) > 0$ for all $t > 0$,
- (c) for all $\epsilon > 0$, there exists $\delta > 0$ such that $\phi(t) \leq \epsilon$, for all $t \in [\epsilon, \epsilon + \delta]$.

Moreover, for each $\epsilon > 0$ we have $\phi(\epsilon) \leq \epsilon$.

Definition 5 (see [17, 21]). A mapping $T : X \rightarrow X$, from a metric space (X, d) into itself, is called Meir-Keeler mapping, whenever, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in X$, if $\epsilon \leq d(x, y) < \epsilon + \delta$, then $d(Tx, Ty) < \epsilon$.

Meir and Keeler [21] proved that every Meir-Keeler mapping on a complete metric space has a unique fixed point.

Definition 6 (see [19]). A simulation function is a mapping $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ which satisfies the following conditions:

- (ζ_1) $\zeta(t, s) < s - t$ for all $t, s > 0$,
 - (ζ_2) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} s_n > 0$, then
- $$\limsup_{n \rightarrow \infty} \zeta(t_n, s_n) < 0. \quad (1)$$

Many examples of simulation functions can be found in [18, 19, 25]. Very recently, Roldán López de Hierro and Shahzad [20] introduced the following category of mappings which are connected to extend the simulation functions.

Definition 7. Let $A \subseteq \mathbb{R}$ be a nonempty subset and let $\varrho : A \times A \rightarrow \mathbb{R}$ be a function. We say that ϱ is R -function if it satisfies the following two conditions:

- (ϱ_1) If $\{a_n\} \subset (0, \infty) \cap A$ is a sequence such that $\varrho(a_{n+1}, a_n) > 0$, for all $n \in \mathbb{N}$, then $\lim_{n \rightarrow \infty} a_n = 0$.
- (ϱ_2) If $\{a_n\}, \{b_n\} \subset (0, \infty) \cap A$ are two sequences which converge to the same limit $L \geq 0$ such that $L < a_n$ and $\varrho(a_n, b_n) > 0$, for all $n \in \mathbb{N}$, then $L = 0$.

We denote by R_A the family of all R -functions whose domain is $A \times A$.

In some cases, for a given function $\varrho : A \times A \rightarrow \mathbb{R}$, we will also consider the following property:

- (ϱ_3) If $\{a_n\}, \{b_n\} \subset (0, \infty) \cap A$ are two sequences such that $\lim_{n \rightarrow \infty} b_n = 0$ and $\varrho(a_n, b_n) > 0$, for all $n \in \mathbb{N}$, then $\lim_{n \rightarrow \infty} a_n = 0$.

In the present research, we will introduce our main result and demonstrate that many of known results which have been invented in the set of metric spaces, with or without w -distances, can be deduced easily from our main result. We draw the respondents' attention to Alegre et al. [3, Theorem 2, Corollary 1] about Meir-Keeler contraction in metric spaces with w -distances which can be acquired and generalized with Theorem 9 (for more detail see Corollary 13).

3. Main Result

The following definition plays the crucial role in our main result.

Definition 8. Let (X, d) be a metric space, let p be w -distance on X , and let $T : X \rightarrow X$ be a mapping. We say that T is (R_p) -contraction, if there exists R -function $\varrho : A \times A \rightarrow \mathbb{R}$ such that $\text{ran}(p) \subseteq A$ and

$$\varrho(p(Tx, Ty), p(x, y)) > 0, \quad (2)$$

for all $x, y \in X$. In such case, we say that T is R_p -contraction with respect to ϱ . The family of all R_p -contractions from (X, d) into itself, with respect to ϱ , is denoted by $R_{X,p,A,\varrho}$ or, when no confusion is possible, by $R_{\varrho,p}$.

In what follows, we assume that p is a symmetric w -distance, in the sense of Definition 3, unless the contrary is asserted. Now we are ready to give the main result.

Theorem 9. *Let (X, d) be a complete metric space, let p be w -distance on X , and let $T : X \rightarrow X$ be R_p -contraction with respect to $\varrho \in R_A$. Assume that, at least, one of the following conditions holds:*

- (a) T is p -continuous; that is, for any sequence $\{x_n\}$, if $p(x_n, z) \rightarrow 0$ then $p(Tx_n, Tz) \rightarrow 0$ as $n \rightarrow \infty$.
- (b) The function ϱ satisfies (ϱ_3).

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be chosen arbitrary. If $x_0 = Tx_0$, then we have nothing to prove. So let $x_0 \neq Tx_0$. Define $x_1 = Tx_0$. Similarly one can define the Picard sequence $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$, such that $x_{n+1} \neq x_n$. Let $a_n = p(x_n, x_{n+1}) > 0$ and divide the proof into three steps.

Step 1. $\{x_n\}$ is p -asymptotically regular; that is, $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0$. T is R_p -contraction with respect to $\varrho \in R_A$. Thus

$$\begin{aligned} \varrho(a_{n+1}, a_n) &= \varrho(p(x_{n+1}, x_{n+2}), p(x_n, x_{n+1})) \\ &= \varrho(p(Tx_n, Tx_{n+1}), p(x_n, x_{n+1})) > 0. \end{aligned} \quad (3)$$

By (ϱ_1) of Definition 7 we have $\lim_{n \rightarrow \infty} a_n = 0$ and desired result is obtained.

Step 2. $\{x_n\}$ is a Cauchy sequence and so it converges to some $z \in X$.

On the contrary, assume that $\{x_n\}$ failed to be a Cauchy sequence. By (iii) of Lemma 2, $\{x_n\}$ failed to be p -Cauchy sequence. Hence, there exists $\epsilon_0 > 0$ such that, for all $k \in \mathbb{N}$, there exist subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ and $m(k)$ can be chosen as the smallest integer such that $m(k) > n(k) \geq k$ and for all $s \in \{m(k) - 1, \dots, n(k), n(k) - 1, \dots, 1\}$

$$p(x_{n(k)}, x_s) \leq \epsilon_0 < p(x_{n(k)}, x_{m(k)}). \quad (4)$$

Obviously, $\lim_{k \rightarrow \infty} p(x_{n(k)}, x_{m(k)}) = \epsilon_0$, because

$$\begin{aligned} \epsilon_0 &< p(x_{n(k)}, x_{m(k)}) \\ &\leq p(x_{n(k)}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) \\ &\leq p(x_{m(k)-1}, x_{m(k)}) + \epsilon_0. \end{aligned} \quad (5)$$

Also, applying the symmetry property we have

$$\begin{aligned} \epsilon_0 &< p(x_{n(k)}, x_{m(k)}) \\ &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)-1}) \\ &\quad + p(x_{m(k)-1}, x_{m(k)}) \\ &\leq 2p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)}, x_{m(k)-1}) \\ &\quad + p(x_{m(k)-1}, x_{m(k)}) \\ &\leq 2p(x_{n(k)}, x_{n(k)-1}) + \epsilon_0 + p(x_{m(k)-1}, x_{m(k)}). \end{aligned} \quad (6)$$

Therefore, $\lim_{k \rightarrow \infty} p(x_{n(k)-1}, x_{m(k)-1}) = \epsilon_0$. Now, taking $L = \epsilon_0$, $a_k = p(x_{n(k)}, x_{m(k)})$, and $b_k = p(x_{n(k)-1}, x_{m(k)-1})$ and by the fact that $L = \epsilon_0 < p(x_{n(k)}, x_{m(k)}) = a_k$, we have

$$\begin{aligned} \varrho(a_k, b_k) &= \varrho(p(x_{n(k)}, x_{m(k)}), p(x_{n(k)-1}, x_{m(k)-1})) \\ &= \varrho(p(Tx_{n(k)-1}, Tx_{m(k)-1}), p(x_{n(k)-1}, x_{m(k)-1})) \\ &> 0. \end{aligned} \quad (7)$$

By (ϱ_2) of Definition 7, we have $L = \epsilon_0 = 0$ and this is a contradiction. Therefore, $\{x_n\}$ is p -Cauchy sequence and by (iii) of Lemma 2 is Cauchy. Thus it converges to some $z \in X$.

Step 3. z is a unique fixed point and $p(z, z) = 0$.

If we consider a sequence $\{\alpha_n\}$, in a sense of Lemma 2, $\{x_n\}$ is p -Cauchy sequence; that is, for all $m, n \in \mathbb{N}$, $m > n$, $p(x_n, x_m) \leq \alpha_n$. Also $x_n \rightarrow z$ as $n \rightarrow \infty$ and $p(x, \cdot)$ is a lower semicontinuous, for all $x \in X$. Hence

$$p(z, x_n) \leq \liminf_{m \rightarrow \infty} p(x_m, x_n) \leq \alpha_n. \quad (8)$$

It means that $p(z, x_n) \rightarrow 0$ as $n \rightarrow \infty$.

Now suppose that (a) holds. Since T is p -continuous mapping, $\lim_{n \rightarrow \infty} p(Tx_n, Tz) = 0$ and for all $x \in X$, specially $x = z$ and $x = Tz$, we have

$$p(z, z) \leq \liminf_{n \rightarrow \infty} p(z, x_{n+1}) = 0, \quad (9)$$

and so

$$\begin{aligned} p(z, Tz) &\leq \liminf_{n \rightarrow \infty} p(x_{n+1}, Tz) = \liminf_{n \rightarrow \infty} p(Tx_n, Tz) \\ &= 0. \end{aligned} \quad (10)$$

Therefore, $p(z, Tz) = p(z, z) = 0$. By (i) of Lemma 2, we have $Tz = z$ and so z is the fixed point of T .

On the other hand, let (b) hold. For all $n \in \mathbb{N}$, suppose that $a_n = p(x_{n+1}, Tz)$ and $b_n = p(x_n, z)$. By (8), we have $\lim_{n \rightarrow \infty} b_n = 0$. In addition, for all $n \in \mathbb{N}$,

$$\begin{aligned} \varrho(a_n, b_n) &= \varrho(p(x_{n+1}, Tz), p(x_n, z)) \\ &= \varrho(p(Tx_n, Tz), p(x_n, z)) > 0. \end{aligned} \quad (11)$$

Henceforth, if $b_n = 0$, then $x_n = z$. So $x_{n+1} = Tx_n = Tz$; thus $a_n = 0$.

Now consider the following set:

$$\Omega = \{n \in \mathbb{N} : a_n = 0\} = \{n \in \mathbb{N} : p(x_{n+1}, Tz) = 0\}. \quad (12)$$

One can consider two cases:

(a₁) Ω is a finite set: in this case, one can choose $n_0 \in \mathbb{N}$, such that $a_n > 0$, for all $n > n_0$, and so $b_n > 0$, for all $n > n_0$. Applying (11) concludes that $\lim_{n \rightarrow \infty} a_n = 0$. It means that $\lim_{n \rightarrow \infty} p(x_{n+1}, Tz) = 0$ and so applying (i) of Lemma 2 yields $Tz = z$.

(a₂) Ω is infinite set: in this case, one can find a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ in Ω . We have

$$p(x_{n(k)+1}, Tz) = 0, \quad (13)$$

for all $k \in \mathbb{N}$. Analogous the argument in the first case and applying (i) of Lemma 2 results $Tz = z$ again.

Moreover, if x, y are two distinct fixed points for T , then by considering $a_n = p(x, x) \geq 0$ we have

$$\begin{aligned} \varrho(a_{n+1}, a_n) &= \varrho(p(x, x), p(x, x)) \\ &= \varrho(p(Tx, Tx), p(x, x)) > 0. \end{aligned} \quad (14)$$

Applying (ϱ_1) deduces that $\lim_{n \rightarrow \infty} a_n = 0$ and so $p(x, x) = 0$. Applying the same argument on $b_n = p(x, y)$ concludes that $p(x, y) = 0$ and (i) of Lemma 2 eventuates $x = y$. $\square$

Example 10. The following items are some examples of R -functions:

- (a) $\varrho(t, s) = s\varphi(s) - t$, where $\varphi : [0, +\infty) \rightarrow [0, 1)$ is a mapping such that $\limsup_{t \rightarrow s^+} \varphi(t) < 1$, for all $s \in (0, +\infty)$,
- (b) $\varrho(t, s) = s\varphi(s) - t$, where $\varphi : [0, +\infty) \rightarrow [0, 1)$ is a mapping such that, for all sequence $\{t_n\} \subset [0, +\infty)$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \varphi(t_n) &= 1 \\ \text{implies } \lim_{n \rightarrow \infty} t_n &= 0, \end{aligned} \quad (15)$$

- (c) $\varrho(t, s) = \varphi(s) - t$, where $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ is L -function in the sense of Definition 4,
- (d) every simulation function $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$,
- (e) $\varrho_{\varphi, \psi}(t, s) = \psi(s) - \varphi(s) - \psi(t)$, where $\psi, \varphi : [0, \infty) \rightarrow [0, \infty)$ are two functions such that ψ is nondecreasing and continuous from the right, φ is lower semi-continuous, and $\varphi^{-1}(\{0\}) = \{0\}$,
- (f) $\varrho(t, s) = s/(t+1) - t$,
- (g) $\varrho(t, s) = s/e^t - t$,
- (h) $\varrho(t, s) = \ln(s+1) - t$.

Proof. It is sufficient to show that (ϱ_1) and (ϱ_2) of Definition 7 and (ϱ_3) hold for all (a)–(h).

(a)

(ϱ_1) Let $\{a_n\} \subset (0, +\infty)$ be a sequence and $\varrho(a_{n+1}, a_n) > 0$. Then

$$0 < \varrho(a_{n+1}, a_n) = a_n \varphi(a_n) - a_{n+1}. \quad (16)$$

It means that

$$\frac{a_{n+1}}{a_n} < \varphi(a_n), \quad (17)$$

and so taking limit on both sides of (17), we have

$$\limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < \limsup_{n \rightarrow \infty} \varphi(a_n) < 1. \quad (18)$$

By ratio test, $\sum_{n=1}^{\infty} a_n < \infty$ and then concludes that $\lim_{n \rightarrow \infty} a_n = 0$.

(ϱ_2) Let $\{a_n\}, \{b_n\} \subset (0, +\infty)$ be two sequences which converge to the same limit $L \geq 0$, $L < a_n$, and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$. Suppose that $L > 0$; then $b_n \varphi(b_n) - a_n > 0$ and so

$$\frac{a_n}{b_n} < \varphi(b_n). \quad (19)$$

By taking limit on both sides of (19), we have

$$1 = \limsup_{n \rightarrow \infty} \frac{a_n}{b_n} \leq \limsup_{n \rightarrow \infty} \varphi(b_n) < 1, \quad (20)$$

which is a contradiction. Thus, $L = 0$.

(ϱ_3) Let $\{a_n\}, \{b_n\} \subset (0, +\infty)$ be two sequences such that $\lim_{n \rightarrow \infty} b_n = 0$ and $\varrho(a_n, b_n) > 0$. Therefore,

$$0 < \varrho(a_n, b_n) = b_n \varphi(b_n) - a_n \leq b_n - a_n, \quad (21)$$

so $0 < a_n < b_n$ which implies that $\lim_{n \rightarrow \infty} a_n = 0$.

(b)

(ϱ_1) Let $\{a_n\} \subset (0, +\infty)$ be a sequence such that $\varrho(a_{n+1}, a_n) > 0$. Therefore,

$$0 < \varrho(a_{n+1}, a_n) = \varphi(a_n) a_n - a_{n+1}. \quad (22)$$

Since $a_n > 0$ and $\varphi(a_n) < 1$, one can see easily

$$a_{n+1} < \varphi(a_n) a_n < a_n. \quad (23)$$

Hence, $\{a_n\}$ is a strictly decreasing sequence of nonnegative real numbers and so converges to $L \geq 0$. On the contrary, assume that $L > 0$. Therefore,

$$0 < L < a_{n+1} < \varphi(a_n) a_n < a_n. \quad (24)$$

Taking limit on both sides of (24) deduces that $\lim_{n \rightarrow \infty} \varphi(a_n) = 1$. Since φ is a Geraghty function, so $L = \lim_{n \rightarrow \infty} a_n = 0$ which contradicts $L > 0$. So $L = 0$.

(ϱ_2) Let $\{a_n\}, \{b_n\} \subset (0, +\infty)$ be two sequences which converge to the same limit $L \geq 0$, $L < a_n$ hold, and $\varrho(a_n, b_n) > 0$ hold. On the contrary, assume that $L > 0$. Therefore,

$$0 < \varrho(a_n, b_n) = \varphi(b_n) b_n - a_n, \quad (25)$$

and so

$$L < a_n < \varphi(b_n) b_n < b_n. \quad (26)$$

Taking limit on both sides of (26) deduces that $\lim_{n \rightarrow \infty} \varphi(b_n) = 1$. Since φ is a Geraghty function, it concludes that $L = \lim_{n \rightarrow \infty} b_n = 0$ which contradicts $L > 0$. So $L = 0$.

(ϱ_3) Let $\{a_n\}, \{b_n\} \subset (0, +\infty)$ be two sequences such that $\lim_{n \rightarrow \infty} b_n = 0$ and $\varrho(a_n, b_n) > 0$. Therefore,

$$0 < \varrho(a_n, b_n) = \varphi(b_n) b_n - a_n \leq b_n - a_n, \quad (27)$$

so $0 < a_n < b_n$ which implies that $\lim_{n \rightarrow \infty} a_n = 0$.

(c)

(ϱ_1) Let $\{a_n\} \subset (0, +\infty)$ be a sequence and let $\varrho(a_{n+1}, a_n) > 0$. Then

$$0 < \varrho(a_{n+1}, a_n) = \varphi(a_n) - a_{n+1}. \quad (28)$$

It means that

$$a_{n+1} < \varphi(a_n) \leq a_n. \quad (29)$$

Therefore, $\{a_n\}$ is a nonincreasing sequence; so it converges to $L \geq 0$. On the contrary, suppose that $L > 0$. By considering $\epsilon = L$, there exists $\delta > 0$ such that, for all $t \in [L, L + \delta)$, $\varphi(t) \leq \epsilon$. Since $\lim_{n \rightarrow \infty} a_n = L$, there exists $n_0 > 0$ such that, for all $n \geq n_0$, we have $L < a_n < L + \delta$. Therefore, $a_{n_0+1} < \varphi(a_{n_0}) \leq L$ and this is a contradiction to $L < a_n < L + \delta$. So $L = 0$.

(ϱ_2) Let $a_n, b_n \in (0, +\infty)$ be two sequences which converge to the same limit $L \geq 0$, $L < a_n$ hold, and $\varrho(a_n, b_n) > 0$ hold. On the contrary, assume $L > 0$. Therefore,

$$0 < \varrho(a_n, b_n) = \varphi(b_n) - a_n. \quad (30)$$

Also

$$L < a_n < \varphi(b_n) \leq b_n. \quad (31)$$

Since φ is L -function, taking $\epsilon = L > 0$, there exists $\delta > 0$ such that

$$\varphi(t) \leq \epsilon = L \quad \forall t \in [L, L + \delta]. \quad (32)$$

Since $\lim_{n \rightarrow \infty} b_n = L$, there exists $n_0 \in \mathbb{N}$ such that $L < b_{n_0} < L + \delta$. Applying (32), one deduces that $\varphi(b_{n_0}) \leq L$ which is a contradiction, because $\varphi(b_{n_0}) \leq L < a_{n_0} < \varphi(b_{n_0})$. So $L = 0$.

(ϱ_3) Let $\{a_n\}, \{b_n\} \subset (0, +\infty)$ be two sequences such that $\lim_{n \rightarrow \infty} b_n = 0$ and $\varrho(a_n, b_n) > 0$. Therefore,

$$0 < \varrho(a_n, b_n) = \varphi(b_n) - a_n \leq b_n - a_n, \quad (33)$$

so $0 < a_n < b_n$ which implies that $\lim_{n \rightarrow \infty} a_n = 0$.

(d) Suppose that ζ is a simulation function.

(ϱ_1) Let $\{a_n\} \subset (0, +\infty)$ be a sequence and let $\zeta(a_{n+1}, a_n) > 0$. Therefore $0 < \zeta(a_{n+1}, a_n) < a_n - a_{n+1}$. Hence $\{a_n\}$ is a strictly decreasing sequence so it converges to $L \geq 0$. On the contrary, suppose that $L > 0$. Let $t_n = a_{n+1}$ and $s_n = a_n$. Then $\{t_n\}$ and $\{s_n\}$ are convergent to L and $t_n < s_n$. By condition (ζ_3) we have

$$0 \leq \limsup_{n \rightarrow \infty} \zeta(a_{n+1}, a_n) = \limsup_{n \rightarrow \infty} \zeta(t_n, s_n) < 0, \quad (34)$$

and this is a contradiction. So $L = 0$.

(ϱ_2) Let $\{a_n\}, \{b_n\} \subset (0, +\infty)$ be two sequences which converge to the same limit $L \geq 0$, $L < a_n$, and $\zeta(a_n, b_n) > 0$. In order to prove that $L = 0$, assume that $L > 0$. On the other hand, (ζ_2) shows that $0 < \zeta(a_n, b_n) = b_n - a_n$ so $L < a_n < b_n$. Applying (ζ_3), one can yield that

$$0 \leq \limsup_{n \rightarrow \infty} \zeta(a_n, b_n) < 0, \quad (35)$$

which is a contradiction. So $L = 0$.

(ϱ_3) Let $\{a_n\}, \{b_n\} \subset (0, +\infty)$ be two sequences such that $\lim_{n \rightarrow \infty} b_n = 0$ and $\zeta(a_n, b_n) > 0$. Since ζ is a simulation function $0 < \zeta(a_n, b_n) < b_n - a_n$. Hence, $0 < a_n < b_n$, which means that $\lim_{n \rightarrow \infty} a_n = 0$.

(e) First of all, we show that, for $t, s \in [0, \infty)$,

$$\begin{aligned} s &> 0, \\ \varrho_{\psi, \varphi} &\geq 0 \implies \\ t &< s. \end{aligned} \quad (36)$$

Let $\varrho_{\psi, \varphi}(t, s) \geq 0$ and $t \geq s$; we are going to show that $s = 0$. As ψ is nondecreasing,

$$\psi(s) \leq \psi(t) \leq \psi(s) - \varphi(s) \leq \psi(s), \quad (37)$$

which implies that $\varphi(s) = 0$. Hence $s = 0$ and this is a contradiction. So we have $t < s$.

(ϱ_1) Let $\{a_n\} \subset (0, +\infty)$ be a sequence and let $\varrho_{\psi, \varphi}(a_{n+1}, a_n) > 0$. By (36), $a_{n+1} < a_n$. Hence $\{a_n\}$ is a strictly decreasing sequence so it converges to $L \geq 0$ where $L = \inf_{n \in \mathbb{N}} \{a_n\}$; hence $L < a_n$ for all $n \in \mathbb{N}$. On the contrary, suppose that $L > 0$. Therefore,

$$0 < \varrho_{\psi, \varphi}(a_{n+1}, a_n) = \psi(a_n) - \varphi(a_n) - \psi(a_{n+1}), \quad (38)$$

and so $0 \leq \varphi(a_n) < \psi(a_n) - \psi(a_{n+1})$. Since ψ is continuous from the right and $\lim_{n \rightarrow \infty} a_n = L$, we deduce that $\lim_{n \rightarrow \infty} \varphi(a_n) = 0$. Since φ is lower semicontinuous,

$$0 \leq \varphi(L) \leq \liminf_{r \rightarrow L} \varphi(r) = \lim_{n \rightarrow \infty} \varphi(a_n) = 0. \quad (39)$$

Hence $\varphi(L) = 0$, so $L = 0$.

(ϱ_2) Let $\{a_n\}, \{b_n\} \subset (0, +\infty)$ be two sequences which converge to the same limit $L \geq 0$, verifying $L < a_n$, and $\varrho_{\psi, \varphi}(a_n, b_n) > 0$. By (36) one can see $L < a_n < b_n$. In order to prove that $L = 0$, assume that $L > 0$. Therefore,

$$0 < \varrho_{\psi, \varphi}(a_n, b_n) = \psi(b_n) - \varphi(b_n) - \psi(a_n), \quad (40)$$

and so $0 \leq \varphi(b_n) < \psi(b_n) - \psi(a_n)$. As ψ is continuous from the right and $\lim_{n \rightarrow \infty} b_n = L$, we deduce that $\lim_{n \rightarrow \infty} \varphi(b_n) = 0$. Since φ is lower semicontinuous,

$$0 \leq \varphi(L) \leq \liminf_{r \rightarrow L} \varphi(r) = \lim_{n \rightarrow \infty} \varphi(b_n) = 0. \quad (41)$$

Hence, $\varphi(L) = 0$, so $L = 0$.

(ϱ_3) Let $\{a_n\}, \{b_n\} \subset (0, +\infty)$ be two sequences such that $\lim_{n \rightarrow \infty} b_n = 0$ and $\varrho_{\psi, \varphi}(a_n, b_n) > 0$. By (36), one can see $0 < a_n < b_n$ and this implies that $\lim_{n \rightarrow \infty} a_n = 0$.

(f)

(ϱ_1) Let $\{a_n\} \subset (0, +\infty)$ be a sequence and let $\varrho(a_{n+1}, a_n) > 0$. It means that

$$0 < \varrho(a_{n+1}, a_n) = \frac{a_n}{a_{n+1} + 1} - a_{n+1}. \quad (42)$$

Hence,

$$0 < a_{n+1} < \frac{a_n}{a_{n+1} + 1}. \quad (43)$$

Therefore,

$$0 < \frac{a_{n+1}}{a_n} < \frac{1}{a_{n+1} + 1} < 1. \quad (44)$$

Suppose that $\lim_{n \rightarrow \infty} a_n = r$. If $r > 0$, then by letting $n \rightarrow \infty$ we deduce that

$$0 < \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1 \leq \frac{1}{r + 1} < 1, \quad (45)$$

which is a contradiction. So $\lim_{n \rightarrow \infty} a_n = 0$.

(ϱ_2) Let $\{a_n\}, \{b_n\} \subset (0, +\infty)$ be two sequences which converge to the same limit $L \geq 0$, $L < a_n$, and $\varrho(a_n, b_n) > 0$, for all $n \in \mathbb{N}$. We have

$$0 < \varrho(a_n, b_n) = \frac{b_n}{a_n + 1} - a_n. \quad (46)$$

Therefore,

$$0 < a_n < \frac{b_n}{a_n + 1}. \quad (47)$$

Hence,

$$0 \leq L < a_n < \frac{b_n}{a_n + 1}. \quad (48)$$

Letting n tend to infinity, we deduce $0 \leq L \leq L/(L + 1)$, so $0 \leq L(L + 1) \leq L$ and hence $0 \leq L^2 \leq 0$. It means that $L = 0$.

(ϱ_3) Let $\{a_n\}, \{b_n\} \subset (0, +\infty)$ be two sequences such that $\{b_n\} \rightarrow 0$ and $\varrho(a_n, b_n) > 0$. Therefore,

$$0 < \varrho(a_n, b_n) = \frac{b_n}{a_n + 1} - a_n, \quad (49)$$

so $0 < a_n < b_n/(a_n + 1)$, which implies that $\{a_n\} \rightarrow 0$.

(g)

(ϱ_1) Let $\{a_n\} \subset (0, +\infty)$ be a sequence and let $\varrho(a_{n+1}, a_n) > 0$. It means that

$$0 < \varrho(a_{n+1}, a_n) = \frac{a_n}{e^{a_{n+1}}} - a_{n+1}. \quad (50)$$

Hence,

$$0 < a_{n+1} < \frac{a_n}{e^{a_{n+1}}}. \quad (51)$$

Therefore,

$$0 < \frac{a_{n+1}}{a_n} < \frac{1}{e^{a_{n+1}}} < 1. \quad (52)$$

Suppose that $\{a_n\} \rightarrow r$. If $r > 0$, then taking limit on both sides of (52), we deduce that

$$0 < \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1 \leq \frac{1}{e^r} < 1, \quad (53)$$

which is a contradiction. So $\{a_n\} \rightarrow 0$.

(ϱ_2) Let $\{a_n\}, \{b_n\} \subset (0, +\infty)$ be two sequences which converge to the same limit $L \geq 0$, $L < a_n$, and $\varrho(a_n, b_n) > 0$, for all $n \in \mathbb{N}$. On the contrary, suppose that $L > 0$. We have

$$0 < \varrho(a_n, b_n) = \frac{b_n}{e^{a_n}} - a_n. \quad (54)$$

Therefore,

$$0 < a_n < \frac{b_n}{e^{a_n}}. \quad (55)$$

Hence,

$$0 < L < a_n < \frac{b_n}{e^{a_n}}. \quad (56)$$

Letting $n \rightarrow \infty$, we deduce that $0 < L \leq L/e^L$, so $0 < Le^L \leq L$ and hence $0 < e^L \leq 1$ and this is a contradiction. So $L = 0$.

(ϱ_3) Let $\{a_n\}, \{b_n\} \subset (0, +\infty)$ be two sequences such that $\lim_{n \rightarrow \infty} b_n = 0$ and $\varrho(a_n, b_n) > 0$. Therefore,

$$0 < \varrho(a_n, b_n) = \frac{b_n}{e^{a_n}} - a_n, \quad (57)$$

so $0 < a_n < b_n/e^{a_n}$ which implies that $\{a_n\} \rightarrow 0$.

(h)

(ϱ_1) Let $\{a_n\} \subset (0, +\infty)$ be a sequence and let $\varrho(a_{n+1}, a_n) > 0$. It means that

$$0 < \varrho(a_{n+1}, a_n) = \ln(a_n + 1) - a_{n+1}. \quad (58)$$

Thus,

$$0 < a_{n+1} < \ln(a_n + 1). \quad (59)$$

Therefore,

$$0 < \frac{a_{n+1}}{a_n} < \frac{\ln(a_n + 1)}{a_n} < 1. \quad (60)$$

Suppose that $\lim_{n \rightarrow \infty} a_n = r$. If $r > 0$, then by letting $n \rightarrow \infty$, we deduce that

$$0 < \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1 \leq \frac{\ln(r+1)}{r} < 1, \quad (61)$$

which is a contradiction. So $\lim_{n \rightarrow \infty} a_n = 0$.

(Q_2) Let $\{a_n\}, \{b_n\} \subset (0, +\infty)$ be two sequences which converge to the same limit $L \geq 0$, $L < a_n$, and $\varrho(a_n, b_n) > 0$, for all $n \in \mathbb{N}$. We have

$$0 < \varrho(a_n, b_n) = \ln(b_n + 1) - a_n. \quad (62)$$

Therefore,

$$0 < a_n < \ln(b_n + 1). \quad (63)$$

Hence,

$$0 \leq L < a_n < \ln(b_n + 1). \quad (64)$$

Letting $n \rightarrow \infty$, we deduce that $0 \leq L \leq \ln(L+1)$ and so $L = 0$.

(Q_3) Let $\{a_n\}, \{b_n\} \subset (0, +\infty)$ be two sequences such that $\lim_{n \rightarrow \infty} b_n = 0$ and $\varrho(a_n, b_n) > 0$. Therefore,

$$0 < \varrho(a_n, b_n) = \ln(b_n + 1) - a_n, \quad (65)$$

so $0 < a_n < \ln(b_n + 1)$ which implies that $\lim_{n \rightarrow \infty} a_n = 0$. $\square$

The following corollary is the generalization of Reich's theorem (see [21, 24]).

Corollary 11. Let (X, d) be a complete metric space, let p be w -distance on X , and let $T : X \rightarrow X$ be a mapping such that

$$p(Tx, Ty) < \varphi(p(x, y)) p(x, y), \quad (66)$$

for all $x, y \in X$, where $\varphi : [0, +\infty) \rightarrow [0, 1)$ is a mapping such that $\limsup_{t \rightarrow s^+} \varphi(t) < 1$, for all $s \in (0, +\infty)$. Then, T has a unique fixed point.

Proof. Defining $\varrho(t, s) = s\varphi(s) - t$, applying (a) of example, and using Theorem 9, one can conclude desired result. $\square$

The following corollary is the generalization of Geraghty fixed point theorem (see [9, Theorem 1.3]).

Corollary 12. Let (X, d) be a complete metric space, let p be w -distance on X , and let $T : X \rightarrow X$ be a mapping such that

$$p(Tx, Ty) < \varphi(p(x, y)) p(x, y), \quad (67)$$

for all $x, y \in X$, where $\varphi : [0, +\infty) \rightarrow [0, 1)$ is a mapping such that, for all sequence $\{t_n\} \subset [0, +\infty)$,

$$\lim_{n \rightarrow \infty} \varphi(t_n) = 1 \quad (68)$$

implies $\lim_{n \rightarrow \infty} t_n = 0$.

Then, T has a unique fixed point.

Proof. Defining $\varrho(t, s) = s\varphi(s) - t$, applying (b) of example, and using Theorem 9, one can conclude desired result. $\square$

The following corollary is the generalization of Meir-Keeler fixed point theorem (also see [24, Theorem 1]).

Corollary 13. Let (X, d) be a complete metric space, let p be w -distance on X , and let $T : X \rightarrow X$ be a mapping such that

$$p(Tx, Ty) < \varphi(p(x, y)), \quad (69)$$

for all $x, y \in X$, where $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ is L -function in a sense of Definition 4. Then T has a unique fixed point.

Proof. Defining $\varrho(t, s) = \varphi(s) - t$, applying (c) of example, and using Theorem 9, one can conclude desired result. $\square$

The following corollary is the generalization of Khojasteh's result [19, Theorem 2.8] (see also [25]).

Corollary 14. Let (X, d) be a complete metric space, let p be w -distance on X , and let $T : X \rightarrow X$ be a mapping such that

$$\zeta(p(Tx, Ty), p(x, y)) \geq 0, \quad (70)$$

for all $x, y \in X$, where ζ is a simulation function. Then T has a unique fixed point.

Proof. Applying $\varrho(t, s) = \zeta(t, s)$, (d) of example and considering Theorem 9, one can conclude desired result. $\square$

Corollary 15. Let (X, d) be a complete metric space, let p be w -distance on X , and let $T : X \rightarrow X$ be a mapping such that

$$\psi(p(Tx, Ty)) < \psi(p(x, y)) - \varphi(p(x, y)), \quad (71)$$

for all $x, y \in X$, where $\psi, \varphi : [0, \infty) \rightarrow [0, \infty)$ are two functions such that ψ is nondecreasing and continuous from the right, φ is lower semicontinuous, and $\varphi^{-1}(\{0\}) = \{0\}$. Then, T has a unique fixed point.

Proof. Considering $\varrho_{\varphi, \psi}(t, s) = \psi(s) - \varphi(s) - \psi(t)$, (e) of example, and applying Theorem 9, one can conclude desired result. $\square$

In the following, we are going to present some new corollaries to show that our results are constructive and new results can be built by R -contractions.

Corollary 16. Let (X, d) be a metric space, let p be w -distance on X , and let $T : X \rightarrow X$ be a self-mapping. Let, for every $x, y \in X$, with $x \neq y$, the following hold:

$$p(Tx, Ty) < \frac{p(x, y)}{p(Tx, Ty) + 1}. \quad (72)$$

Then, T is R_p -contraction.

Proof. Defining $\varrho(t, s) = s/(t+1) - t$, applying (f) of example, and using Theorem 9, one can conclude desired result. $\square$

Corollary 17. Let (X, d) be a metric space, let p be w -distance on X , and let $T : X \rightarrow X$ be a self-mapping. Let, for every $x, y \in X$, with $x \neq y$, the following hold:

$$p(Tx, Ty) < \frac{p(x, y)}{e^{p(Tx, Ty)}}. \quad (73)$$

Then, T is R_p -contraction.

Proof. Considering $\varrho(t, s) = s/e^t - t$, applying (g) of example, and using Theorem 9, one can conclude desired result. $\square$

Corollary 18. Let (X, d) be a metric space, let p be w -distance on X , and let $T : X \rightarrow X$ be a self-mapping. Let for every $x, y \in X$, with $x \neq y$,

$$p(Tx, Ty) < \ln(p(x, y) + 1). \quad (74)$$

Then, T is R_p -contraction.

Proof. Defining $\varrho(t, s) = \ln(s + 1) - t$, applying (h) of example, and using Theorem 9, one can conclude desired result. $\square$

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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